Boundedly rational departure time choice in a dynamic continuum user equilibrium model for an urban city

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Abstract

Based on Wardrop's first principle, the perfectly rational dynamic user equilibrium is widely used to study dynamic traffic assignment problems. However, due to imperfect travel information and a certain "inertia" in decision-making, the boundedly rational dynamic user equilibrium is more suitable to describe realistic travel behavior. In this study, we consider the departure time choice problem incorporating the concept of bounded rationality. The continuum modeling approach is applied, in which the road network within the modeling region is assumed to be sufficiently dense and can be viewed as a continuum. We describe the traffic flow with the reactive dynamic continuum user equilibrium model and formulate the boundedly rational departure time problem as a variational inequality problem. We prove the existence of the solution to our boundedly rational reactive dynamic continuum user equilibrium model under particular assumptions and provide an intuitive and graphical illustration to demonstrate the non-uniqueness of the solution. Numerical examples are conducted to demonstrate the characteristics of this model and the non-uniqueness of the solution.

Key Words: dynamic continuum user equilibrium, bounded rationality, departure time choice, existence, uniqueness

1 Introduction

Since the pioneering work of Merchant (1978a,b), the problem of dynamic traffic assignment (DTA) has received much attention. The two fundamental components of DTA are the traffic flow component and the travel choice principle (Szeto and Lo, 2006). The traffic flow component describes the propagation of the traffic flow in the traffic system. The travel choice

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principle describes the traveler's propensity to travel, in which three major problems are considered: the route choice (RC), the departure time choice (DTC) and the simultaneous departure time and route choice (SDTRC). Based on different travel choice principles, the DTA problem can generally be classified into two categories: dynamic system optimal problems and dynamic user equilibrium (DUE) problems. In dynamic system optimal problems, all travelers in the traffic system cooperate and make their travel choices to minimize the total travel cost of the system. In DUE problems, if any two travelers have the same departure time, origin and destination, their travel costs should be equal and minimized. The Wardrop user equilibrium (Wardrop, 1952) is a commonly used travel choice principle that was originally proposed to study the static traffic assignment (STA) problem and then extended to the DTA problem.

In classical DUE problems, travelers make their travel choices in a perfectly rational way. However, many surveys and experiments (Nakayama, 2001; Zhu, 2010) show that people do not usually choose the least costly route or departure time due to the lack of perfect travel information and inability to obtain the optimal decision in a complex situation. Thus, the perfectly rational DUE is not entirely in line with real-life driving behavior and empirical observation, and therefore does not accurately describe realistic traffic flow.

Bounded rationality was developed to relax perfect rationality. The concept was first proposed by Simon (1957) to describe people when making a choice, and has subsequently become widely used in economics and psychology. Mahmassani and Chang (1987) were the first to use bounded rationality to describe travel behavior in the STA problem. In their study, they defined an indifference band that consists of a range of acceptable travel costs and established a boundedly rational user equilibrium (BR-UE) model. Since then, bounded rationality has been gradually incorporated into various static traffic problems, and many different BR-UE models for STA problems have been derived. For example, Sivak (2002) studied traffic safety by incorporating the concept of bounded rationality. Han and Timmermans (2006) studied traveler choice behavior under uncertainty and bounded rationality. Luo et al. (2010) investigated congestion pricing strategies in static networks under boundedly rational route choice behavior.

Di et al. (2013) formulated the BR-UE problem as a nonlinear complementary problem and constructed a solution set on a traffic network with fixed demand.

The concept of bounded rationality has also been widely used in DTA problems in recent decades. In the early studies, bounded rationality was simulated in laboratory experiments. However, these studies were imprecise due to the lack of a complete mathematical model. Later, Ridwan (2004) tried to apply the theory of fuzzy systems to the study of bounded rationality. Szeto and Lo (2006) proposed a mathematical model for route-choice boundedly rational dynamic user equilibrium (BR-DUE). Ge and Zhou (2012) considered the route-choice BR-DUE model with endogenously determined tolerances by allowing the width of the indifference band to depend on time and the actual path departure rates. Han et al. (2015) analyzed the simultaneous route-and-departure-time BR-DUE, and made significant contributions to the model formulation, analysis of existence, solution characterization and heuristic numerical computation of such problems. Di and Liu (2016) provided a comprehensive survey of the models of boundedly rational route choice behavior. They divided these models into two categories: substantive boundedly rational models (Di et al., 2013; Han et al., 2015) and procedural boundedly rational models (Zhu, 2010; Gao et al., 2011). Guo et al. (2018) considered the day-to-day departure time choice under bounded rationality in a bottleneck model. Yu et al. (2020) presented a double day-to-day DTA model with travel choices while incorporating incomplete and imperfect information as well as bounded rationality.

In the study of traffic equilibrium problems within transportation systems, two primary methods can be found in the literature: discrete modeling and continuum modeling. Discrete modeling is typically employed for in-depth planning and analysis of transportation systems (Sheffi, 1985). On the other hand, continuum modeling is utilized during the early stages of planning and modeling dense transportation systems in regional studies. In these cases, planners focus on the overall trends and patterns of user distribution, travel choices, and their adaptations to policy changes in the transportation system at a macroscopic level, rather than on specific details (Blumenfeld, 1977; Vaughan, 1987; Ho and Wong, 2007; Du et al., 2013).

During this initial phase, there is often a lack of sufficient data to establish a dense network for detailed analysis using discrete modeling. Gathering such data can be time-consuming and labor-intensive, and adequate resources are typically not available at this stage. In these situations, creating a detailed transportation network for analysis is challenging. From a regional perspective, land use and corresponding traffic demand are spatially continuous, which makes the continuum modeling approach more effective. This method assists urban and transport planners in exploring system design philosophies and concepts at the macroscopic level. Examples include catchment areas of competing facilities, cordon locations for congestion charging, additional central business district (CBD) locations, and transportation system expansion in various parts of the city.

Recently, the continuum modeling approach has been adapted for use in dynamic paradigms, where travelers are assumed to make perfectly rational travel decisions. Two equilibrium concepts have been considered in this context: reactive dynamic continuum user equilibrium (RDUE) and predictive dynamic continuum user equilibrium (PDUE). The RDUE model posits that travelers consistently select routes that minimize their instantaneous travel costs based on instantaneous traffic conditions, e.g., real-time travel information (Hughes, 2002; Huang et al., 2009; Yang et al., 2019). In contrast, the PDUE model assumes that travelers possess complete knowledge of the modeled domain and can accurately predict future traffic conditions. As a result, they choose routes that minimize their actual travel costs (Hoogendoorn and Bovy, 2004; Du et al., 2013; Yang et al., 2022).

However, the assumption of perfect rationality faces a similar challenge as the early stages of the discrete modeling approach, as empirical evidence suggests that travelers do not always make rational and optimal decisions in complex, real-world decision-making processes (Simon, 1957; Nakayama, 2001; Zhu, 2010). This highly restrictive assumption limits the effectiveness of continuum modeling in accurately describing complex travel decisions. As a result, it is crucial to examine the analytical properties of dynamic continuum models under the context of boundedly rational decisions. Such examination enhances the understanding of the formation of spatiotemporal congestion patterns, including peak demand levels and timings around sensitive areas and strategic facilities, such as central business districts (CBDs) and airports within cities, as well as their sensitivity to essential global modeling parameters at the regional level. This research aims to strengthen the theoretical foundation of the continuum modeling approach for urban and city planning and provide insights into the development of bounded rationality concepts in the discrete modeling approach, capitalizing on the rigorous mathematical study of relationships between essential parameters, and the existence and uniqueness properties of the solution, in transportation models at the macroscopic level facilitated by the continuum representation.

In this study, we make the first attempt to integrate the concept of bounded rationality into the continuum modeling approach. We formulate a boundedly rational departure time choice in reactive dynamic continuum user equilibrium (BR-DTC-RDUE-C) problem and rigorously investigate its theoretical properties. To illustrate our approach, we consider bounded rationality in departure time choice while incorporating the reactive dynamic continuum user equilibrium (RDUE-C) model to describe traffic flow. The density of travelers is governed by a conservation law, while travel direction (route choice) is determined by solving an eikonal equation. We first provide a mathematical definition of the BR-DUE principle and formulate an equivalent variational inequality. Next, we offer a mathematical analysis of the existence and uniqueness of the solution to our model. Although the existence of a solution can be proven under specific conditions, uniqueness is not guaranteed in all cases. Finally, we present numerical examples to demonstrate the characteristics of this model and the non-uniqueness of the solution.

This study makes three contributions. First, to the best of our knowledge, we make the first attempt to use the continuum modeling approach to study the BR-DUE problem and establish a related BR-DUE model. Second, we theoretically analyze the existence and uniqueness of the solution to the BR-DTC-RDUE-C model and show that the solution depends on the distribution of the indifference band. Third, we test several numerical examples and demonstrate the characteristics of this model and the non-uniqueness of its solution, such as the fact that the BR can ease traffic congestion.

The rest of this paper is organized as follows. Section 2 gives the formulation of the BR-DTC-DUE-C model. Section 3 considers the existence and uniqueness of the solution to the model. The solution algorithms are introduced in Section 4. Section 5 presents numerical examples to demonstrate the characteristics of the model and the solution. Finally, Section 6 presents our conclusions.

2 Model formulations

In this section, we describe the formulation of our model. Recall that there are two fundamental components of DTA, the traffic flow component and the travel choice principle. We introduce the formulation of the first component in Section 2.1. For travel choices, we first fix the departure time temporarily and consider the travel route choice in Section 2.2. Then, the concept of bounded rationality is considered in the departure time choice, and the BR-DT-DUE-C problem is discussed and proved in Sections 2.3 and 2.4. The complete DTA model is concluded in Section 2.5.

For the convenience of readers, Table 1 lists the notations frequently used in the paper.

2.1 Traffic flow models

We consider a two-dimensional bounded modeling region with a highly dense road network, such as an urban city. We use the continuum modeling approach to approximate the modeling region as a continuum $\Omega \subset R^2$ and assume that travelers can move freely within this continuum. Let $O \subset \Omega$ be the origin area and $D \subset \Omega$ be the destination area. Both O and D are assumed to be closed sets, i.e., the boundary is contained in the set, and travelers can use any point in O/D to enter/exit the modeling region Ω. Let Γ_i be the outer boundary of Ω and Γ_c be the boundary of D.

We denote the density of travelers at the location $(x, y) \in \Omega$ and at time $t \in [0, T]$ as

Table 1: List of notations

Notations			
Ω	modeling region	O	origin area
D	destination area	Γ_c	boundary of D
Γ_i	boundary of Ω	$[0, T]$	modeling time interval
ρ	density of travelers	$\boldsymbol{f}$	flow vector
f_1	flow flux in x direction	f_2	flow flux in y direction
$\boldsymbol{v}$	velocity vector	U	speed
u_1	speed in x direction	u_2	speed in y direction
q	travel demand	Q	total travel demand
Λ	feasible set of travel demand	ϕ	the instantaneous travel cost
c	local travel cost	I	travel time
p	schedule delay cost	l	instantaneous total cost
$\hat{l}$	minimum total cost	ϵ	tolerance
ϕ_a	actual travel cost	l_a	actual total cost
t_a	actual arrival time	Δ	measure of work start time flexibility
Φ	accumulated instantaneous travel cost	Ψ	accumulated instantaneous total cost
Φ_a	accumulated actual travel cost	Ψ_a	accumulated actual total cost
ρ_0	initial density	ϕ_c	cost of entering D
$\gamma_{1,2}$	parameters	κ	value of time
N_x	grid point in x direction	N_y	grid point in y direction
N_t	grid point in t direction		

$\rho(x, y, t)$, where $[0, T]$ is the modeling time interval. Similar to the mass conservation law in fluid dynamics, the density can be governed dynamically by the following conservation equation:

$$\rho_t(x, y, t) + \nabla \cdot \boldsymbol{f}(x, y, t) = q(x, y, t), \quad \forall (x, y) \in \Omega, \ t \in [0, T]. \tag{1}$$

Here $\rho_t = \partial\rho/\partial t$, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, $\boldsymbol{f}(x, y, t) = \big(f_1(x, y, t), f_2(x, y, t)\big)$ is the flow vector at location (x, y) at time t where $f_1(x, y, t)$ and $f_2(x, y, t)$ are the flow fluxes in x and y directions, respectively. This flow vector is defined as:

$$\boldsymbol{f}(x, y, t) = \rho(x, y, t)\boldsymbol{v}(x, y, t), \quad \forall (x, y) \in \Omega, \quad t \in [0, T]. \tag{2}$$

where $\boldsymbol{v} = (u_1(x, y, t), u_2(x, y, t))$ is the velocity vector at location (x, y) at time t and $u_1(x, y, t)$ and $u_2(x, y, t)$ are the speeds in the x and y directions, respectively. We denote the speed along

$\boldsymbol{v}$ at location (x, y) at time t as $U(x, y, t)$, which is the norm of the velocity vector, i.e.,

$$U(x, y, t) = \|\boldsymbol{v}(x, y, t)\|.$$

Thus, the corresponding flow intensity is defined as $\|\boldsymbol{f}\| = \rho U$. In this paper, we assume that the speed U depends on the density ρ and can be computed by a given speed-density relationship.

On the right-hand side of equation (1), $q(x, y, t)$ is the travel demand at location (x, y) and at time t, which is a time-varying non-negative square-integrable function of $\Omega \times [0, T]$. We denote the total travel demand at any location (x, y) as $Q(x, y)$, and thus

$$\int_0^T q(x, y, t)\mathrm{dt} = Q(x, y). \tag{3}$$

In this study, we assume that the total travel demand $Q(x, y)$ at any location (x, y) is fixed and given. However, as travelers may have different departure time choices, the corresponding function $q(x, y, t)$ is variable in t. We denote $\boldsymbol{q} = \{q(x, y, t),\ \forall (x, y) \in \Omega,\ t \in [0, T]\}$. Because the basics assumption (3) should be satisfied, we denote the feasible set of the travel demand $\boldsymbol{q}$ as

$$\Lambda = \left\{ \boldsymbol{q} \in L^2(\Omega \times [0, T]) :\ q(x, y, t) \geq 0,\ \int_0^T q(x, y, t)\mathrm{dt} = Q(x, y),\ \ \forall (x, y) \in \Omega \right\}. \tag{4}$$

To update the density dynamically according to equation (1), the following two travel choices need to be specified:

1. Based on the given origin-destination pairs, construct a suitable route-choice strategy to choose the moving direction of $\boldsymbol{v}(x, y, t)$. Then, the flux $\boldsymbol{f}(x, y, t)$ can be determined.

2. Choose the departure time and thus determine $\boldsymbol{q} \in \Lambda$.

It is obvious that these two travel choices influence each other. In the next sections, we introduce the departure time and route choice step by step.

2.2 Travel route choices

In this subsection, we assume for the moment that the traveler's departure time is fixed, i.e., the traffic demand $\boldsymbol{q}$ is given, and consider the route-choice strategies. When the given $\boldsymbol{q}$ changes, the route choice changes accordingly. We briefly review the formulation of the RDUE-C model (Huang et al., 2009; Yang et al., 2019).

When making route choices, travelers consider the travel cost and try to minimize it. We denote the travel cost potential incurred by a traveler who departs from (x, y) at time t to travel to destination D using the constructed path-choice strategy $\phi(x, y, t)$. In RDUE-C models, travelers always choose routes to minimize their instantaneous travel cost and change moving directions in a reactive manner. Here the travel cost ϕ is defined as the instantaneous travel cost and can be computed by the following eikonal equation:

$$||\nabla\phi(x, y, t)|| = c(x, y, t), \quad \forall(x, y) \in \Omega, \quad t \in [0, T], \tag{5}$$

where $c(x, y, t)$ is the local travel cost per unit distance of travel at location (x, y) at time t, which may depend on the traffic conditions ρ or the preferences of travelers. Furthermore, this equation is subject to the following boundary condition:

$$\phi(x, y, t) = \phi_c, \qquad \forall(x, y) \in \Gamma_c$$

where ϕ_c is the value of ϕ on the boundary of the destination and represents the cost incurred by the traveler when reaching the destination. According to Du et al. (2013), the related actual travel cost, denoted as ϕ_a, can be computed by the following time-dependent Hamilton-Jacobi (HJ) equation:

$$\frac{1}{U}(\phi_a)_t - (\phi_a)_x u_1 + (\phi_a)_y u_2 = -c. \tag{6}$$

where $(\phi_a)_t = \partial\phi_a(x, y, t)/\partial t$, $(\phi_a)_x = \partial\phi_a(x, y, t)/\partial x$,$(\phi_a)_y = \partial\phi_a(x, y, t)/\partial y$. Notice that both U and c may depend on the density ρ, and equations (5)/(6) should be solved together with equation (1). Hence, the travel cost ϕ also depends on the departure time choice $\boldsymbol{q}$, and the rigorous formula should be $\phi = \phi(x, y, t, \boldsymbol{q})$. However, to simplify the notations, we omit the term $\boldsymbol{q}$ in $\phi(x, y, t, \boldsymbol{q})$. The same rule applies to other variables, such as $\boldsymbol{v}$, ϕ_a and $\boldsymbol{f}$.

Yang et al. (2019) proved that the instantaneous travel cost is minimized and the RDUE principle can be achieved if the route-choice strategy satisfies the following requirement:

$$\boldsymbol{v}(x, y, t) // -\nabla\phi(x, y, t), \quad \forall (x, y) \in \Omega, \ \ t \in [0, T]. \tag{7}$$

where "//" indicates that the two vectors are parallel. Based on this route-choice strategy, the flow flux in the RDUE-C model can be computed by the following equation:

$$\boldsymbol{f}(x, y, t) = -\rho(x, y, t)U(x, y, t)\frac{\nabla\phi(x, y, t)}{\|\nabla\phi(x, y, t)\|}, \quad \forall (x, y) \in \Omega, \ t \in [0, T]. \tag{8}$$

2.3 Schedule delay cost

Once the route-choice strategy and the velocity vectors $\boldsymbol{v}(x, y, t)$ at each point are given by (7), the total travel time of a traveler who departs from any point (x, y) at any time t to reach the destination can be computed. In the RDUE-C model, we consider the instantaneous travel time, denoted as $I(x, y, t)$, which can be computed by

$$I_x u_1 + I_y u_2 = -1. \tag{9}$$

For any traveler who departs from point (x, y) at time t, the arrival time to the destination becomes

$$t_a(x, y, t) := t + I(x, y, t).$$

Let the time interval $[t^* - \triangle, t^* + \triangle]$ be the desired arrival time interval for all travelers, where t^* is the center of the period and $\triangle \geq 0$ is a measure of work start time flexibility. We introduce the following schedule delay cost $p(x, y, t)$, which describes the penalty for early or late arrival:

$$p(x, y, t) = \begin{cases} \gamma_1(t^* - \Delta - t_a(x, y, t)), & t_a(x, y, t) < t^* - \Delta, \\ 0, & t^* - \Delta \leq t_a(x, y, t) \leq t^* + \Delta, \\ \gamma_2(t_a(x, y, t) - (t^* + \Delta)), & t_a(x, y, t) > t^* + \Delta, \end{cases} \tag{10}$$

where $\gamma_1, \gamma_2 > 0$ are two given parameters in accordance with previous empirical results.

2.4 Boundedly rational departure time choice dynamic user equilibrium principle

In this subsection, we consider the departure time choice problem. Recall that the choice of $\boldsymbol{q}$ affects the route choice. For each given $\boldsymbol{q}$, we apply the route-choice strategy discussed in Section 2.2. We further denote the total cost of a traveler who departs from (x, y) at time t to travel to a destination using the corresponding route-choice strategy as $l(x, y, t, \boldsymbol{q})$, which is defined as the summation of the travel cost incurred during travel and the penalty based on the arrival time:

$$l(x, y, t, \boldsymbol{q}) := \phi(x, y, t) + p(x, y, t). \tag{11}$$

In turn, the total cost function l affects the choice of $\boldsymbol{q}$. In general, we seek a good choice for $\boldsymbol{q}^* \in \Lambda$ and the corresponding route-choice strategy such that the total cost $l(x, y, t, \boldsymbol{q}^*)$ is small. More rigorous definitions and formulations are given as follows.

We now define the minimum total cost for travelers in the modeling period. In our continuum model, the variables are defined in measure space; thus, we require the measure-theoretic analogue of the infimum of a set of numbers. For any measurable function $g : [0, T] \to R$, the essential infimum of $g(\cdot)$ on $[0, T]$ is defined by

$$\operatorname*{essinf}_{t\in[0,T]} \left\{g(t)\right\} = \sup\left\{z \in R : \nu(\{t \in [0, T] : g(t) < z\}) = 0\right\},$$

where $\nu(A)$ is the measure of set A. Thus, the essential infimum is the minimum value of $g(\cdot)$ over all t except a set with zero measure.

Thus, for any $\boldsymbol{q} \in \Lambda$, the minimum total cost $\hat{l}(x, y, \boldsymbol{q})$ for travelers at location (x, y) in the modeling period can be defined by the following equation:

$$\hat{l}\,(x, y, \boldsymbol{q}) = \operatorname*{essinf}_{t\in[0,T]} \left\{l\,(x, y, t, \boldsymbol{q})\right\}. \tag{12}$$

Before introducing the BR-DTC-RDUE-C model, we first give the definition of DTC-RDUE-C:

Definition 2.1. *The departure time choice dynamic user equilibrium is satisfied if for any* $(x, y) \in \Omega, t \in [0, T]$ *we have:*

$$l\,(x, y, t, \boldsymbol{q}^*) = \hat{l}\,(x, y, \boldsymbol{q}^*)\,, \quad \text{if } q^*(x, y, t) > 0, \tag{13}$$

i.e., for any location $(x, y) \in \Omega$*, if the travel demand (departure rate) is positive at time t, the instantaneous total cost is minimized.*

The above definition is established based on Wardrop's first principle in which travelers choose the departure time in a perfectly rational way. The concept of bounded rationality is a relaxation of perfect rationality. For BR-RDUE problems, we establish the following new definition.

Definition 2.2. *The boundedly rational departure time choice dynamic user equilibrium principle is satisfied if for any* $(x, y) \in \Omega,\ t \in [0,\ T]$*, we have:*

$$l\,(x, y, t, \boldsymbol{q}^*) \in \left[\hat{l}\,(x, y, \boldsymbol{q}^*)\,, \hat{l}\,(x, y, \boldsymbol{q}^*) + \epsilon(x, y, t)\right], \quad \text{if } q^*(x, y, t) > 0, \tag{14}$$

where $\epsilon(x, y, t)$ *is the tolerance function that represents the range of acceptable difference in the total cost to the traveler who departs from location* (x, y) *at time t.*

The BR-DTC-RDUE condition in equation (14) is defined to ensure that the total costs incurred by travelers who depart from the same place but at different times belong to an "indifference band." As with the DTC-RDUE problem, we try to find an equivalent solvable variational inequality (VI) for the BR-DTC-RDUE problem. We first define a new operator:

$$l^{\epsilon}(x, y, t, \boldsymbol{q}) = \max\left\{l(x, y, t, \boldsymbol{q}), \hat{l}\,(x, y, \boldsymbol{q}) + \epsilon(x, y, t)\right\} - \epsilon(x, y, t) + \operatorname*{essinf}_{t \in [0,T]}\{\epsilon(x, y, t)\}. \tag{15}$$

Then, for the BR-DTC-RDUE problem defined above, an equivalent VI formulation can be proved as in the following theorem.

Theorem 2.1. *The boundedly rational dynamic user equilibrium condition in* ***Definition 2.2*** *is equivalent to the following VI problem: Find* $\boldsymbol{q}^* \in \Lambda$*, such that for all* $\boldsymbol{q} \in \Lambda$*, we have*

$$\iint_{\Omega} \int_0^T l^{\epsilon}\,(x, y, t, \boldsymbol{q}^*)\,(q(x, y, t) - q^*(x, y, t))\,\mathrm{d}\nu(t)\mathrm{d}\Omega \geq 0. \tag{16}$$

Proof. (Necessity.) Suppose that $\boldsymbol{q}^* \in \Lambda$ is the travel demand that satisfies the boundedly rational dynamic user equilibrium condition (14). We define

$$\mu^\epsilon(x,y,\boldsymbol{q}^*) = \operatorname*{essinf}_{t\in[0,T]} \Big\{ l^\epsilon(x,y,t,\boldsymbol{q}^*) \Big\}. \tag{17}$$

According to the definition of $l^\epsilon(x,y,t,\boldsymbol{q}^*)$, it is obvious that

$$\begin{aligned}
\mu^\epsilon(x,y,\boldsymbol{q}^*) &= \operatorname*{essinf}_{t\in[0,T]} \left\{ \max\{\, l(x,y,t,\boldsymbol{q}^*), \hat{l}\,(x,y,\boldsymbol{q}^*) + \epsilon(x,y,t)\} - \epsilon(x,y,t) + \operatorname*{essinf}_{t\in[0,T]} \{\epsilon(x,y,t)\} \right\}, \\
&\geq \operatorname*{essinf}_{t\in[0,T]} \left\{ \hat{l}\,(x,y,\boldsymbol{q}^*) + \epsilon(x,y,t) - \epsilon(x,y,t) + \operatorname*{essinf}_{t\in[0,T]} \{\epsilon(x,y,t)\} \right\}, \\
&= \operatorname*{essinf}_{t\in[0,T]} \left\{ \hat{l}\,(x,y,\boldsymbol{q}^*) + \operatorname*{essinf}_{t\in[0,T]} \{\epsilon(x,y,t)\} \right\}, \\
&= \hat{l}\,(x,y,\boldsymbol{q}^*) + \operatorname*{essinf}_{t\in[0,T]} \{\epsilon(x,y,t)\}.
\end{aligned} \tag{18}$$

However, according to the definition of $\hat{l}\,(x,y,\boldsymbol{q})$ and the fact that the set $[0,T]$ is closed, there exists $t_1 \in T$ such that $l(x,y,t_1,\boldsymbol{q}^*) = \hat{l}\,(x,y,\boldsymbol{q}^*)$. Then we have

$$l^\epsilon(x,y,t_1,\boldsymbol{q}^*) = \hat{l}\,(x,y,\boldsymbol{q}^*) + \operatorname*{essinf}_{t\in[0,T]} \{\, \epsilon(x,y,t)\}. \tag{19}$$

Combining equations (18) and (19), we conclude that

$$\mu^\epsilon(x,y,\boldsymbol{q}^*) = \hat{l}(x,y,\boldsymbol{q}^*) + \operatorname*{essinf}_{t\in[0,T]} \{\, \epsilon(x,y,t)\}. \tag{20}$$

For any feasible travel demand $\boldsymbol{q} \in \Lambda$, any location $(x,y) \in \Omega$ and time $t \in [0,T]$, if $q(x,y,t) - q^*(x,y,t) < 0$, we have

$$\begin{aligned}
0 \leq q(x,y,t) < q^*(x,y,t) \;\Rightarrow\; & l(x,y,t,\boldsymbol{q}^*) \leq \hat{l}(x,y,\boldsymbol{q}^*) + \epsilon(x,y,t) \\
\Rightarrow\; & l^\epsilon(x,y,t,\boldsymbol{q}^*) = \mu^\epsilon(x,y,\boldsymbol{q}^*).
\end{aligned} \tag{21}$$

With the above equation and the fact that $l^\epsilon(x,y,t,\boldsymbol{q}^*) - \mu^\epsilon(x,y,\boldsymbol{q}^*) \geq 0$, the following equation is satisfied:

$$\left[l^\epsilon(x,y,t,\boldsymbol{q}^*) - \mu^\epsilon(x,y,\boldsymbol{q}^*)\right]\left[q(x,y,t) - q^*(x,y,t)\right] \geq 0. \tag{22}$$

Integrating the above equation over space and time, we get

$$\iint_{\Omega}\int_0^T \left(l^{\epsilon}(x,y,t,\boldsymbol{q}^*) - \mu^{\epsilon}(x,y,\boldsymbol{q}^*)\right)\left(q(x,y,t) - q^*(x,y,t)\right) d\nu(t) d\Omega \geq 0. \tag{23}$$

As $\mu^{\epsilon}(x,y,\boldsymbol{q}^*)$ is independent of time, and the travel demand satisfies

$$\int_0^T \left(q(x,y,t) - q^*(x,y,t)\right) d\nu(t) = 0,$$

we have

$$\iint_{\Omega}\int_0^T \mu^{\epsilon}(x,y,\boldsymbol{q}^*)\left(q(x,y,t) - q^*(x,y,t)\right) d\nu(t) d\Omega = 0.$$

Thus, equation (23) reduces to

$$\iint_{\Omega}\int_0^T l^{\epsilon}(x,y,t,\boldsymbol{q}^*)\left(q(x,y,t) - q^*(x,y,t)\right) d\nu(t) d\Omega \geq 0,$$

so the VI equation (16) follows.

(Sufficiency.) Let $\boldsymbol{q}^*$ be a solution for the VI problem. For the case $q^*(x,y,t) > 0$, we first claim that

$$\forall\, q^*(x,y,t) > 0, \;\; l^{\epsilon}(x,y,t,\boldsymbol{q}^*) = \mu^{\epsilon}(x,y,\boldsymbol{q}^*). \tag{24}$$

We can prove the above claim by contradiction. Assume that this claim is not satisfied for the following set,

$$S_1 = \left\{(x,y,t) \in \Omega \times [0,T] : \; q^*(x,y,t) > 0, \; l^{\epsilon}(x,y,t,\boldsymbol{q}^*) - \mu^{\epsilon}(x,y,\boldsymbol{q}^*) > 0\right\}. \tag{25}$$

and this set has a positive measure. AS $l^{\epsilon}(x,y,t,\boldsymbol{q}^*) - \mu^{\epsilon}(x,y,\boldsymbol{q}^*)$ is a measurable function, there exists a sufficiently small value $\varepsilon > 0$ such that the subset

$$S_1(\varepsilon) = \left\{(x,y,t) \in S_1 : \; l^{\epsilon}(x,y,t,\boldsymbol{q}^*) - \mu^{\epsilon}(x,y,\boldsymbol{q}^*) > 2\varepsilon\right\} \tag{26}$$

has a positive measure. Again, as $q^*(x,y,t)$ is a measurable function, there exists a sufficiently small value $\delta > 0$ such that the subset

$$S_1(\varepsilon,\delta) = \left\{(x,y,t) \in S_1(\varepsilon) : \; q^*(x,y,t) > \delta\right\} \tag{27}$$

has a positive measure. Then, we can find a subset $\Omega_a \times T_a \subset S_1(\varepsilon, \delta)$ with $\nu(\Omega_a) \neq 0$ and $\nu(T_a) \neq 0$. According to the definition of $\mu^\epsilon(x, y, \boldsymbol{q}^*)$, the subset

$$S_2(\varepsilon) = \left\{(x, y, t) : \ (x, y) \in \Omega_a, \ l^\epsilon(x, y, t, \boldsymbol{q}^*) < \mu^\epsilon(x, y, \boldsymbol{q}^*) + \varepsilon\right\} \tag{28}$$

also has a positive measure. Similarly, we also can find subset $\Omega_b \times T_b \subset S_2(\varepsilon)$ with $\nu(\Omega_b) \neq 0$ and $\nu(T_b) \neq 0$, and note that $\Omega_b \subset \Omega_a$ and $T_a \cap T_b = \varnothing$. Letting

$$\alpha_0 = \min\{\nu(T_a)), \ \nu(T_b)\}, \tag{29}$$

for any $\alpha \in (0, \alpha_0)$, we can find the subsets $T_a(\alpha) \subset T_a$ and $T_b(\alpha) \subset T_b$ with $\nu(T_a(\alpha)) = \nu(T_b(\alpha)) = \alpha$.

A special $\boldsymbol{q} \in \Lambda$ that contradicts equation (16) can be constructed as follows:

$$q(x, y, t) = \begin{cases} q^*(x, y, t) - \delta, & \forall (x, y) \in \Omega_b, \ t \in T_a(\alpha), \\ q^*(x, y, t) + \delta, & \forall (x, y) \in \Omega_b, \ t \in T_b(\alpha), \\ q^*(x, y, t), & \text{otherwise}. \end{cases} \tag{30}$$

If $(x, y) \in \Omega_b$ and $t \in T_a(\alpha)$, from equations (27) and (30) it can be shown that

$$q^*(x, y, t) > \delta \Rightarrow q(x, y, t) = q^*(x, y, t) - \delta > 0. \tag{31}$$

If $(x, y) \in \Omega_b$ and $t \in T_b(\alpha)$, from equations (28) and (30) it can be shown that

$$q^*(x, y, t) > 0 \Rightarrow q(x, y, t) = q^*(x, y, t) + \delta > 0. \tag{32}$$

Otherwise, from equation (30), we have

$$q(x, y, t) = q^*(x, y, t) \geq 0. \tag{33}$$

Moreover, for any $(x, y) \in \Omega$,

$$\begin{aligned} \int_0^T q(x, y, t)\mathrm{d}t &= \int_{[0,T]\backslash T_a(\alpha)\backslash T_b(\alpha)} q(x, y, t)\mathrm{d}t + \int_{T_a(\alpha)} q(x, y, t)\mathrm{d}t + \int_{T_b(\alpha)} q(x, y, t)\mathrm{d}t \\ &= \int_{[0,T]\backslash T_a(\alpha)\backslash T_b(\alpha)} q^*(x, y, t)\mathrm{d}t + \int_{T_a(\alpha)} \left[q^*(x, y, t) - \delta\right]\mathrm{d}t + \int_{T_b(\alpha)} \left[q^*(x, y, t) + \delta\right]\mathrm{d}t \\ &= Q(x, y). \end{aligned} \tag{34}$$

Thus, $q(x, y, t)$ constructed in equation (30) is within the feasible set Λ. Then, with the constructed $q(x, y, t)$ in equation (30), we obtain:

$$\begin{aligned}
&\iint_{\Omega}\int_0^T l^{\epsilon}(x, y, t, \boldsymbol{q}^*)(q(x, y, t) - q^*(x, y, t))dtd\Omega \\
&= \iint_{\Omega_b}\int_{T_a(\alpha)\cup T_b(\alpha)} l^{\epsilon}(x, y, t, \boldsymbol{q}^*)(q(x, y, t) - q^*(x, y, t))dtd\Omega \\
&= \iint_{\Omega_b}\left[\int_{T_a(\alpha)} -\delta l^{\epsilon}(x, y, t, \boldsymbol{q}^*)dt + \int_{T_b(\alpha)} \delta l^{\epsilon}(x, y, t, \boldsymbol{q}^*)dt\right]d\Omega \\
&\leq \iint_{\Omega_b}\left[\int_{T_a(\alpha)} -\delta(\mu^{\epsilon}(x, y, \boldsymbol{q}^*) + 2\varepsilon))dt + \int_{T_b(\alpha)} \delta(\mu^{\epsilon}(x, y, \boldsymbol{q}^*) + \varepsilon))dt\right]d\Omega \\
&= \iint_{\Omega_b}(-\delta\alpha(\mu^{\epsilon}(x, y, \boldsymbol{q}^*) + 2\varepsilon)) + \delta\alpha(\mu^{\epsilon}(x, y, \boldsymbol{q}^*) + \varepsilon))d\Omega \\
&= \iint_{\Omega_a}(-\delta\alpha)\varepsilon)d\Omega \\
&= -\delta\alpha|\Omega_b|\varepsilon < 0.
\end{aligned} \tag{35}$$

This contradicts (16) and thus the $q(x, y, t) > 0$ case in equation (24) is proved by contradiction.

Now from equation (24), we have

$$\begin{aligned}
&q^*(x, y, t) > 0 \Rightarrow l^{\epsilon}(x, y, t, \boldsymbol{q}^*) = \mu^{\epsilon}(x, y, \boldsymbol{q}^*) \\
&\Rightarrow \max\left\{ l(x, y, t, \boldsymbol{q}^*), \hat{l}(x, y, \boldsymbol{q}^*) + \epsilon(x, y, t)\right\} - \epsilon(x, y, t) + \min_{s\in[0,T]}\{\epsilon(x, y, s)\} = \mu^{\epsilon}(x, y, \boldsymbol{q}^*) \\
&\Rightarrow \max\left\{ l(x, y, t, \boldsymbol{q}^*), \hat{l}(x, y, \boldsymbol{q}^*) + \epsilon(x, y, t)\right\} - \epsilon(x, y, t) + \min_{s\in[0,T]}\{\epsilon(x, y, s)\} \\
&\qquad = \hat{l}(x, y, \boldsymbol{q}^*) + \operatorname*{essinf}_{t\in[0,T]}\left\{\epsilon(x, y, t)\right\} \\
&\Rightarrow \max\left\{ l(x, y, t, \boldsymbol{q}^*), \hat{l}(x, y, \boldsymbol{q}^*) + \epsilon(x, y, t)\right\} - \epsilon(x, y, t) = \hat{l}(x, y, \boldsymbol{q}^*) \\
&\Rightarrow \hat{l}(x, y, \boldsymbol{q}^*) \leq l(x, y, t, \boldsymbol{q}^*) \leq \hat{l}(x, y, \boldsymbol{q}^*) + \epsilon(x, y, t).
\end{aligned} \tag{36}$$

Thus, $\boldsymbol{q}^* \in \Lambda$ satisfies the dynamic user equilibrium condition with the departure time choice consideration defined in equation (14). □

2.5 The complete model

Combining the traffic flow equations and travel choices strategies discussed in the previous sections, the complete BR-DTC-RDUE model becomes: find $\boldsymbol{q}^* \in \Lambda$, such that for all $\boldsymbol{q} \in \Lambda$, we have

$$\iint_{\Omega}\int_0^T l^{\epsilon}(x, y, t, \boldsymbol{q}^*)\,(q(x, y, t) - q^*(x, y, t))\,\mathrm{d}\nu(t)\mathrm{d}\Omega \geq 0, \tag{37}$$

where the operator $l^{\epsilon}(x, y, t, \boldsymbol{q}^{*})$ is defined in (15) and can be obtained by solving the following RDUE-C problem:

$$
\begin{cases}
\rho_t(x, y, t) + \nabla \cdot \boldsymbol{f}(x, y, t) = q^{*}(x, y, t) & \forall (x, y) \in \Omega, t \in [0, T], \\
\boldsymbol{f}(x, y, t) = -\rho(x, y, t) U(x, y, t) \dfrac{\nabla \phi(x, y, t)}{\|\nabla \phi(x, y, t)\|} & \forall (x, y) \in \Omega, t \in [0, T], \\
\rho(x, y, 0) = \rho_0(x, y) & \forall (x, y) \in \Omega.
\end{cases} \tag{38}
$$

In RDUE problems, ϕ can be computed by solving the following static eikonal equation at each fixed time level of (38)

$$
\begin{cases}
\|\nabla \phi(x, y, t)\| = c(x, y, t), & \forall (x, y) \in \Omega, \quad t \in [0, T], \\
\phi(x, y, t) = \phi_c, & \forall (x, y) \in \Gamma_c.
\end{cases} \tag{39}
$$

3 The existence and uniqueness of the solution to the BR-DTC-RDUE-C problem

In this section, we consider the existence and uniqueness of the solution to the BR-DTC-RDUE-C problem, which has two parts: the existence and uniqueness of the solution to the RDUE-C model, and the existence and uniqueness of the solution to the VI problem. Section 3.1 studies the existence of the solution of the RDUE-C model under certain conditions of travel demand and the initial condition. In Section 3.2, we show that the operator l^{ϵ} is well-defined and continuous with respect to travel demand, and then prove the existence of the solution to the VI problem. Finally, Section 3.3 provides an example to show that the uniqueness of the BR-DTC-RDUE-C model does not hold.

3.1 The existence of a solution to the RDUE-C model

In this subsection we consider the RDUE-C model which consists of a conservation law (CL) part (38) and an eikonal equation part (39). For simplicity, we define the travel cost as the value of the travel time. In this case, the local cost c can be defined as

$$
c(x, y, t) = \frac{\kappa}{U(x, y, t)},
$$

where $1/U$ is the travel time per unit of distance and κ represents the value of a unit of time.

The eikonal equation is a special static HJ equation. In the following, we first give the definition of semi-concave, the most fundamental regularity property of the HJ equation solution, and then show the existence of a solution to the eikonal equation using a theorem.

Definition 3.1. *A map $w : E \to R$, with E being open and convex, is semi-concave if there is a constant C, such that one of the following conditions is satisfied:*

1. *$D^2 w \leq C I_d$ in the sense of distribution,*
2. *$\langle \boldsymbol{p} - \boldsymbol{q}, \boldsymbol{x} - \boldsymbol{y} \rangle \leq C|\boldsymbol{x} - \boldsymbol{y}|^2$ for any $\boldsymbol{x}, \boldsymbol{y} \in E, \boldsymbol{p} \in D^+_{\boldsymbol{x}} w(\boldsymbol{x})$ and $\boldsymbol{q} \in D^+_{\boldsymbol{x}} w(\boldsymbol{y})$, where $D^+_{\boldsymbol{x}}$ denotes the super-differential of w with respect to the variable $\boldsymbol{x}$, defined by*
$$D^+_{\boldsymbol{x}} w(\boldsymbol{x}) = \Big\{ \boldsymbol{p} \in R^2 : \limsup_{\boldsymbol{y} \to \boldsymbol{x}} \frac{w(\boldsymbol{y}) - w(\boldsymbol{x}) - \langle \boldsymbol{p}, \boldsymbol{y} - \boldsymbol{x} \rangle}{|\boldsymbol{y} - \boldsymbol{x}|} \leq 0 \Big\}. \tag{40}$$
where $D^2 w$ is the second-order derivative of w and $\langle \cdot, \cdot \rangle$ is the inner product.

Theorem 3.1. *If $c \in W^{1,\infty}(\Omega)$ is semi-concave and $\phi_c \in C(\Omega)$, then there exists a viscosity solution ϕ to eikonal equation (39), and $\phi \in W^{1,\infty}(\Omega)$ is semi-concave.*

Proof. See Lions (1982). □

Next, we consider the existence of the solution to the following linear CL:
$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\boldsymbol{A} \rho) = B(\boldsymbol{x}, t), & \forall \boldsymbol{x} \in \Omega,\ \forall t \in [0, T] \\ \rho(\boldsymbol{x}, 0) = \rho_0(\boldsymbol{x}), & \forall \boldsymbol{x} \in \Omega \end{cases} \tag{41}$$
where $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{A}(\boldsymbol{x}, t) = (A_1(\boldsymbol{x}, t), A_2(\boldsymbol{x}, t))$. We assume that $A_1(\cdot, t)$ and $A_2(\cdot, t)$ satisfy the following assumptions:

1. $A_i(\cdot, t),\ i = 1, 2$ is bounded for almost every t, i.e., for almost every t, there is a constant C, such that
$$|A_i(\boldsymbol{x}, t)| \leq C,\ \ \forall \boldsymbol{x} \in \Omega,\ \ i = 1, 2. \tag{42}$$
2. $\boldsymbol{A}(\boldsymbol{x}, t)$ satisfies the one-sided Lipschitz condition
$$\langle \boldsymbol{A}(\boldsymbol{x}, t) - \boldsymbol{A}(\boldsymbol{y}, t), \boldsymbol{x} - \boldsymbol{y} \rangle \geq -m(t)|\boldsymbol{x} - \boldsymbol{y}|^2,\ \ \forall \boldsymbol{x}, \boldsymbol{y} \in \Omega, \tag{43}$$
where $m \in L^1[0, T]$, $m(t) \geq 0$ a.e. in $[0, T]$, and $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_2 y_2$ and $|\boldsymbol{x}|^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$.

Next, we review the theorem on the existence of the solution to the above linear CL from Conway (1967),Petrova and Popov (1999) and Bouchut and Crippa (2006).

Theorem 3.2. *In bounded domain Ω, if $\boldsymbol{A}$ satisfies the above assumptions, $\rho_0(\boldsymbol{x}) \in L^2(\Omega)$, and $B(\boldsymbol{x}, t) \in L^2(\Omega \times [0, T])$, then there exists a solution $\rho \in L^2(\Omega \times [0, T])$ to equation (41).*

Now, we are ready to prove the existence of a solution to CL (38).

Theorem 3.3. *The conservation law (38) in our RDUE-C model has a solution $\rho \in L^2(\Omega \times [0, T])$ if the speed is bounded and smooth.*

Proof. To prove this theorem, we only need to show that the coefficients A_1 and A_2 in our model satisfy the conditions (42) and (43). In our problem, we have

$$A(\boldsymbol{x}, t) = -\frac{U(\boldsymbol{x}, t)}{||\nabla\phi(\boldsymbol{x}, t)||}\nabla\phi(\boldsymbol{x}, t) = -U^2(\boldsymbol{x}, t)\nabla\phi(\boldsymbol{x}, t)/\kappa. \tag{44}$$

From the analysis of the eikonal equation, we know that ϕ is Lipschitz continuous and semi-concave. Thus, condition (42) is satisfied. From the equivalent definition of semi-concavity, we have

$$\langle \boldsymbol{p} - \boldsymbol{q}, \boldsymbol{x} - \boldsymbol{y}\rangle \leq C|\boldsymbol{x} - \boldsymbol{y}|^2, \tag{45}$$

where $C > 0$, $\boldsymbol{p} \in D^+_{\boldsymbol{x}}\phi(\boldsymbol{x}, t)$, and $\boldsymbol{q} \in D^+_{\boldsymbol{x}}(\boldsymbol{y}, t)$ as $\nabla\phi(\boldsymbol{x}, t) \in D^+_{\boldsymbol{x}}(\boldsymbol{x}, t)$. We then have

$$\langle \nabla\phi(\boldsymbol{x}, t) - \nabla\phi(\boldsymbol{y}, t), \boldsymbol{x} - \boldsymbol{y}\rangle \leq C|x - y|^2. \tag{46}$$

Because $A(\boldsymbol{x}, t) = -U^2(\boldsymbol{x}, t)\nabla\phi(\boldsymbol{x}, t)/\kappa$, we obtain

$$\begin{aligned}
&\langle \boldsymbol{A}(\boldsymbol{x}, t) - \boldsymbol{A}(\boldsymbol{y}, t), \boldsymbol{x} - \boldsymbol{y}\rangle = \langle -U^2(\boldsymbol{x}, t)\nabla\phi(\boldsymbol{x}, t)/\kappa - U^2(\boldsymbol{y}, t)\nabla\phi(\boldsymbol{y}, t)/\kappa, \boldsymbol{x} - \boldsymbol{y}\rangle \\
&= \frac{-U^2(\boldsymbol{x}, t)}{\kappa}\langle \nabla\phi(\boldsymbol{x}, t), \boldsymbol{x} - \boldsymbol{y}\rangle + \frac{U^2(\boldsymbol{y}, t)}{\kappa}\langle \nabla\phi(\boldsymbol{y}, t), \boldsymbol{x} - \boldsymbol{y}\rangle \\
&= \frac{-U^2(\boldsymbol{x}, t)}{\kappa}\langle \nabla\phi(\boldsymbol{x}, t) - \nabla\phi(\boldsymbol{y}, t), \boldsymbol{x} - \boldsymbol{y}\rangle - (\frac{U^2(\boldsymbol{x}, t)}{\kappa} - \frac{U^2(\boldsymbol{y}, t)}{\kappa})\langle \nabla\phi(\boldsymbol{y}, t), \boldsymbol{x} - \boldsymbol{y}\rangle \\
&\geq -\frac{U^2(\boldsymbol{x}, t)}{\kappa}C|\boldsymbol{x} - \boldsymbol{y}|^2 - (\frac{U^2(\boldsymbol{x}, t)}{\kappa} - \frac{U^2(\boldsymbol{y}, t)}{\kappa})\langle \nabla\phi(\boldsymbol{y}, t), \boldsymbol{x} - \boldsymbol{y}\rangle \\
&\geq -C^{'}|\boldsymbol{x} - \boldsymbol{y}|^2,
\end{aligned} \tag{47}$$

where the last inequality holds because the speed $U(x, t)$ is bounded and smooth and $\phi \in W^{1,\infty}(\Omega)$. Thus, the one-sided Lipschitz condition holds for $m(t) = C^{'}$. According to Theorem 3.2, the conservation law (38) has the solution $\rho \in L^2(\Omega \times [0, T])$; thus, the theorem holds. □

3.2 The existence of a solution to the VI problem

In this subsection, we show the existence of a solution to the VI problem. The first step is to show that the operator l^ϵ is continuous with respect to travel demand $\boldsymbol{q}$.

From the analysis of the eikonal equation, the solution $\phi \in W^{1,\infty}$ belongs to space $L^2(\Omega \times [0, T])$. Because the penalty cost is a piece-wise linear function of ϕ, it is obvious that the total cost $l(\cdot, \boldsymbol{q})$ belongs to space $L^2(\Omega \times [0, T])$. We next show that operator $l^\epsilon(\cdot, \boldsymbol{q})$ also belongs to space $L^2(\Omega \times [0, T])$.

Lemma 3.1. *If the tolerance $\epsilon(x, y, t)$ is uniformly bounded, i.e., there exists a constant C such that*

$$0 \leq \sup\left\{\epsilon(x, y, t : \ \forall (x, y) \in \Omega, \ \forall t \in [0, T]\right\} \leq C, \tag{48}$$

then the operator $l^\epsilon(\cdot, \boldsymbol{q})$ belongs to space $L^2(\Omega \times [0, T])$.

Proof. According to the definition of l^ϵ, we have

$$\begin{aligned}
\int_\Omega \int_0^T [l^\epsilon(x, y, t)]^2 dt d\Omega &\leq \int_\Omega \int_0^T [l(x, y, t) + C]^2 dt d\Omega \\
&= \int_\Omega \int_0^T ([l(x, y, t)]^2 + 2Cl(x, y, t) + C^2) dt d\Omega \\
&\leq +\infty.
\end{aligned} \tag{49}$$

Therefore, $l^\epsilon(x, y, t) \in L^2(\Omega \times [0, T])$ □

Theorem 3.4. *If the tolerance ϵ is continuous on $\Omega \times [0, T]$, then the operator l^ϵ*

$$l^\epsilon : \Lambda \to L^2(\Omega \times [0, T]), \quad \boldsymbol{q} \to l^\epsilon(\cdot, \boldsymbol{q}), \tag{50}$$

is continuous.

Proof. Firstly, if the travel demand $\boldsymbol{q}^1, \boldsymbol{q}^2 \in \Lambda$ and $||\boldsymbol{q}^1 - \boldsymbol{q}^2||_{L^2(\Omega \times [0,T])} \to 0$, we claim that

$$||\hat{l}^\epsilon(\cdot, \boldsymbol{q}^1) - \hat{l}^\epsilon(\cdot, \boldsymbol{q}^2)||_{L^2(\Omega)} \to 0. \tag{51}$$

According to the stability and regularity of the eikonal equation, the travel cost ϕ is continuous with respect to $\boldsymbol{q}$ with respect to the L^2 norm, and $\phi(x, y, t)$ is Lipschitz continuous with respect to variables x, y, t, respectively. According to the definition of $l(x, y, t, \boldsymbol{q})$ where the penalty cost function is a piece-wise linear continuous function, the operator $l(\cdot, \boldsymbol{q})$ is continuous, and $l(x, y, t, \boldsymbol{q})$ is Lipschitz continuous with respect to variables x, y, t.

Thus, there exists a constant $C > 0$, such that

$$|l(x, y, t, \boldsymbol{q}) - l(x_1, y_1, t_1, \boldsymbol{q})| \leq C(|x - x_1| + |y - y_1| + |t - t_1|). \tag{52}$$

Given any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\boldsymbol{q}^1, \boldsymbol{q}^2 \in \Lambda$

$$||\boldsymbol{q}^1 - \boldsymbol{q}^2||_{L^2(\Omega\times[0,T])} < \delta \ \rightarrow \ ||l(\cdot, \boldsymbol{q}^1) - l(\cdot, \boldsymbol{q}^2)||_{L^2(\Omega\times[0,T])} < \frac{\varepsilon^{5/2}}{\sqrt{532C^3}}. \tag{53}$$

Next, we show that for any $(x, y, t) \in \Omega \times [0, T]$, the following inequality holds:

$$|l(x, y, t, \boldsymbol{q}^1) - l(x, y, t, \boldsymbol{q}^2)| < \varepsilon. \tag{54}$$

Without loss of generality, we assume $l(x, y, x, \boldsymbol{q}^1) \leq l(x, y, x, \boldsymbol{q}^2)$, and prove it by contradiction. Assume that

$$l(x, y, t, \boldsymbol{q}^1) < l(x, y, t, \boldsymbol{q}^2) - \varepsilon. \tag{55}$$

Then for any $(x^{'}, y^{'}, t^{'}) \in [x - \frac{\varepsilon}{6C}, x] \times [y - \frac{\varepsilon}{6C}, y] \times [t - \frac{\varepsilon}{6C}, t]$, according to equation (47),

$$\begin{aligned} l(x^{'}, y^{'}, t^{'}, \boldsymbol{q}^1) &\leq l(x, y, t, \boldsymbol{q}^1) + C(|x - x^{'}| + |y - y^{'}| + |t - t^{'}|), \\ &< l(x, y, t, \boldsymbol{q}^2) - \varepsilon + C(|x - x^{'}| + |y - y^{'}| + |t - t^{'}|), \\ &\leq l(x, y, t, \boldsymbol{q}^2) - \varepsilon + 3C\frac{\varepsilon}{6C}, \\ &\leq l(x, y, t, \boldsymbol{q}^2) - \frac{1}{2}\varepsilon. \end{aligned} \tag{56}$$

Thus, we have

$$\begin{aligned} ||l(\cdot, \boldsymbol{q}^1) - l(\cdot, \boldsymbol{q}^2)||^2_{L^2(\Omega\times[0,T])} &= \int_\Omega \int_0^T |l(x, y, t, \boldsymbol{q}^1) - l(x, y, t, \boldsymbol{q}^2)|^2 dtdxdy, \\ &\geq \int_{x-\frac{\varepsilon}{2C}}^{x} \int_{y-\frac{\varepsilon}{2C}}^{y} \int_{t-\frac{\varepsilon}{2C}}^{t} |l(x, y, t, \boldsymbol{q}^1) - l(x, y, t, \boldsymbol{q}^2)|^2 dtdxdy, \\ &\geq (\frac{\varepsilon}{6C})^3 (\frac{\varepsilon}{2})^2. \end{aligned} \tag{57}$$

This is contradictory to equation (53); thus, equation (54) holds. Next, we claim that for any $(x, y) \in \Omega$

$$|\hat{l}(x, y, \boldsymbol{q}^1) - \hat{l}(x, y, \boldsymbol{q}^2)| < \varepsilon. \tag{58}$$

Without loss of generality, we assume that $\hat{l}(x, y, \boldsymbol{q}^1) < \hat{l}(x, y, \boldsymbol{q}^2)$. If equation (58) is not satisfied, then

$$\hat{l}(x, y, \boldsymbol{q}^1) < \hat{l}(x, y, \boldsymbol{q}^2) - \varepsilon. \tag{59}$$

Assuming that $\hat{l}(x, y, \boldsymbol{q}^1) = l(x, y, \hat{t}, \boldsymbol{q}^1)$, then again, according to equation (54), we have

$$l(x, y, \hat{t}, \boldsymbol{q}^1) > l(x, y, \hat{t}, \boldsymbol{q}^2) - \varepsilon. \tag{60}$$

Combining equations (59) and (60), we have $l(x, y, \hat{t}, \boldsymbol{q}^2) < \hat{l}(x, y, \boldsymbol{q}^2)$, which is contradicted by the definition of $\hat{l}(x, y, \boldsymbol{q}^2)$; thus, equation (58) hold.

Accordingly, by definition, we have

$$\begin{aligned}
&\left|l^{\epsilon}(x, y, t, \boldsymbol{q}^1) - l^{\epsilon}(x, y, t, \boldsymbol{q}^2)\right| \\
=&\left|\max\left\{l(x, y, t, \boldsymbol{q}^1), \hat{l}(x, y, \boldsymbol{q}^1) + \epsilon(x, y, t)\right\} - \max\left\{l(x, y, t, \boldsymbol{q}^2), \hat{l}(x, y, \boldsymbol{q}^2) + \epsilon(x, y, t)\right\}\right|, \\
=&\begin{cases}
(i) \quad \left|l(x, y, t, \boldsymbol{q}^1) - l(x, y, t, \boldsymbol{q}^2)\right|, \\
(ii) \quad \left|l(x, y, t, \boldsymbol{q}^1) - \hat{l}(x, y, \boldsymbol{q}^2) - \epsilon(x, y, t)\right|, \\
(iii) \quad \left|\hat{l}(x, y, \boldsymbol{q}^1) + \epsilon(x, y, t) - l(x, y, t, \boldsymbol{q}^2)\right|, \\
(iv) \quad \left|\hat{l}(x, y, \boldsymbol{q}^1) - \hat{l}(x, y, \boldsymbol{q}^2)\right|.
\end{cases}
\end{aligned} \tag{61}$$

We next consider case (ii) in equation (61), in which $l(x, y, t, \boldsymbol{q}^1) \geq \hat{l}(x, y, \boldsymbol{q}^1) + \epsilon(x, y, t)$ and $l(x, y, t, \boldsymbol{q}^2) \leq \hat{l}(x, y, \boldsymbol{q}^2) + \epsilon(x, y, t)$.

If $l(x, y, t, \boldsymbol{q}^1) \geq \hat{l}(x, y, \boldsymbol{q}^2) + \epsilon(x, y, t)$, then

$$\left|l(x, y, t, \boldsymbol{q}^1) - \hat{l}(x, y, \boldsymbol{q}^2) - \epsilon(x, y, t)\right| \leq l(x, y, t, \boldsymbol{q}^1) - l(x, y, t, \boldsymbol{q}^2). \tag{62}$$

Otherwise, if $l(x, y, t, \boldsymbol{q}^1) < \hat{l}(x, y, \boldsymbol{q}^2) + \epsilon(x, y, t)$, then

$$\left|l(x, y, t, \boldsymbol{q}^1) - \hat{l}(x, y, \boldsymbol{q}^2) - \epsilon(x, y, t)\right| \leq \left|\hat{l}(x, y, \boldsymbol{q}^1) - \hat{l}(x, y, \boldsymbol{q}^2)\right|. \tag{63}$$

Thus, according to the above two equations, we have

$$\left|l(x,y,t,\boldsymbol{q}^1) - \hat{l}(x,y,\boldsymbol{q}^2) - \epsilon(x,y,t)\right| \leq \max\left\{l(x,y,t,\boldsymbol{q}^1) - l(x,y,t,\boldsymbol{q}^2), \left|\hat{l}(x,y,\boldsymbol{q}^1) - \hat{l}(x,y,\boldsymbol{q}^2)\right|\right\}. \tag{64}$$

Similarly, the following inequality for case (iii) also holds

$$\left|\hat{l}(x,y,\boldsymbol{q}^1) + \epsilon(x,y,t) - l(x,y,t,\boldsymbol{q}^2)\right| \leq \max\left\{l(x,y,t,\boldsymbol{q}^1) - l(x,y,t,\boldsymbol{q}^2), \left|\hat{l}(x,y,\boldsymbol{q}^1) - \hat{l}(x,y,\boldsymbol{q}^2)\right|\right\}. \tag{65}$$

According to equations (61), (64) and (65), we have

$$\begin{aligned}\left|l^\epsilon(x,y,t,\boldsymbol{q}^1) - l^\epsilon(x,y,t,\boldsymbol{q}^2)\right| &\leq \max\left\{l(x,y,t,\boldsymbol{q}^1) - l(x,y,t,\boldsymbol{q}^2), \left|\hat{l}(x,y,\boldsymbol{q}^1) - \hat{l}(x,y,\boldsymbol{q}^2)\right|\right\} \\ &\leq \varepsilon.\end{aligned} \tag{66}$$

Therefore, if $\boldsymbol{q}^n$ is a sequence that converges to $\boldsymbol{q}^*$ in the L^2 norm, then

$$\begin{aligned}||l^\epsilon(\cdot,\boldsymbol{q}^n) - l^\epsilon(\cdot,\boldsymbol{q}^*)||^2_{L^2(\Omega\times[0,T])} &= \int_\Omega \int_0^T (l^\epsilon(x,y,t,\boldsymbol{q}^n) - l^\epsilon(x,y,t,\boldsymbol{q}^*))^2 dt dx dy \\ &\leq |\Omega| T \varepsilon^2.\end{aligned} \tag{67}$$

Thus, the operator l^ϵ is continuous. □

Next, we present the existence of a solution to the VI problem following Theorem

Theorem 3.5. *If the travel demand $q(x,y,t)$ is uniformly bounded on $\Omega\times[0,T]$, the operator l^ϵ is continuous with respect to $\mathbf{q}$, and the tolerance $\epsilon(\cdot):\Omega\times[0,T]\to R_+$ is bounded, then a solution exists for the VI problem.*

Proof. First, because the travel demand has a uniform upper bound, we denote it as C_1. We next construct the finite dimensional set Λ^n as an approximation of the feasible set Λ. For simplicity of introduction and denotation in the following proof, we assume that the region is a rectangle, and denote $\Omega = [0,X]\times[0,Y]$. For each $n \geq 1$, we divide the domain $[0,X]\times[0,Y]\times[0,T]$ into 2^n sub-intervals in each direction, and denote

$$0 = x_0 < x_1 < \cdots < x_{2^n} = X,$$

$$0 = y_0 < y_1 < \cdots < y_{2^n} = Y,$$

$$0 = t_0 < t_1 < \cdots < t_{2^n} = T.$$

Then, the sequence of the finite dimensional set is defined as

$$\Lambda^n = \Big\{ \boldsymbol{q}^n : q(x,y,t) \leq C_1, \forall (x,y,t) \in \Omega \times [0,T], q(x,y,t) \text{ is constant on} \\ [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [t_{k-1}, t_k], \int_T q(x,y,t)dt = Q(x_i, y_j), \forall (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \Big\}. \tag{68}$$

According to Han et al. (2015), the set Λ^n is convex and compact in Hilbert space $L^2(\Omega \times [0,T])$. Thus, for each $n \geq 1$, according to Schauder fixed-point theorem, (Evans (1998), Section 9.2, Theorem 3), there exists $\boldsymbol{q}^{n,*} \in \Lambda^n$ such that

$$\langle l^\epsilon(\cdot, \boldsymbol{q}^{n,*}), \boldsymbol{q}^n - \boldsymbol{q}^{n,*} \rangle \geq 0, \quad \forall \boldsymbol{q}^n \in \Lambda^n. \tag{69}$$

where $\langle \cdot, \cdot \rangle$ is the inner product. Because $\boldsymbol{q}^{n,*}$ is uniformly bounded, the sequence $\boldsymbol{q}^{n,*}$ is uniformly bounded and satisfies equation (69). According to the Banach–Alaoglu theorem, as L^2 space is a reflexive space, the bounded subsets are weakly sequentially precompact. Thus, there is a subsequence in the space $L^2(\Omega \times [0,T])$. For simplicity, we denote this subsequence as $\boldsymbol{q}^{n,*}$, and denote its limit as $\boldsymbol{q}^*$. It is obvious that $\boldsymbol{q}^*$ belongs to Λ. However, for any $\boldsymbol{q} \in \Lambda$, there exists a piecewise constant approximation $\{\boldsymbol{q}^n\}$, which converges to $\boldsymbol{q}$ in the $\|\cdot\|_{L^2}$ norm. Because the operator $l^\epsilon(\cdot, \boldsymbol{q})$ is continuous with respect to $\boldsymbol{q}$ with respect to the L^2 norm, and there is continuity of the inner product in Hilbert space, we pass equation (69) to the limit (let $n \to \infty$) and obtain

$$\lim_{n\to\infty} \langle l^\epsilon(\cdot, \boldsymbol{q}^{n,*}), \boldsymbol{q}^n - \boldsymbol{q}^{n,*} \rangle = \langle l^\epsilon(\cdot, \boldsymbol{q}^*), \boldsymbol{q} - \boldsymbol{q}^* \rangle \geq 0, \quad \forall \boldsymbol{q} \in \Lambda. \tag{70}$$

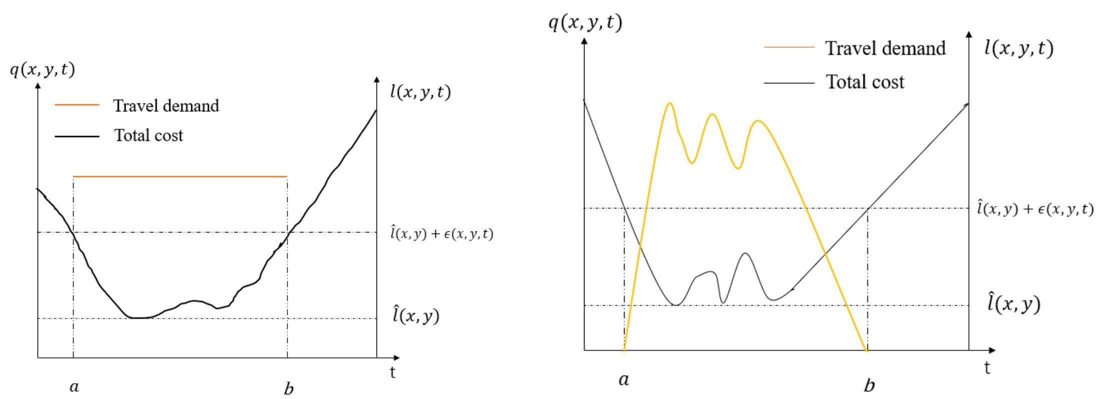
Thus, the existence holds. □

3.3 Uniqueness

On the one hand, according to Theorem 3.5, if q is uniformly bounded, a solution to the VI problem. On the other hand, for the BR-DUE problem, some researchers have shown that the solution is usually not unique. According to Definition 2.2, for any fixed location (x,y), the travel demand should satisfy constraint (3), and if the travel demand is positive at time t, then the related total cost should be located in indifference band. Under these two constraints, the feasible solution is not unique, and is dependent on the distribution of the travel demand in the indifference band. Next, we give an intuitive and graphical illustration of the non-uniqueness for a boundedly rational DUE problem. For any fixed location (x, y), Figure 1 depicts two types of the temporal distribution of boundedly rational DUE solutions $q(x, y, t)$ and the corresponding total cost $l(x, y, t)$. In Figure 1(a), the travel demand is uniformly distributed on $[a, b]$. In Figure 1(b), the travel demand satisfies a non-uniform distribution and is defined as

$$q(x, y, t) = Q(x, y)\frac{[\hat{l}(x, y) + \epsilon(x, y, t) - l(x, y, t)]_+}{\int_0^T [\hat{l}(x, y) + \epsilon(x, y, t) - l(x, y, t)]_+ dt}. \tag{71}$$

where $[a]_+ = max(a, 0)$.



(a) uniform distribution (b) non-uniform distribution

Fig. 1: An illustration of a BR-RDUE solution and the associated total cost.

Either of the specific uniform and non-uniform distributions, among the infinite possibilities, could have some physical meaning. For the uniform distribution, there is no difference in choice preference at any point within the indifference band, and thus there is no reason to prefer one point over any other in a short-run decision. Then, in the long run, the chances of a choice falling on any point in the band are likely to be equal, as there is no inherent preference for one point over any others within the band. Hence, a uniform distribution would probably be the long-run, or expected, outcome in the probabilistic context. The specific non-uniform distribution defined in equation (71) may also imply a certain probabilistic outcome. In the long run, the points that have lower costs would probably eventually be experienced by travelers and attract more people to these points in the long run. Therefore, although we argue that there is no difference among people's preferences within the indifference band in the short run, people will still inform their choices through accumulated travel experience, good or bad, in the long run. In addition, to derive the unique solution, we must determine the distribution of the travel demand in the indifference band.

Remark 3.1. *There exists a constant C, such that for any $\epsilon \geq C$, the traveler can depart at any time, i.e., the "indifference band" included in the overall time period.*

4 Solution procedure

In this section, we introduce the numerical methods and the complete solution procedure to solve the established BR-DTC-RDUE model. For simplicity, we consider rectangular computational regions in the numerical tests and use classical finite difference methods. Nevertheless, our model imposes no restriction on the shape of the computational domain. For a more complicated domain, one simply needs to replace the finite difference method with other methods, such as discontinuous Galerkin methods, and follow the same solution procedure discussed in this section.

We divide the computational domain Ω into $N_x \times N_y$ grid points and denote the (i, j)-th point as (x_i, y_j). We further denote the number of grid points in time as N_t and denote the n-th time level as t_n. For any function $u(x, y, t)$ defined on $\Omega \times [0, T]$, we approximate it with discrete grid point values $\{u^n_{i,j},\ i = 1, \cdots, N_x,\ j = 1, \cdots, N_y,\ n = 1, \cdots, N_t\}$, where $u^n_{i,j}$ is an approximation for $u(x_i, y_j, t_n)$. The whole BR-DTC-DUE model comprises several parts, and we apply the following methods to solve each part.

- The projection method is used to solve the VI problem (37).
- The Lax-Friedrichs scheme is used to solve the conservation law (38).
- The fast sweeping method is used to solve the eikonal equation (39).
- The Lax-Friedrichs scheme is used to solve the HJ equation (6).

In this study, we omit the formulations of the above numerical schemes. For more details, please refer to Yang et al. (2019, 2022).

Notice that different parts of the model interact with each other and cannot be solved independently. We therefore use an iteration method to solve the complete model. Starting from an initial guess about the traffic demand, we solve the RDUE model to get the cost functions. Then we solve the VI problem to update the traffic demand function. We repeat this procedure until we obtain the convergent solutions. In the following table, we introduce the detailed solution procedure (Figure 2), where the relative gap function is defined as

$$RGAP_{discrete} = \frac{\sum_{i\in\{1,\ldots,N_x\}} \sum_{k\in\{1,\ldots,N_y\}} \sum_{n\in\{1,\ldots,N_t\}} q_{ik}^n \left(l_{ik}^n - \hat{l}_{ik}\right)}{\sum_{i\in\{1,\ldots,N_x\}} \sum_{k\in\{1,\ldots,N_y\}} Q_{ik}\hat{l}_{ik}}. \tag{72}$$

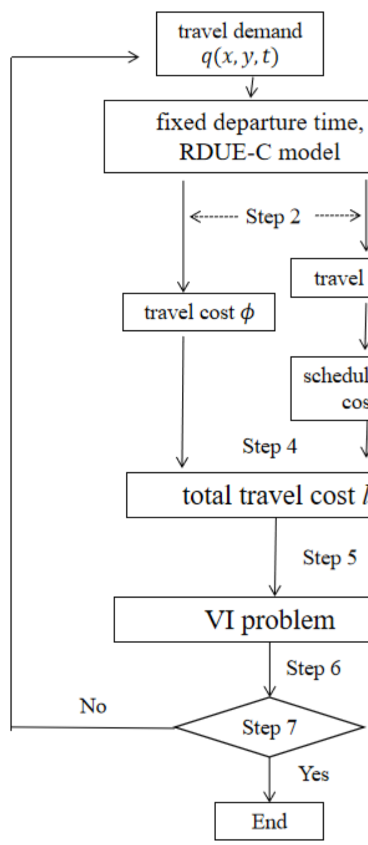


Fig. 2: Flowchart of the solution procedure

Algorithm 1 Solution procedure for BR-DTC-RDUE model

Step 1. According to equation (3), set an initial travel demand $\boldsymbol{q}^k$ based on $Q(x, y)$ and set $k = 1$, where the initial travel demand follows a distribution such as a uniform distribution, the Gaussian distribution, or any stochastic distribution.

Step 2. Compute the travel cost $\phi(x, y, t)$ and travel time $I(x, y, t)$ by solving the RDUE-C model defined in Section 2.1.

Step 3. Compute the schedule delay cost $p(x, y, t)$ by using equation (10).

Step 4. Compute the total cost $l(x, y, t)$ by using equation (11).

Step 5. Compute the operator $l^{\epsilon}(x, y, t, \boldsymbol{q}^k)$ by using equation (15).

Step 6. Update the travel demand $\boldsymbol{q}^{k+1}$ by solving the VI problem based on the given distribution in the indifference band.

Step 7. Compute the relative gap function. If $\frac{|\boldsymbol{q}^{k+1}-\boldsymbol{q}^k|}{\boldsymbol{q}^k} \leq \varepsilon_1$ and $RGAP_{discrete} \leq \varepsilon_2$, stop; Otherwise set $k = k + 1$ and go to Step 2.

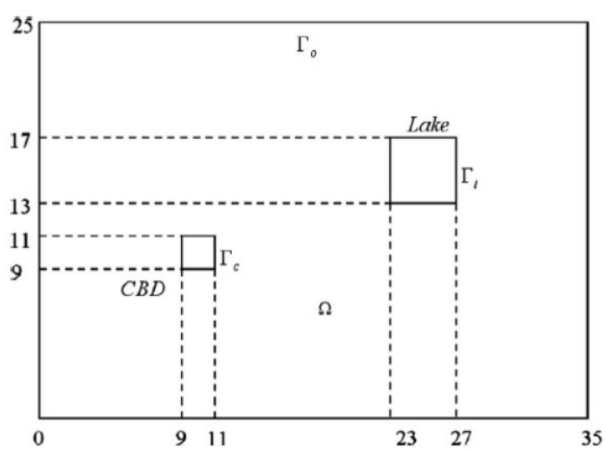


Fig. 3: The modeling domain

5 Numerical examples

In this section, we provide a numerical example to demonstrate the correctness and characteristics of the model and our algorithms. We give the problem settings in Section 5.1 and then show the numerical results in Section 5.2.

5.1 Problem settings

As shown in Figure 3, we consider a rectangular modeling region, which is 35 km long and 25 km wide with a single central business district (CBD) and an obstacle within it. The center of the CBD is located at (10 km, 10 km) with a diameter of 2 km, and the obstacle is located at (25 km, 15 km) with a diameter of 4 km. We consider the activities of travelers traveling from their home (origin) to the CBD (destination).

In this numerical example, it is assumed that no traffic is present at the beginning of the modeling period and the travel cost at the boundary of the CBD is zero, i.e.,

$$\rho_0(x,y) = 0, \forall (x,y) \in \Omega, \qquad \phi_c(x,y,t) = 0,\ \forall (x,y) \in \Gamma_c,\ t \in [0,T].$$

The modeling period $[0,T] = [0\text{ h}, 6\text{ h}]$. It is considered that travelers, regardless of their residential location, have a similar desired arrival time as they head to the CBD. Thus, the desired arrival time for this numerical example is defined as $t^* = 2.8$ h. For the schedule-delay cost function, the parameters γ_1, γ_2 and Δ are taken as 48 \$/h, 108\$/h and 0.2 h, respectively. In this numerical example, the speed function is defined as

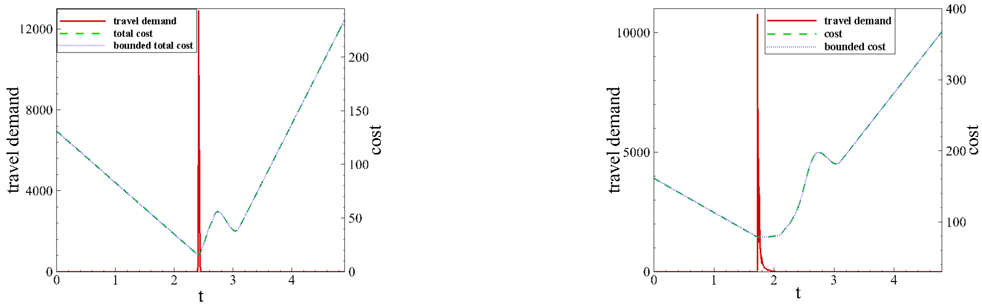
$$U(x,y,t) = U_f(x,y)\left\{1 - \exp\left[\frac{C}{U_f(x,y)}(1 - \frac{\rho_j}{\rho(x,y)})\right]\right\}, \quad \forall (x,y) \in \Omega,\ t \in [0,T], \tag{73}$$

where $U_f(x,y) = 56\left[1 + 4 \times 10^{-3} d(x,y)\right]$ km/h is the free-flow speed when $d(x,y)$ is the distance from the location (x,y) to the center of the CBD, $\rho_j = 6000$ veh/km^2 is the traffic jam density, and $C = 8$ km/h is the backward congested wave parameter. Here functions $U_f(x,y)$ and $d(x,y)$ are chosen such that the free-flow speed in the domain further from the CBD is higher due to fewer junctions.

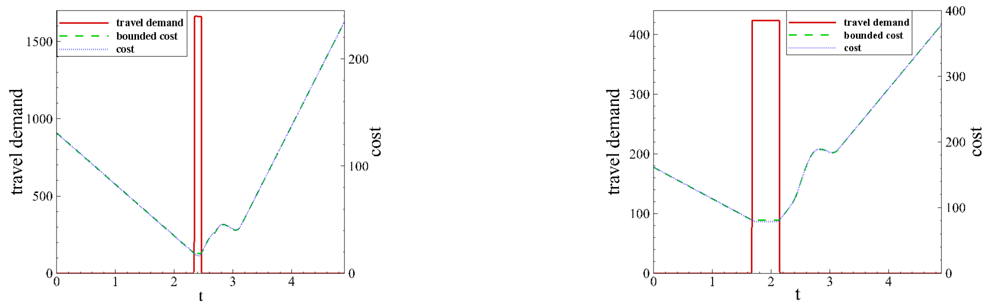
5.2 Numerical results

We present the numerical results in this subsection. According to the analysis in Section 3, the uniqueness of the solution to the BR-DTC-RDUE model does not hold. In our numerical example, to derive a unique solution, we assume that the travel demand (departure rate) satisfies a given distribution in the indifference band. In the following results, the travel demand satisfies the uniform distribution in Figures 4–13 and satisfies the non-uniform distribution in Figure 14.

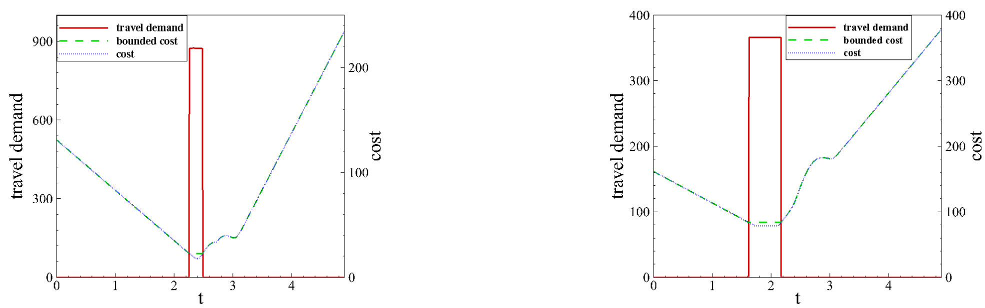
We first consider the BR-DTC-RDUE-C model with different choices of ϵ. Figure 4 shows the travel demand $q(x, y, t)$, the total cost $l(x, y, t, \boldsymbol{q})$ and the bounded total cost $l^{\epsilon}(x, y, t, \boldsymbol{q})$ at locations $(x, y) = (15 \text{ km}, 10 \text{ km})$ and $(x, y) = (34 \text{ km}, 24 \text{ km})$, respectively. In the case where $\epsilon = 0$, we can see that travel demand is concentrated at the time when the total cost is minimized. Otherwise, the travel demand is always located within the time region where the total cost belongs to the indifference band. It can be concluded that travelers always choose a departure time such that the total costs are "indifferent," and thus, the boundedly rational dynamic user equilibrium departure time choice principle is satisfied. As the desired arrival time interval is $[2.6 \text{ h}, 3.0 \text{ h}]$, all traveler departure times at different locations are earlier than 3 h. Additionally, as the distance to the CBD from location $(34 \text{ km}, 24 \text{ km})$ is greater than that from location $(15 \text{ km}, 10 \text{ km})$, the departure time from $(34 \text{ km}, 24 \text{ km})$ is always earlier. At the beginning and ending period, the modeling region is uncongested, and therefore the instantaneous travel cost $\phi(x, y, t)$ and travel time $I(x, y, t)$ should be relatively constant in this period. Therefore, the total cost l is mainly affected by the early- and late-arrival penalties, which are linearly dependent on the departure time. Because the time values for early or late arrivals are $\gamma_1 = 48$ \$/h and $\gamma_2 = 108$ \$ /h, respectively, we observe from Figure 4 that the total cost $l(x, y, t)$ decreases at a constant rate of 48 \$/h in the beginning and increases at a constant rate of 108 \$/h at the ending period. Finally, the value of the departure rate decreases as the tolerance ϵ increases, which demonstrates that the boundedly rational concept can disperse the departure rate.



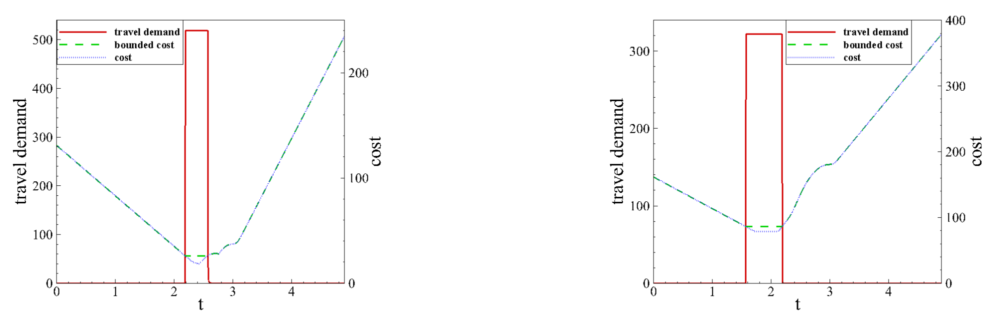
(a) $\epsilon = 0$



(b) $\epsilon = 2.5$

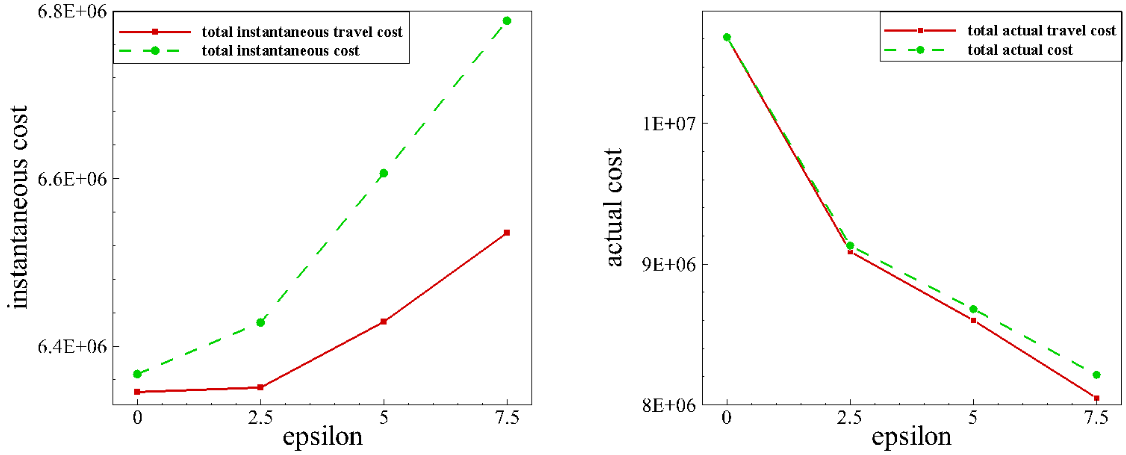


(c) $\epsilon = 5.0$



(d) $\epsilon = 7.5$

Fig. 4: The travel demand and total travel cost of traveler with different ϵ at location $(15, 10)$ (left column) and $(34, 24)$ (right column).



(a) instantaneous cost (b) actual cost

Fig. 5: The accumulated instantaneous/actual travel cost Φ/Φ_a and accumulated instantaneous/actual total cost Ψ/Ψ_a under different ϵ.

Next, we define the accumulated instantaneous travel cost and the total cost in the whole city and the entire time period as

$$\Phi = \int_0^T \iint_\Omega q(x, y, t)\phi(x, y, t) d\Omega dt, \tag{74}$$

$$\Psi = \int_0^T \iint_\Omega q(x, y, t)l(x, y, t) d\Omega dt. \tag{75}$$

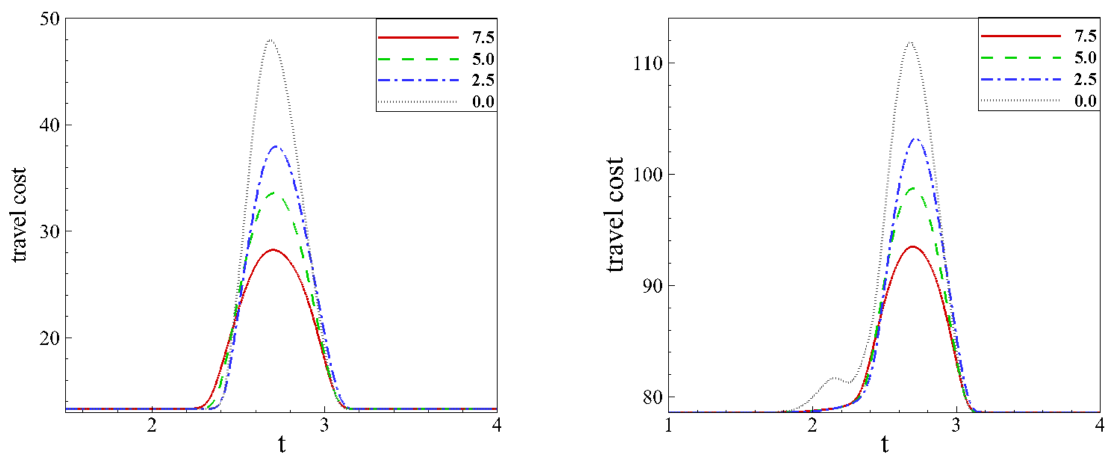
Similarly, the accumulated actual travel cost and total cost in the whole city and entire time period are defined as

$$\Phi_a = \int_0^T \iint_\Omega q(x, y, t)\phi_a(x, y, t) d\Omega dt, \tag{76}$$

$$\Psi_a = \int_0^T \iint_\Omega q(x, y, t)l_a(x, y, t) d\Omega dt. \tag{77}$$

where ϕ_a is the actual travel cost, which can be computed using the time-dependent HJ equation, and l_a is the related actual total cost.

Figure 5 shows the accumulated instantaneous/actual travel cost Φ/Φ_a and the accumulated instantaneous/actual total cost Ψ/Ψ_a under different ϵ. From Figure 5(a), we find that the accumulated instantaneous travel cost Φ and accumulated instantaneous total cost Ψ increase as ϵ increases. This occurs because in BR-DTC-RDUE-C models, travelers choose their route and departure time according to instantaneous information. Under the boundedly rational influence, there is a high degree of uncertainties in the travelers' decisions. Additionally, as the tolerance increases, the dispersion of the departure rate increases. From Figure 5(b), we can see that the relevant accumulated actual costs (Φ_a and Ψ_a) decrease as ϵ increase. According to the difference between the instantaneous and actual costs, it is demonstrated that inaccuracy is high when travelers base their choices on instantaneous information.

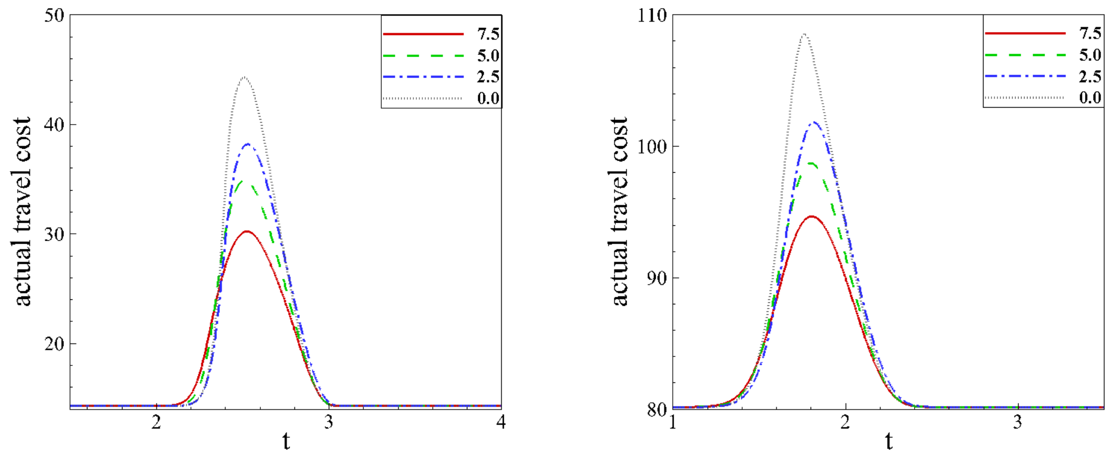


(a) $x = 15$ km, $y = 10$ km

(b) $x = 34$ km, $y = 24$ km

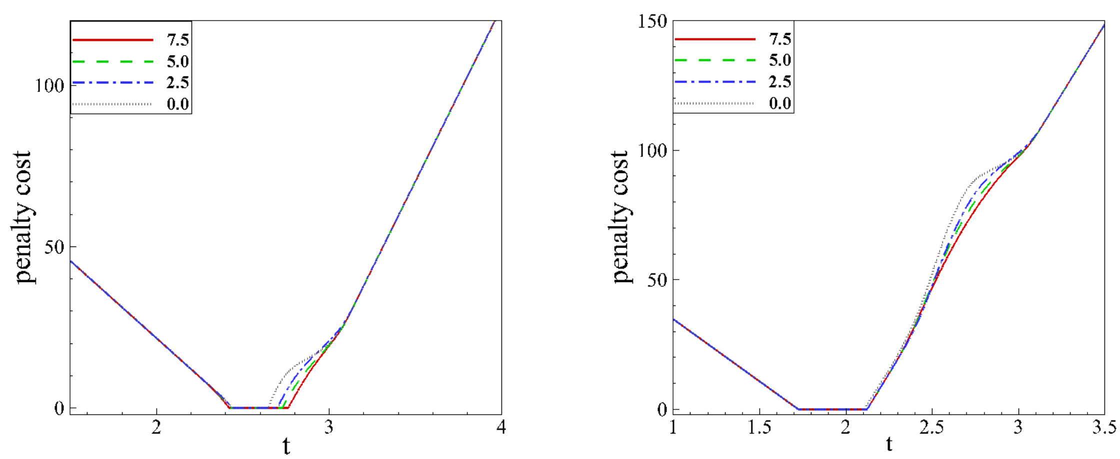
Fig. 6: The instantaneous travel cost of traveler with different ϵ at (15 km, 10 km) and (34 km, 24 km), respectively.

Next, we look into the detailed traffic-related costs at location (15 km, 10 km) and (34 km, 24 km) at different times. Figures 6 and 7 show the instantaneous/actual travel cost for travelers with different ϵ at (15 km, 10 km) and (34 km, 24 km), respectively. Comparing Figures 6(a) and 7(a) (or 6(b) and 7(b)), the shapes of the curves of instantaneous travel cost and actual travel cost are very similar, but the times when the travel cost is maximized are different, especially for the location at (34 km, 24 km). From all these sub-figures, we see that the peak value of instantaneous or actual travel cost decreases as the tolerance increases, and greater tolerance can lead to a less congested traffic system.



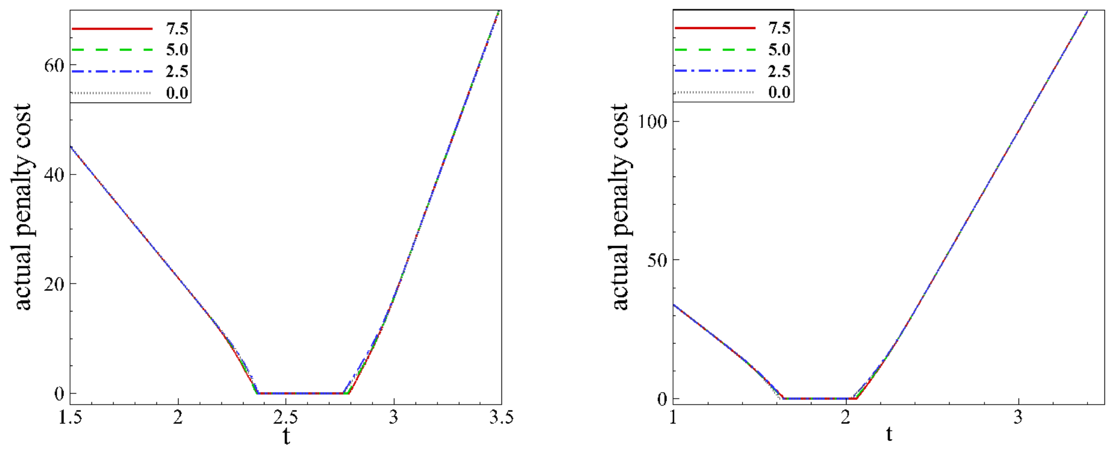
(a) x = 15 km, y = 10 km (b) x = 34 km, y = 24 km

Fig. 7: The actual travel cost of traveler with different ϵ at (15 km, 10 km) and (34 km, 24 km), respectively.



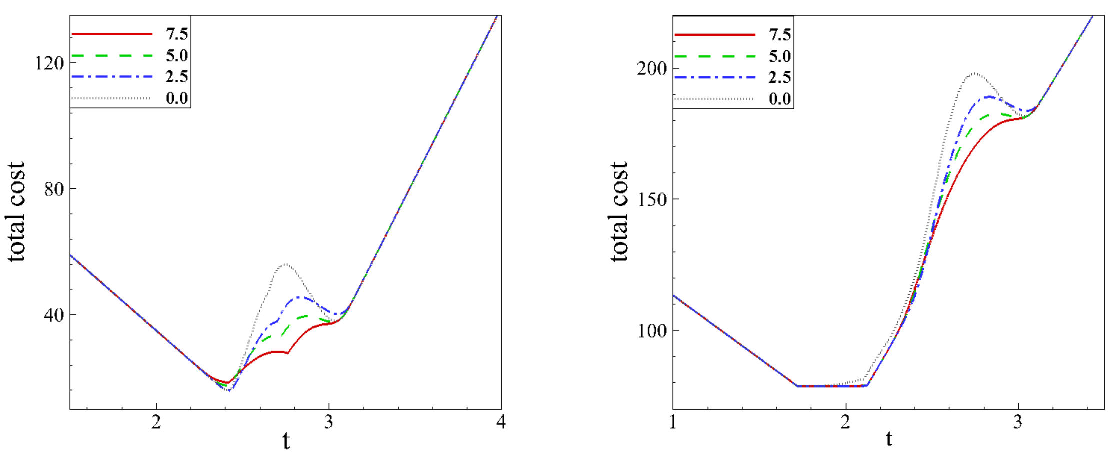
(a) x = 15 km, y = 10 km (b) x = 34 km, y = 24 km

Fig. 8: The instantaneous penalty cost of traveler with different ϵ at (15 km, 10 km) and (34 km, 24 km), respectively.



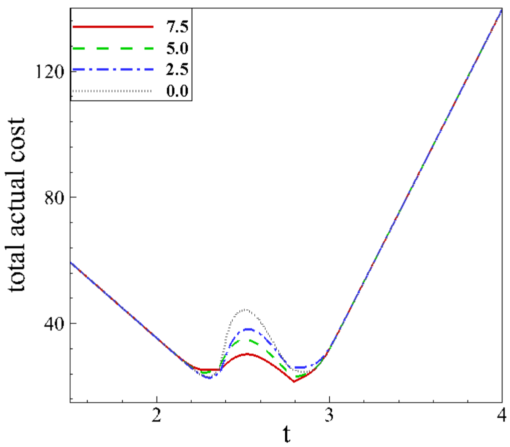
(a) $x = 15$ km, $y = 10$ km (b) $x = 34$ km, $y = 24$ km

Fig. 9: The actual penalty cost of with different ϵ at (15 km, 10 km) and (34 km, 24 km), respectively.

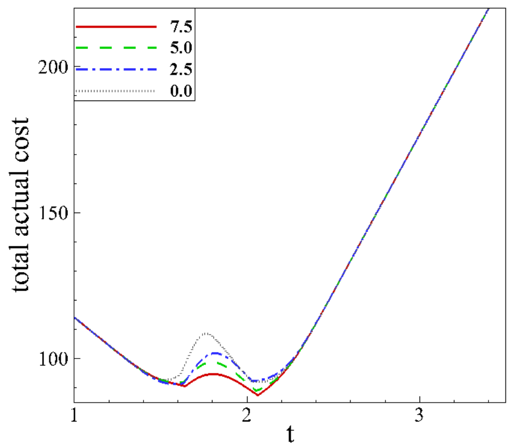


(a) $x = 15$ km, $y = 10$ km (b) $x = 34$ km, $y = 24$ km

Fig. 10: The instantaneous total cost of traveler with different ϵ at (15 km, 10 km) and (34 km, 24 km), respectively.



(a) $x = 15$ km, $y = 10$ km



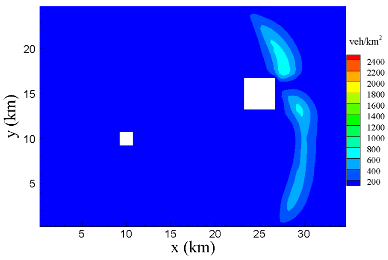
(b) $x = 34$ km, $y = 24$ km

Fig. 11: The actual total cost of traveler with different ϵ at (15 km, 10 km) and (34 km, 24 km), respectively.

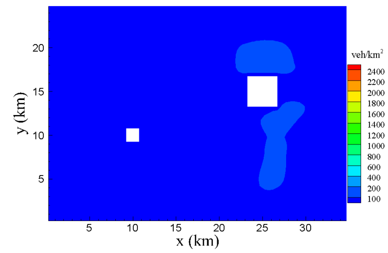
Figures 8 and 9 show the instantaneous/actual penalty cost of travelers with different ϵ at (15 km, 10 km) and (34 km, 24 km), respectively. These sub-figures show that the instantaneous and actual penalty costs are nearly the same, and that in the BR-DTC-RDUE model, only inaccurate instantaneous information affects the traveler's travel cost.

Combining the travel cost and penalty cost, Figures 10 and 11 show the instantaneous/actual total cost for travelers with different ϵ at (15 km, 10 km) and (34 km, 24 km), respectively. Coupled with the plot of travel demand, this shows that when the travel demand is positive, the instantaneous total cost with a greater tolerance is large and equal to the instantaneous total cost with a smaller tolerance, but the actual total cost has an opposite effect. These results are also in accordance with the results in Figure 5; thus, bounded rationality can reduce congestion.

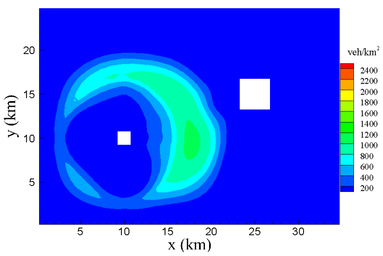
Figure 12 shows the density plots for travelers with $\epsilon = 0$ and $\epsilon = 7.5$ at different times, respectively. As shown in Figure 12(a1) (Figure 12 (b1)) at $t = 2$ h, travelers who reside further away have already departed because of the longer distance and hence longer travel time. At $t = 2.4$ h, travelers who reside near the CBD start to join the traffic system (Figure 12(a2) and 12(b2)). At $t = 2.8$ h, all travelers have nearly reached their destination, resulting in a



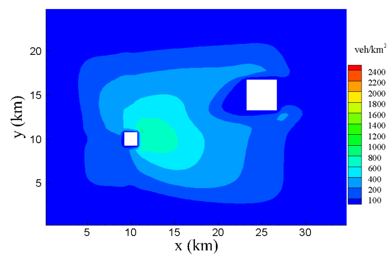
(a1) $t = 2.0, \epsilon = 0$



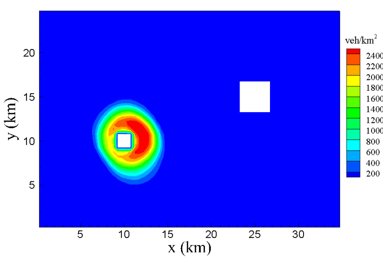
(b1) $t = 2.0, \epsilon = 7.5$



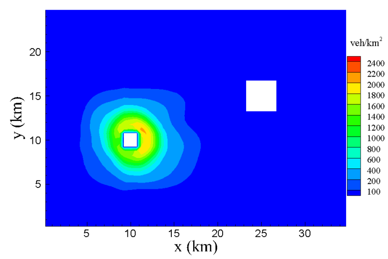
(a2) $t = 2.4, \epsilon = 0$



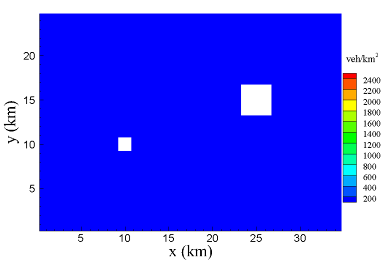
(b2) $t = 2.4, \epsilon = 7.5$



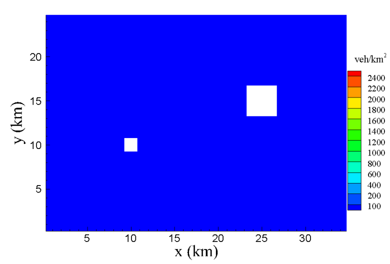
(a3) $t = 2.8, \epsilon = 0$



(b3) $t = 2.8, \epsilon = 7.5$



(a4) $t = 3.2, \epsilon = 0$



(b4) $t = 3.2, \epsilon = 7.5$

Fig. 12: The density plots for first group traveler with $\epsilon = 0$ and $\epsilon = 7.5$ at different time, respectively.

high density of this type of traveler in the vicinity of the CBD (Figure 12(a3) and 12(b3)). At $t = 3.2$ h, most travelers have entered the CBD and left the traffic system (Figure 12(a4) and 12(b4)). Comparing the density plots at $\epsilon = 0$ and $\epsilon = 7.5$, when $\epsilon = 7.5$, travelers depart earlier than they do when $\epsilon = 0$, resulting in less congestion in the city.

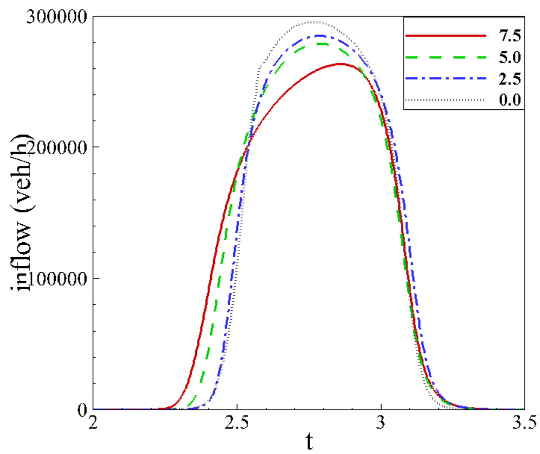


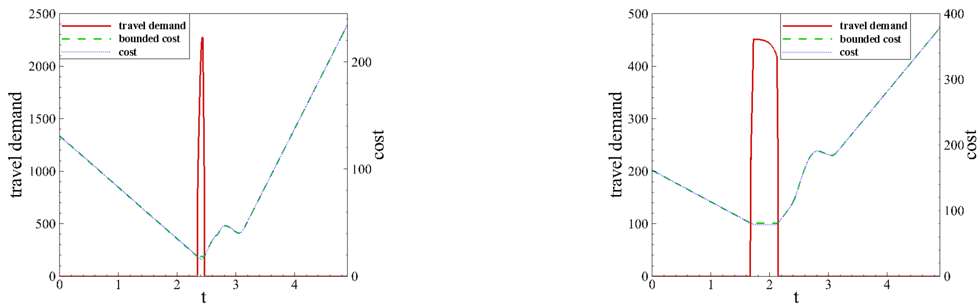
Fig. 13: The inflow plot.

We consider the total flow to the CBD through Γ_c (consisting of the inflow when vehicles travel to the CBD) defined as

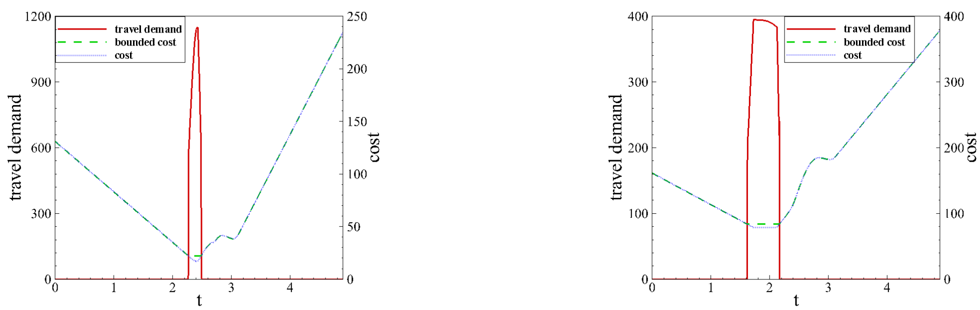
$$f_{CBD}(t) = \oint_{\Gamma_c} \boldsymbol{f}(x, y, t) \cdot \boldsymbol{n}(x, y, t) ds \tag{78}$$

where $\boldsymbol{n}$ is the unit normal vector pointing toward the CBD.

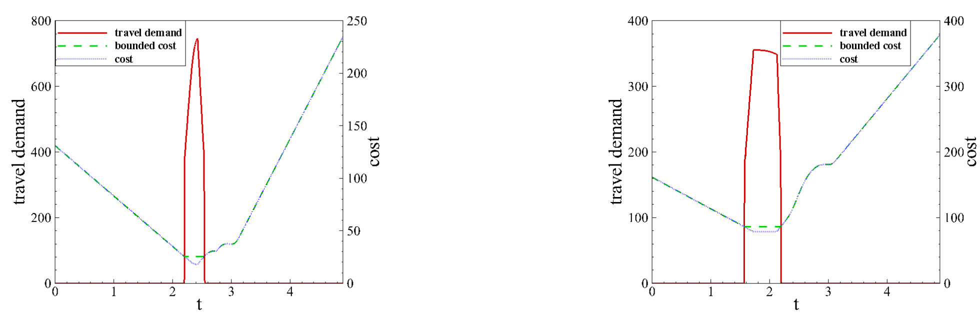
Figure 13 shows a plot of the inflow into the CBD. From this figure, we can see that the larger the ϵ, the earlier the travelers depart. Therefore, with reference to Figure 4, this indicates that the traffic-related cost of the system is reduced under these conditions, which means that application of the boundedly rational concept can reduce the city's congestion. Moreover, the peaks of the inflow plots decrease as the tolerance increases, confirming that there is a decrease in the city's congestion. Finally, by using the BR-DTC-RDUE-C model, we find that the inflow plots show large differences for different ϵ, and travelers tend to enter the CBD earlier when the tolerance is larger. This causes some travelers to switch from late to early arrival.



(a) $\epsilon = 2.5$



(b) $\epsilon = 5.0$



(c) $\epsilon = 7.5$

Fig. 14: The travel demand and total cost of traveler with different ϵ at location (15 km, 10 km) (left column) and (34 km, 24 km) (right column) with non-uniform distribution.

Figure 14 shows the travel demand, total cost and bounded total cost for travelers with different ϵ at locations $(15\ \text{km}, 10\ \text{km})$ (left column) and $(34\ \text{km}, 24\ \text{km})$ (right column) with non-uniform distribution (71). From this figure, greater travel demand is located in the time region where the total cost is smaller. Compared with Figure 4, these two solutions also satisfy the DTC principle in definition 2.2. Thus, the uniqueness of the solution to the BR-RDUE-C model does not hold

6 Conclusion

This paper considers the departure time choice dynamic user equilibrium problem and incorporates the concept of bounded rationality using the continuum modeling approach. The boundedly rational dynamic continuum user equilibrium model is formulated, after which the existence and uniqueness of the solution to this model are discussed. We first use the RDUE-C model to describe the traffic flow, in which travelers choose their routes to minimize the instantaneous travel cost. Then, we define the boundedly rational departure time choice dynamic user equilibrium, and prove that the BR departure time choice DUE model is equivalent to a VI problem by constructing a new travel cost operator. Next, we consider the existence and uniqueness of the solution to the BR-DTC-RDUE-C model. The existence is shown by Schauder's fixed-point existence theorem, and the non-uniqueness is demonstrated by an intuitive and graphical illustration. Finally, in our numerical examples, we test the BR-DTC-RDUE-C model with varying tolerance . We find that both the total actual travel cost and the total actual cost decrease as increases. However, we show that the solution depends on the distribution on the "indifference band." Both the uniform and non-uniform distribution solutions are shown in our numerical results. In this study, we considered the existence and uniqueness of the solution to the BR-DTC-RDUE-C model only. The existence and uniqueness of the solution to BR-DTC-PDUE-C model are more difficult to analyze because these models consist of coupled partial differential equations (the CL and HJ equations) and a forward–backward structure. In our future work, we can consider the day-to-day DTA problem incorporating bounded rationality. In addition, She and Ouyang (2021) use the continuum model to investigate emerging self-organized unmanned aerial vehicle traffic flow in low-altitude three-dimensional (3D) airspace. Based on their work, we can extend our continuum BR model to the 3D case.

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