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Probability rate optimization of positive Markov jump linear systems via DC programming

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Abstract
We investigate the stabilization problem of positive Markov jump linear systems by optimizing their transition probability rates. By using the convex property of posynomials and the standard mathematical programming that deals with the difference in convex functions, we show that transition probability rate synthesis problems can be solved via difference-of-convex (DC) programming. A numerical example is used to illustrate the effectiveness of our results.

KEYWORDS
DC programming, Markov process, positive linear systems, transition probability rate design

1 | INTRODUCTION

Dynamical systems whose state trajectories remain in a nonnegative orthant for any given nonnegative initial state are called *positive systems* [1]. Owing to this special feature, the results of the nonnegative matrix can be utilized to develop unique but efficient approaches to address analysis and synthesis problems [2–5]. On the basis of these achievements, several applications have been realized in recent years. Examples include drug therapy [6], biochemistry [7], transportation [8], epidemic processes [9, 10], and product development [11].

In practical problems, there is a common sense that neither the structure nor the parameters of real physical systems can remain unchanged at all times. This is because physical systems are vulnerable to abrupt changes such as network topology changes, device failures, and time-varying parameters. To describe this stochastic switching feature of positive systems, *positive Markov jump linear systems* [12] are presented as an efficient tool for modeling and control, where the switching features of all modes are characterized by *Markov processes*. Many analysis and synthesis results can be found in the literature [12–21]. However, the assumption underlying this research was that the probability transition rates of positive Markov jump linear systems are fixed *a priori*. In fact, system stabilization based on designing the probability transition rates is also necessary—particularly in handling practical problems. One of these applications is the scheme design of networks-in-package [22] where the communication process is modeled by positive Markov jump linear systems to optimally design the switched communication probability rates of the networks-in-package.

Analysis results for the transition probability rates indicate that positive Markov jump linear systems can be exponentially mean-stable even if all the subsystems are unstable [12]. Conversely, owing to the influence of the probability rates, a system with stable subsystems can be exponentially unstable. Thus, the development of a framework for the transition probability rate synthesis problem is not trivial. In a previous study on the uncertainty of transition probability rates [23], a sufficient condition for designing the desired controller gains and transition rate matrices was provided for continuous-time Markov jump linear systems with time delays. Feng et al. [24] showed that the conditions for designing probability rates are formulated in terms of linear matrix inequalities with equality constraints. In Cao et al. [25], hybrid sliding mode control of Markov jump linear systems via probability rate design was studied. In Song et al. [26], a framework for integrally designing transition probability rates and an output feedback controller was proposed. However, the aforementioned results regarding transition probability rate design are inevitably reduced to solving nonconvex optimization problems [24]. In addition, the aforementioned results are all based on general Markov jump linear systems; for positive systems, the synthesis problem remains unsolved.

In this study, according to the stability results presented in Ogura and Preciado [20], we resort to a special class of nonconvex optimization problems called *DC programming* problems [27], which deal with functions in the form of the difference between two convex functions. By laying some regularity assumptions on the transition probability rates, we show that the problem of optimizing positive Markov jump linear systems by tuning the transition probability rates can be reduced to standard DC programming problems, which are solved using the specific DC programming algorithm [28].

The following notation is used in this study. Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ denote sets of real, nonnegative, and positive numbers, respectively. The sets of corresponding vectors of size n are denoted as $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$. Let $\mathbb{1}_n$ denote the vector with all entries equal to 1 and I represents the identity matrix. We define the entry-wise exponential operation of vector $v \in \mathbb{R}^n$ as $\exp[v] = [\exp v_1, \dots, \exp v_n]^\top$ and the entry-wise logarithm operation as $\log[v] = [\log v_1, \dots, \log v_n]^\top$. Let $v \in \mathbb{R}^n$, $\|v\|_1 = \sum_{i=1}^n |v_i|$ denote the 1-norm of a vector and let the vector ∞-norm be defined as $\|v\|_\infty = \max_i |v_i|$. Given $v : [0,\infty) \to \mathbb{R}^n$, the $\mathscr{L}_1$-norm of a function $v(t)$ is denoted as $\|v\|_{\mathscr{L}_1} = \int_0^\infty \|v(t)\|_1$, and the $\mathscr{L}_\infty$-norm is defined as $\|v\|_{\mathscr{L}_\infty} = \operatorname{esssup}_{t\geq 0}\|v(t)\|_\infty$. $\|\cdot\|$ denotes the Euclidean norm of the vector. $E[\cdot]$ denotes the mathematical expectation operator. The real matrix A is said to be nonnegative (positive) and is denoted as $A \geq 0$ ($A > 0$) if all entries of A are nonnegative (positive). The notion $B < A$ ($B \leq A$) is defined as $B - A < 0$ ($B - A \leq 0$).

2 | PROBLEM FORMULATION

Consider the following time-invariant linear system:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ z(t) = Cx(t) + Dw(t), \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times s}$, $C \in \mathbb{R}^{r\times n}$, and $D \in \mathbb{R}^{r\times s}$. We say that Σ is a *positive linear system* [29] if its initial state $x(0)$ is nonnegative, input $w(\cdot)$ is nonnegative, A is a Metzler matrix, and matrices B, C, and D are nonnegative. Suppose that there is a set of systems given by (1) and the switching law is governed by the *Markov process*. Then, we obtain a positive Markov jump linear system

$$\Sigma_\sigma : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}w(t) \\ z(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}w(t), \end{cases} \tag{2}$$

where $\sigma = \{\sigma(t)\}_{t\geq 0}$ is a time-homogeneous Markov process that takes values from the finite set $S = \{1, \dots, N\}$. We say that Σ_σ is positive if the subsystems (A_i, B_i, C_i, D_i) are positive for all $i \in S$. Let $\Pi(\theta) = [\pi(\theta)_{ij}]_{i,j\in S}$ denote the infinitesimal generator of $\sigma(t)$, where θ is a parameter for tuning the probability rate matrix belonging to a set $\Theta \subset \mathbb{R}^\ell$. Then, the parameterized probability rate of the system Σ_σ is given by

$$\Pr\{\sigma(t+h) = j \mid \sigma(t) = i\} = \begin{cases} \pi_{ij}(\theta)h + o(h), & \text{if } j \neq i, \\ 1 + \pi_{ii}(\theta)h + o(h), & \text{if } j = i, \end{cases}$$

where $o(h)$ is a little-o notation defined by $\lim_{h\to 0} o(h)/h = 0$, and $\pi_{ij}(\theta) > 0$, $j \neq i$, represents the parameterized transition rate from mode i to mode j, which obeys the rule

$$\pi_{ii}(\theta) + \sum_{j=1,\, i\neq j}^{N} \pi_{ij}(\theta) = 0. \tag{3}$$

For any $t \geq 0$ and $x_0 \in \mathbb{R}^n_+$, we let $x(t; x_0)$ denote the trajectory of the system Σ_σ at time t with the initial condition $x(0) = x_0$. In this study, we focused on the following stability measurements:

Definition 1. We say that Σ_σ defined in (2) is exponentially mean-stable [30] if $\alpha > 0$ and $\beta > 0$ exist such that

$$E[\|x(t; x_0)\|] \leq \alpha e^{-\beta t}\|x(0)\|$$

for any $t \geq 0$, x_0, and $\sigma(0)$.

Definition 2 ([20]). For an exponentially mean-stable system Σ_σ and the initial conditions $\sigma(0), x(0) = 0$, and $w \in \mathscr{L}_1$, the $\mathscr{L}_1$ gain of the system Σ_σ is defined as

$$\|\Sigma_\sigma\|_1 = \sup_{w \in \mathscr{L}_1} \frac{\|E[z]\|_{\mathscr{L}_1}}{\|w\|_{\mathscr{L}_1}}.$$

Definition 3. For an exponentially mean-stable system Σ_σ and the initial conditions $\sigma(0), x(0) = 0$, and $w \in \mathscr{L}_\infty$, the $\mathscr{L}_\infty$ gain of the system Σ_σ is defined as

$$\|\Sigma_\sigma\|_\infty = \sup_{w \in \mathscr{L}_\infty} \frac{\|E[z]\|_{\mathscr{L}_\infty}}{\|w\|_{\mathscr{L}_\infty}}.$$

Remark 1. In this study, our goal is to stabilize the system Σ_σ by optimizing its transition probability matrix defined by (2). We assume that the set of parameters θ is adjustable within the given intervals and that the adjustment of the parameters incurs the corresponding cost. Such a setting can be used to characterize many practical problems. For example, in the prevention and control of the epidemic spreading process [9], a group of people can take two states: living at home and at the workplace. Because these two scenarios have different population densities, the corresponding epidemic spread rates are different. Experience with preventing COVID-19 has revealed that the infection outbreak can be suppressed by scheduling the time in the workplace, which is represented by the transition probability rates in Σ_σ. Although people can choose to work at home, this can reduce their productivity. Therefore, one of the optimization problems can be summarized as determining the ratio of each mode to maximize the performance of the entire system.

If the constraint on the transition rate tuning cost is given, the goal is to minimize the given norm of the system Σ_σ. The optimization problem considered in this study can be stated as follows.

Problem 1 ($\mathcal{L}_1$ gain optimization). Let $L : \mathbb{R}_+^\ell \to \mathbb{R}_+$ denote the cost function of θ and $\bar{L}$ denote a bound on the constraint value. Find the parameter θ that minimizes the $\mathscr{L}_1$-gain of Σ_σ while satisfying the constraint $L(\theta) \leq \bar{L}$; that is, solve the following optimization problem:

$$\begin{aligned} &\underset{\theta \in \Theta}{\text{minimize}} \ \|\Sigma_\sigma\|_1 \\ &\text{subject to } L(\theta) \leq \bar{L} \\ &\qquad\qquad \Sigma_\sigma \text{ is exponentially mean-stable, (3),} \end{aligned}$$

Problem 2 ($\mathcal{L}_\infty$ gain optimization). Let $L : \mathbb{R}_+^\ell \to \mathbb{R}_+$ denote the cost function of θ and $\bar{L}$ denote a bound on the constraint value. Find the parameter θ that minimizes the $\mathscr{L}_\infty$-gain of Σ_σ while satisfying the constraint $L(\theta) \leq \bar{L}$; that is, solve the following optimization problem:

$$\begin{aligned} &\underset{\theta \in \Theta}{\text{minimize}} \ \|\Sigma_\sigma\|_\infty \\ &\text{subject to } L(\theta) \leq \bar{L} \\ &\qquad\qquad \Sigma_\sigma \text{ is exponentially mean-stable, (3).} \end{aligned}$$

3 | MAIN RESULTS

As preliminary knowledge for deriving the main results, we begin by introducing a class of nonnegative nonlinear functions called *posynomials*.

Definition 4 ([31]). Let $x_1, \dots, x_n$ denote n real positive variables and $x = [x_1, \dots, x_n]^\top$ be a vector with components x_i. A real-valued function f of x with the form $f(x) = cx_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$, where $c > 0$ and $a_i \in \mathbb{R}$, is called a monomial function. The sum of one or more monomials of the form $f(x) = \sum_{k=1} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}$, where $c_k > 0$ is called a posynomial function.

Lemma 1 ([31]). *If $f : \mathbb{R}_{++}^n \to \mathbb{R}_{++}$ is a posynomial function, the function $F : \mathbb{R}^n \to \mathbb{R} : z \mapsto \log f(\exp[z])$ is convex through the entry-wise exponential transformation $x = \exp[z]$, $z \in \mathbb{R}^n$.*

The convex property shown in Lemma 1 allows us to make the connection with mathematical programming dealing with difference-of-convex (DC) functions, which is called *DC programming*.

Definition 5 ([32]). Let C be a closed convex subset of $\mathbb{R}^n$. A real-valued function $f : C \to \mathbb{R}$ is called a *DC function* on C if there exist two convex functions $g, h : C \to \mathbb{R}$ such that f can be expressed in the form $f(x) = g(x) - h(x)$.

Definition 5 indicates that DC functions are a special class of nonlinear functions. It is widely known that every continuous function can be approximated by a DC function with the desired precision.

We are now ready to state the formal definitions of the DC programs.

Definition 6. Let C be a closed convex subset of $\mathbb{R}^n$. Let $f_0(x) = g_0(x) - h_0(x)$ and $f_i(x) = g_i(x) - h_i(x)$ $(i = 1, \dots, m)$ be DC functions defined on C. The optimization problem of the form

$$\begin{aligned} &\underset{x \in C}{\text{minimize}} \ f_0(x) \\ &\text{subject to } f_i(x) \leq 0, \ i = 1, \dots, m, \end{aligned}$$

is called a DC program.

From Definition 6, both the objective function and constraints can be the differences between any two convex functions. To present the specific form of our results, the form of the DC functions used in this study is defined as the difference between two posynomial functions.

Proposition 1. *Suppose that functions $g_0, h_0, \dots, g_m, h_m$ are posynomials. The following optimization problem is a standard DC programming problem:*

$$\begin{aligned} \underset{z\in\mathbb{R}^n}{\text{minimize}}\quad & \log g_0(\exp[z]) - \log h_0(\exp[z]) \\ \text{subject to}\quad & \log g_i(\exp[z]) - \log h_i(\exp[z]) \le 0 \\ & i = 1, \dots, m. \end{aligned}$$

Proof. According to Lemma 1, $\log g_i(\exp[z])$ and $\log h_i(\exp[z])$ are convex functions for all $i = 0, 1, \dots, m$. Therefore, Proposition 1 is a standard DC programming problem. □

Remark 2. Because DC programming deals with the difference between convex functions, it is regarded as a special nonconvex optimization method that lies between convex optimization and general nonconvex optimization for dealing with a special class of functions. Regarding problem-solving performance, heuristic methods can find good approximate solutions quickly; however, it is difficult to guarantee that the solution is close to the global optimal solution. Global methods always find the optimal solution but can take a long time [33]. However, the results of the DC algorithm indicated that it is common to obtain global optimal solutions, and this algorithm proved to be more robust and more efficient than related standard methods—particularly in large-scale settings [28]. Moreover, owing to the local optimality conditions and DC duality, the DC algorithm has linear convergence for general DC programs [27], which leads to more efficient procedures. Easy-to-use MATLAB packages [34] for constructing and solving DC problems are available.

As necessary, the assumptions regarding the parameters and coefficient matrices in the system Σ_σ are as follows:

Assumption 1.

1. There exists a set of scalars α_i such that $\tilde{A}_i = A_i + \alpha_i I (i \in S)$ are nonnegative.
2. $\pi_{ij(i\neq j)}(\theta)$ is a posynomial, and

$$\pi_{ii}(\theta) = -\sum_{j=1,i\neq j}^{N} \pi_{ij}(\theta).$$

3. The transition probability rate tuning cost function $L(\theta)$ is assumed to be the difference of posynomials in θ, for example, $L(\theta) = L_1(\theta) - L_2(\theta)$, where L_1 and L_1 are polynomials of θ.
4. There exists a set of differences of posynomials $\psi_1(\theta) = \bar{\psi}_1(\theta) - \underline{\psi}_1(\theta), \dots, \psi_s(\theta) = \bar{\psi}_s(\theta) - \underline{\psi}_s(\theta)$ such that the parameter constraint satisfies

$$\Theta = \{\theta \in \mathbb{R}^{n_\theta} : \psi_1(\theta) \le 0, \dots, \psi_s(\theta) \le 0\}. \quad (4)$$

According to Definition 6, DC programming is not feasible for equality constraints. Therefore, we introduced a sufficiently small scalar $\epsilon > 0$ to reshape the probability rate constraint (3) into

$$\begin{aligned} \pi_{ii}(\theta) + \sum_{j=1,i\neq j}^{N} \pi_{ij}(\theta) \le \epsilon, \\ \pi_{ii}(\theta) + \sum_{j=1,i\neq j}^{N} \pi_{ij}(\theta) \ge -\epsilon. \end{aligned} \quad (5)$$

With the introduction of preliminary knowledge and assumptions, the following theorems demonstrate that the aforementioned optimization problems can be reduced to DC programming problems:

Theorem 1. *Let δ^* be the solution of the following DC problem:*

$$\begin{aligned} \underset{\delta,\xi_i\in\mathbb{R}^n}{\text{minimize}}\quad & \zeta \\ \text{subject to}\quad & \log\left[v_i^\top \tilde{A}_i + \sum_{i\neq j}^{N} \pi_{ij}(\exp[\delta]) v_j^\top + \mathbb{1}_r^\top C_i\right] \\ & - \log\left[v_i^\top \alpha_i I - \pi_{ii}(\exp[\delta]) v_i^\top\right] \le 0, \\ & \log\left[v_i^\top B_i + \mathbb{1}_r^\top D_i\right] - \zeta \mathbb{1}_s^\top \le 0, \\ & \log L(\exp[\delta]) - \log \bar{L} \le 0, \\ & \log\left[\sum_{i\neq j}^{N} \pi_{ij}(\exp[\delta])\right] \le \log[\epsilon - \pi_{ii}(\exp[\delta])], \\ & \log[\epsilon - \pi_{ii}(\exp[\delta])] \le \log\left[\sum_{i\neq j}^{N} \pi_{ij}(\exp[\delta])\right], \\ & \log \bar{\psi}_\ell(\exp[\delta]) - \log \underline{\psi}_\ell(\exp[\delta]) \le 0, \\ & \ell = 1, \dots, s. \end{aligned}$$

where $v_i = \exp[\xi_i]$ and $\gamma = \exp\zeta$. Then, the solution to Problem 1 is given by

$$\theta^* = \exp[\delta^*]. \quad (6)$$

The following propositions illustrate the necessary and sufficient conditions for exponential mean stability with $\mathscr{L}_1$ and $\mathscr{L}_\infty$ performance of positive Markov jump linear systems.

Proposition 2 ([20]). *Assume that system Σ_σ is positive. For any $\gamma > 0$, Σ_σ is mean-stable and $\|\Sigma_\sigma\|_1 < \gamma$ if and only if there exist positive vectors $v_1, \dots, v_N \in \mathbb{R}^n$ such* *that*

$$\begin{aligned} &v_i^\top A_i + \sum_{j=1}^{N} \pi_{ij}(\theta) v_j^\top + \mathbb{1}_r^\top C_i < 0 \\ &v_i^\top B_i + \mathbb{1}_r^\top D_i < \gamma \mathbb{1}_s^\top, (3) \end{aligned} \tag{7}$$

for every $i \in \{1, \ldots, N\}$.

Proof of Theorem 1. According to Problem 1 and Proposition 2, the optimization problem can be constructed by adding a tuning cost constraint. Then, the specified problem is to find θ to minimize the $\mathscr{L}_1$-norm γ while satisfying the conditions of (3) and (4). Therefore, we obtain the following optimization problem:

$$\underset{\gamma>0, v_i \in \mathbb{R}_{++}^n}{\text{minimize}} \quad \gamma \tag{8a}$$

$$\begin{aligned} &L(\theta) \leq \bar{L} \\ &(4), (5), (7). \end{aligned} \tag{8b}$$

Because constraint (7) includes the product of $\pi_{ij}(\theta)$ and v_i, it is clear that $\sum_{j=1}^{N} \pi_{ij}(\theta) v_j^\top$ reduces inequality (7) to the bilinear inequality constraint, which reduces problem (8) to the NP-hardness problem appearing in Feng et al. [24]. The difficulty in solving (8) lies in determining a stable combination of θ and v_i. To overcome this difficulty, we adopt the log–log convexity of the posynomials in Lemma 1 and the introduced DC programming knowledge to reduce the problem (8) to Theorem 1.

According to Definition 4 and Assumption 1, the product of $\pi_{ij}(\theta)$ and v_i is a posynomial function. Constraint (7) can be decomposed into the difference between two posynomials $(v_i^\top \tilde{A}_i + \sum_{i \neq j}^{N} \pi_{ij}(\theta) v_j^\top + \mathbb{1}_r^\top C_i) - (v_i^\top \alpha_i I - \pi_{ii}(\theta) v_i^\top) \leq 0$ according to (1) and (2) in Assumption 1. By using Lemma 1, the above function can be transformed into the difference between the two DC functions. Similarly, constraint (5) can be reduced to the form of Theorem 1. The object function (8a) γ subtracts 0 directly satisfies the definition of DC functions in Definition 5. Because constraints (4) are defined in the difference of posynomials, we can obtain the DC function constraints using Lemma 1. Thus, we complete the proof of Theorem 1. □

Theorem 2. *Let δ^* be the solution of the following DC problem:*

$$\begin{aligned} &\underset{\delta, \xi_i \in \mathbb{R}^n}{\textit{minimize}} \ \zeta \\ &\textit{subject to} \ \log\left[\tilde{A}_i v_i + \sum_{i \neq j}^{N} \pi_{ij}(\exp[\delta]) v_j^\top + B_i \mathbb{1}_s\right] \\ &\quad - \log\left[v_i^\top \alpha_i I - \pi_{ii}(\exp[\delta]) v_i^\top\right] \leq 0, \\ &\quad \log\left[C_i v_i + D_i \mathbb{1}_s\right] - \zeta \mathbb{1}_r \leq 0, \\ &\quad \log L(\exp[\delta]) - \log \bar{L} \leq 0, \\ &\quad \log\left[\sum_{i \neq j}^{N} \pi_{ij}(\exp[\delta])\right] \leq \log[\epsilon - \pi_{ii}(\exp[\delta])], \\ &\quad \log[\epsilon - \pi_{ii}(\exp[\delta])] \leq \log\left[\sum_{i \neq j}^{N} \pi_{ij}(\exp[\delta])\right], \\ &\quad \log \bar{\psi}_\ell(\exp[\delta]) - \log \underline{\psi}_\ell(\exp[\delta]) \leq 0, \\ &\qquad \ell = 1, \ldots, s. \end{aligned}$$

where $v_i = \exp[\xi_i]$ and $\gamma = \exp\zeta$. Then, the solution to Problem 2 is given by (6).

Proposition 3. *Assume that the system Σ_σ is positive. For any $\gamma > 0$, Σ_σ is mean-stable and $\|\Sigma_\sigma\|_\infty < \gamma$ if and only if there exist positive vectors $v_1, \ldots, v_N \in \mathbb{R}^n$ such that*

$$A_i v_i + \sum_{j=1}^{N} \pi_{ij}(\theta) v_j^\top + B_i \mathbb{1}_s < 0 \quad C_i v_i + D_i \mathbb{1}_s < \gamma \mathbb{1}_r, (3) \tag{9}$$

for every $i \in \{1, \ldots, N\}$.

Proof of Theorem 2. According to Problem 2 and Proposition 3, the optimization problem can be constructed by adding a tuning cost constraint. Then, the specified problem is to find the θ that minimizes the $\mathscr{L}_\infty$-norm γ while satisfying the conditions of (3) and (4). Therefore, we obtain the following optimization problem:

$$\underset{\gamma>0, v_i \in \mathbb{R}_{++}^n}{\text{minimize}} \ \gamma \tag{10a}$$

$$\begin{aligned} &L(\theta) \leq \bar{L} \\ &(4), (5), (9). \end{aligned} \tag{10b}$$

The proof of reducing Problem (10) to Theorem 2 is the same as that of Theorem 1; therefore, it is omitted.

Remark 3 (Construction of π_{ii}). According to Equation (3), the precise form of $\pi_{ii}(\theta)$ can be determined by utilizing functions $\pi_{ij}(\theta)$. However, such an approach proves ineffective in constructing a DC problem. To overcome this, we introduce an unknown strictly positive parameter$-\pi_{ii}$ that remains independent of the parameters θ in functions π_{ij}. By doing so, we can effectively solve optimization problems in Theorems 1 and 2.

4 | SIMULATION

In this section, an example of stabilizing the epidemic spreading process [9] by optimizing the transition probability rates is used to validate the results.

Consider a weighted, directed network defined by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ denotes the set of n nodes within the network and $\mathcal{E} = \{e_1, \ldots, e_m\} \subseteq \mathcal{V} \times \mathcal{V}$ represents the set of directed edges. The adjacency matrix $A \in \mathbb{R}^{n \times n}$ of graph $\mathcal{G}$ is given by

$$A_{ij} = \begin{cases} 1, & \text{if } (j, i) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}$$

In this simulation example, we allow the epidemic spreading network A_i to have a set of scenarios that take values in the finite discrete set $i \in \{1, \dots, N\}$. Because the scenarios have been determined, for example, home or workplace, we let the recovery rate vector $\delta_i \in \mathbb{R}_+^n$ and infection rate vector $\beta_i \in \mathbb{R}_+^n$ be fixed for all i. Thus, the fixed environmental measurement vector $B(\beta_i, \delta_i)$, that is, the element in the ith row of the input matrix in the system Σ_σ, can impact the population density in area i, affecting the infection level x_i. If the switching feature of the epidemic spreading process is modeled by the Markov process, this process can be expressed by the following positive Markov jump linear system:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \\ z(t) = C_{\sigma(t)}x(t), \end{cases} \tag{11}$$

where $\sigma = \{\sigma(t)\}_{t \geq 0}$ is a time-homogeneous Markov process that takes values from the finite discrete set $S = \{1, \dots, N\}$. $u(t)$ denotes the input population vector. Because β_i and δ_i are fixed, A_i and B_i are constant for all i. The output measurement C_i was also set to be constant for computing the overall performance of the network. In this example, let the epidemic network have two scenarios: A_1 and A_2, each with a size of 10. The probability transition rate matrix is given as

$$\Pi(\theta) = \begin{bmatrix} -\theta_1 & \theta_1 \\ \theta_2 & -\theta_2 \end{bmatrix}, \tag{12}$$

where θ_1 and θ_2 are tuned within the interval [1, 10]. Consider the following constrained cost function:

$$\theta_1 + k\theta_2 \leq \bar{L}, \tag{13}$$

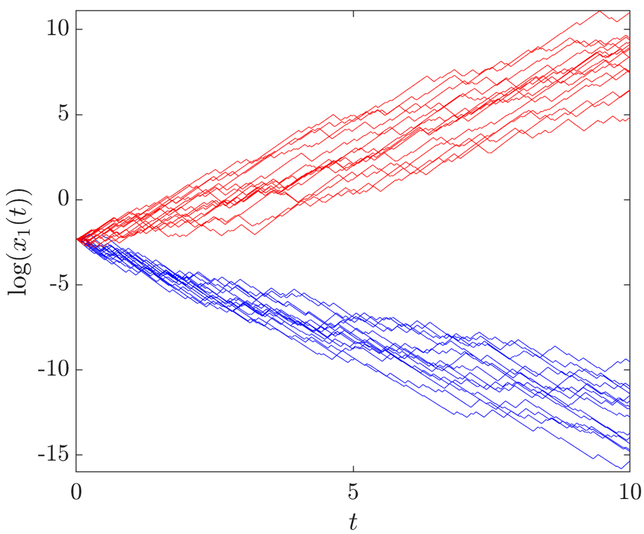

FIGURE 1 Twenty realizations of the state variable $\log x_1(t)$ of the epidemic spreading process. Red line: original. Blue line: stabilized. [Color figure can be viewed at wileyonlinelibrary.com]

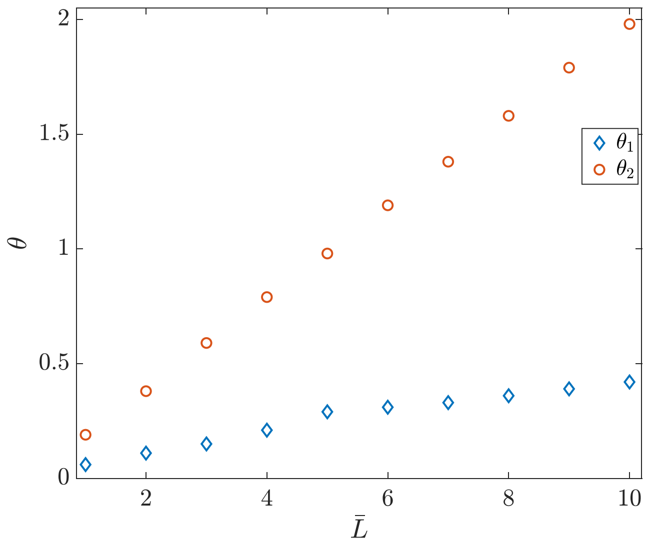

FIGURE 2 Optimized transition probability rates θ_1 and θ_2 with respect to the cost constraint $\bar{L}$ in the interval [1,10]. [Color figure can be viewed at wileyonlinelibrary.com]

where $k (k > 1)$ is a constant characterizing the weight for tuning the sojourn time of the subsystem and $\bar{L}$ is defined as the total cost of tuning the probability transition matrix. In the simulation example, we set $\bar{L} = 1, k = 5$, and $\theta_1, \theta_2 \in [0.01, 0.2]$. By adopting Theorem 1, we obtained the minimized $\mathscr{L}_1$-gain $\gamma^* = 8.76$ and a stabilizing probability rate matrix

$$\Pi(\theta^*) = \begin{bmatrix} -0.06 & 0.06 \\ 0.19 & -0.19 \end{bmatrix}.$$

The infection probability of node 1 versus the time before and after stabilization is presented in Figure 1. We also changed the value of the total cost $\bar{L}$ within the interval [1,10] and plotted the tendency of the optimized probability rates, as shown in Figure 2.

5 | CONCLUSIONS

We investigated the optimization of the probability rate matrix for positive Markov jump linear systems. Using the convex property of the difference between convex functions and existing linear programming-based stability results, we showed that the optimization problems were solved using DC programming. Finally, the effectiveness of the proposed framework was verified using simulation examples. Future research can focus on the optimization of physical networks with intervenable random factors.

AUTHOR CONTRIBUTIONS

Chengyan Zhao: Conceptualization; formal analysis; funding acquisition; investigation; methodology; writing—original draft. **Bohao Zhu:** Formal analysis;

investigation; writing—original draft; writing—review and editing. **Masaki Ogura:** Conceptualization; funding acquisition; project administration; supervision; writing—review and editing. **James Lam:** Conceptualization; funding acquisition; project administration; supervision; writing—review and editing.

CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

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