Transformation Groups
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Transformation Groups

ORIGINAL PAPER

On the Product Functor on Inner forms of the General Linear Group Over A Non-Archimedean Local Field

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Abstract

Let G_n be an inner form of a general linear group over a non-Archimedean local field. We fix an arbitrary irreducible representation σ of G_n. Building on the work of Lapid-Mínguez on the irreducibility of parabolic inductions, we show how to define a full subcategory of the category of smooth representations of some G_m, on which the parabolic induction functor $\tau \mapsto \tau \times \sigma$ is fully-faithful. A key ingredient of our proof for the fully-faithfulness is constructions of indecomposable representations of length 2. Such result for a special situation has been previously applied in proving the local non-tempered Gan-Gross-Prasad conjecture for non-Archimedean general linear groups. In this article, we apply the fully-faithful result to prove a certain big derivative arising from Jacquet functor satisfies the property that its socle is irreducible and has multiplicity one in the Jordan-Hölder sequence of the big derivative.

1 Introduction

Let F be a non-Archimedean local field and let D be a finite-dimensional F-central division algebra. Let $G_n = \mathrm{GL}_n(D)$ be the general linear group over D. Let $\mathrm{Alg}(G_n)$ be the category of smooth representations of G_n over $\mathbb{C}$. The parabolic induction is an important tool in constructing representations and plays a central role in the Zelevinsky classification of irreducible representations of $\mathrm{GL}_n(F)$ [53]. Recently, Aizenbud-Lapid and Lapid-Mínguez [3, 33–36] extensively study the irreducibility of parabolic inductions, with rich connections to combinatorics and geometry.

This paper focuses on some homological aspects of parabolic inductions. The main purpose is to elaborate some observations and results in [15], which we use functorial properties of parabolic inductions for proving the local non-tempered Gan-Gross-Prasad conjecture [21]. Our main result addresses the remark in [15, Section 9.2].

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We first explain the main object– the product functor. We denote by $\times$ the normalized parabolic induction. For a fixed irreducible representation π and a full subcategory $\mathcal{A}$ of $\mathrm{Alg}(G_n)$, define

$$\times_{\pi,\mathcal{A}} : \mathcal{A} \to \mathrm{Alg}(G_{n+k}),$$

given by $\times_{\pi,\mathcal{A}}(\omega) = \pi \times \omega$. Here we regard $\times_{\pi,\mathcal{A}}$ as a functor such that for a morphism $f : \pi' \to \pi''$ in $\mathcal{A}$, $\times_{\pi,\mathcal{A}}(f) = \mathrm{Id}_\pi \times f$, the one induced from the parabolic induction (see Section 3.1 for more precise descriptions).

Some general results about the product functor with respect to smooth duals and cohomological duals are given in Section 3.

We briefly recall the Zelevinsky theory [53] for $D = F$, see Section 2.1 for more notations. A *segment* takes the form $[a, b]_\rho$ for a supercuspidal representation ρ of some G_m and $a, b \in \mathbb{C}$ with $b - a \in \mathbb{Z}_{\geq 0}$. Zelevinsky [53] associates each segment Δ with a representation $\langle \Delta \rangle$, called a *segment representation*. A *multisegment* is a multiset of segments. Let Mult be the set of multisegments. For $\mathfrak{m} \in \mathrm{Mult}$, let $\langle \mathfrak{m} \rangle$ be the associated Zelevinsky module [53].

The irreducibility of the parabolic induction is extensively studied in [33]. A first question is that for a given irreducible representation π of G_n, how one can find another irreducible representation π' of G_m such that $\pi \times \pi'$ is also irreducible. One way to do so is via 'building from the (basic) segment case'. The precise meaning is as follows. Set

$$\mathcal{M}_\pi = \{\mathfrak{n} \in \mathrm{Mult} : \langle \Delta \rangle \times \pi \text{ is irreducible } \forall \Delta \in \mathfrak{n}\}.$$

Then, for any $\mathfrak{n} \in \mathcal{M}_\pi$, $\langle \mathfrak{n} \rangle \times \pi$ is irreducible [33, Proposition 6.1]. The converse is not true in general i.e. if $\langle \mathfrak{n} \rangle \times \pi$ is irreducible, it is not necessary that $\mathfrak{n} \in \mathcal{M}_\pi$.

We write $\mathfrak{m}_1 \leq_Z \mathfrak{m}_2$ if $\mathfrak{m}_1$ is obtained from $\mathfrak{m}_2$ by a sequence of intersection-union operations (see Section 2.2). Our observation is that the set $\mathcal{M}_\pi$ is closed under intersection-union operations in the following sense:

Theorem 1.1 *(=Theorem 4.1) Let π be an irreducible representation of G_n. For $\mathfrak{n} \in \mathcal{M}_\pi$, if $\mathfrak{n}'$ is another multisegment with $\mathfrak{n}' \leq_Z \mathfrak{n}$, then $\mathfrak{n}' \in \mathcal{M}_\pi$.*

Our proof for Theorem 1.1 uses properties from intertwining operators on $\square$-irreducible representations. Another possible approach for proving Theorem 1.1 is to use the combinatorial criteria of Lapid-Mínguez in [33, Proposition 5.12].

We now set $\mathcal{A}_\pi = \mathrm{Alg}_\pi(G_n)$ to be the full subcategory of $\mathrm{Alg}(G_n)$ whose objects are of finite length and have all simple composition factors isomorphic to $\langle \mathfrak{m} \rangle$ for some $\mathfrak{m} \in \mathcal{M}_\pi$. The significance of Theorem 1.1 is that one can obtain plenty examples of extensions from the set $\mathcal{M}_\pi$ and so $\mathcal{A}_\pi$ is not semisimple in most of cases. Indeed, those extensions are preserved under $\times_{\pi,\mathcal{A}_\pi}$, shown in Proposition 5.7 and Theorem 9.2. This in turn implies our main result:

Theorem 1.2 *(=Theorem 10.2) Let π be an irreducible representation of G_n. Then $\times_{\pi,\mathcal{A}_\pi}$ is a fully-faithful functor.*

Chan [15] deals with the case that π is a Speh representation and $\mathcal{A}$ is some subcategory coming from the irreducibility of the product between a cuspidal representation and π.

A key new ingredient in the proof of Theorem 1.2 is a construction of extensions between two irreducible representations. This differs from the approach used in [15], although we also need a basic case (when π is also a segment representation) from [15]. The main idea comes from a study of first extensions in the graded Hecke algebra case in [12]. Roughly speaking, those extensions for two non-isomorphic representations come from Zelevinsky standard modules, and those for two isomorphic representations reduce to the tempered case. However, we remark that we do not have a concrete classification for indecomposable modules of length 2.

For the self-extension case, we actually have more general statement:

Theorem 1.3 *(=Theorem 9.2) Let π_1 and π_2 be irreducible representations of G_k and G_l respectively such that $\pi_1 \times \pi_2$ is still irreducible. Suppose λ is an indecomposable representation of length 2 with both simple composition factors isomorphic to π_2. Then $\pi_1 \times \lambda$ is also indecomposable.*

Perhaps an interesting point of Theorem 1.3 is that the parabolic induction does not preserve indecomposability in general. In other words, some non-trivial self-extensions can be trivialized under parabolic inductions (see Remark 9.3).

Theorem 1.3 concerns about indecomposable modules of length 2. Our proof relies on some constructions of those modules. One important ingredient is analogous properties in the affine highest weight category introduced by Kleshchev [32] (also see [29]), see the proofs in Section 5. Roughly speaking such ingredient reduces to the computations of Ext-groups for tempered modules. Such Ext-groups are now better understood due to the work on discrete series by Silberger, Meyer, Opdam-Solleveld [39, 44, 49] using analytic methods and by [11] using algebraic methods; and more general case [45] via R-groups. We also refer the reader to [12] for more discussions.

Recent articles [3, 33–36] study the conditions of irreducibility for more general multisegments. In particular, when one of the multisegments arises from a so-called $\square$-irreducible representation, there are some precise conjectures connecting to the geometry of nilpotent orbits due to Geiß-Leclerc-Schröer and Lapid-Mínguez [23, 34, 35]. Thus one may hope for a version of Theorem 1.2 for replacing the segment case by other interesting classes of representations such as Speh, ladder or even $\square$-irreducible representations. One main problem goes back to understand the analog of the set $\mathcal{M}_\pi$ in Theorem 1.1 and so $\mathrm{Alg}_\pi(G_n)$ in Theorem 1.2. In [25], Gurevich-Lapid introduce a new class of representations parabolically induced from ladder representations, and so it is natural to ask if the extensions arising from those standard representations can be used to define a suitable analogue of $\mathrm{Alg}_\pi(G_n)$.

We now consider a Jacquet functor version of above discussions. For an irreducible representation π of G_k and for an admissible representation τ of G_n, define

$$\mathbb{D}_\pi(\tau) := \mathrm{Hom}_{G_k}(\pi, \tau_{N^-}),$$

where N^- is the opposite unipotent radical of the standard parabolic subgroup in G_n containing $G_k \times G_{n-k}$. Here τ_{N^-} is viewed as a G_k-module via the embedding $G_k \hookrightarrow G_k \times G_{n-k}$ to the first factor. We shall call such $\mathbb{D}_\pi$ to be a big derivative, and $\mathbb{D}_\pi(\tau)$ has a natural G_{n-k}-module structure (also see Definition 8.1).

The big derivative $\mathbb{D}_\pi$ is the right adjoint functor of the product functor, if we consider the functors are for the category of all smooth representations. However, this is not entirely correct if we restrict the functor to the full subcategory $\mathrm{Alg}_\pi(G_n)$ defined above. Nevertheless, there are some interesting cases that $\mathbb{D}_\pi$ forms an adjoint functor for $\times_{\pi,\mathrm{Alg}_\pi(G_n)}$. For example, the case considered in [15] works. In those cases, we could deduce that the big derivative is irreducible (see Lemma 11.3 for a precise statement). This is consequently applied to prove:

Theorem 1.4 *(c.f. Theorem 12.7) Suppose $D = F$. Let π be a generic irreducible representation of G_k. Let τ be any irreducible representation of G_n such that $\mathbb{D}_\pi(\tau) \neq 0$. Then $\mathbb{D}_\pi(\tau)$ satisfies the socle irreducible property (i.e. $\mathbb{D}_\pi(\tau)$ has unique simple submodule and such simple submodule appears with multiplicity one in the Jordan-Hölder sequence of $\mathbb{D}_\pi(\tau)$).*

When π is a cuspidal representation, the analogous statement for Theorem 1.4 for the affine Hecke algebra of type A is shown in the work of Grojnowski-Vazirani [24] by exploiting the explicit structure of a principal series.

The irreducibility part of the socle in Theorem 1.4 is shown by Kang-Kashiwara-Kim-Oh [30] (also see [3]) in a greater generality on $\square$-irreducible representations. For the irreducibility part of the socle, some variants of more specific cases using Gelfand-Kazhdan method are also shown by Aizenbud-Gourevitch [1].

Our emphasis on Theorem 1.4 is on the application of the product functor, which gives a basic case in Proposition 11.5. An advantage of this method is that one does not have to compute some internal structures of some modules. Hence, it has a higher potential for other applications such as the one in [15]. We shall also show how to extend the socle irreducible result to the case of generic representations (i.e. full version of Theorem 1.4) in the appendix.

As an analog of the problem of studying the irreducibility of parabolic inductions, one may ask the irreducibility of big derivatives. The product functor provides a technique on such problem as shown in the article while Theorem 1.4 provides some concrete examples.

A more classical viewpoint on studying parabolic inductions and Jacquet functors is on the Grothendieck group of the category of smooth representations of $\mathrm{GL}_n(F)$'s, in which those functors give a Hopf algebra structure [26, 51–54]. We hope this work could emphasis on some interesting higher structures associated to parabolic inductions and Jacquet functors.

2 Notations

2.1 Basic Notations

Let F be a non-Archimedean local field and let D be a finite-dimensional F-central division algebra. Let $G_n = \mathrm{GL}_n(D)$, the general linear group over D. The group G_0 is viewed as the trivial group. For $g \in G_n$, let $\nu(g) = |\mathrm{Nrd}(g)|_F$, where $\mathrm{Nrd} : G_n \to F^\times$ is the reduced norm and $|.|_F$ is the absolute value on F. All the representations we consider are smooth over $\mathbb{C}$ and we usually omit the adjectives 'smooth' and 'over $\mathbb{C}$'.

For a representation π of G_n, we write $\deg(\pi)$ for n. We shall usually not distinguish representations in the same isomorphism class.

For a supercuspidal representation ρ of G_n, let s_ρ be the unique value in $\mathbb{R}_{>0}$ such that $\rho \times \nu^{\pm s_\rho}\rho$ is reducible. Set $\nu_\rho = \nu^{s_\rho}$. For $a, b \in \mathbb{Z}$ with $b - a \in \mathbb{Z}_{\geq 0}$ and a supercuspidal representation ρ, a *segment* $[a, b]_\rho$ is the set $\left\{\nu_\rho^a\rho, \nu_\rho^{a+1}\rho, \ldots, \nu_\rho^b\rho\right\}$. We consider two segments $[a, b]_\rho$ and $[a', b']_{\rho'}$ are equal if $\nu_\rho^a\rho \cong \nu_{\rho'}^{a'}\rho'$ and $\nu_\rho^b\rho \cong \nu_\rho^{b'}\rho'$. If $\rho = 1$ is the trivial representation of G_1, we may simply write $[a, b]$ for $[a, b]_1$. We also consider the empty set as a segment and also set $[a, a-1]_\rho = \emptyset$. The *absolute length* $l_{abs}([a, b]_\rho)$ of a segment $[a, b]_\rho$ is $(b - a + 1)\deg(\rho)$. A *multisegment* is a multiset of non-empty segments, and we also consider the empty set as a multisegment. For a multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$, define $l_{abs}(\mathfrak{m}) = l_{abs}(\Delta_1) + \ldots + l_{abs}(\Delta_k)$, called the *length* of $\mathfrak{m}$.

We introduce the following notations:

- $\mathrm{Irr}(G_n)$ = the set of irreducible smooth representations of G_n and $\mathrm{Irr} = \cup_n \mathrm{Irr}(G_n)$;
- $\mathrm{Alg}(G_n)$ = the category of smooth representations of G_n;
- $\mathrm{Alg}_f(G_n)$ = the full subcategory of $\mathrm{Alg}(G_n)$ of all the smooth G_n-representations of finite length;
- let Alg_f be the set of smooth representations of some G_n (in other words, it is the set of all objects in $\mathrm{Alg}_f(G_n)$ for some n);
- Seg_n = the set of segments of absolute length n; and $\mathrm{Seg} = \cup_n \mathrm{Seg}_n$;
- Mult_n = the set of multisegments of length n; and $\mathrm{Mult} = \cup_n \mathrm{Mult}_n$;
- for $\pi \in \mathrm{Alg}_f$, $\mathrm{JH}(\pi)$ = the set of simple composition factors in π (i.e. multiplicities are not counted);
- $^\vee : \mathrm{Alg}(G_n) \to \mathrm{Alg}(G_n)$ is the smooth dual contravariant functor;
- for $\Delta = [a, b]_\rho \in \mathrm{Seg}$, define

$$\Delta^\vee = [-b, -a]_{\rho^\vee};$$

  for $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_r\} \in \mathrm{Mult}$,

$$\mathfrak{m}^\vee = \left\{\Delta_1^\vee, \ldots, \Delta_r^\vee\right\}$$

- for two supercuspidal representations ρ_1, ρ_2, we write $\rho_1 < \rho_2$ if there exists a positive integer c such that $\rho_2 \cong \nu_{\rho_1}^c\rho_1$. We write $\rho_1 \leq \rho_2$ if either $\rho_1 < \rho_2$ or $\rho_1 \cong \rho_2$;
- for a segment $\Delta = [a, b]_\rho$, write $a(\Delta) = \nu_\rho^a\rho$ and $b(\Delta) = \nu_\rho^b\rho$;
- for $\pi \in \mathrm{Irr}$, we write $\mathrm{csupp}(\pi) = \{\sigma_1, \ldots, \sigma_r\}$ to be the unique multiset of supercuspidal representations such that π is a composition factor of $\sigma_1 \times \ldots \times \sigma_r$;
- for $\pi \in \mathrm{Alg}_f$, let $\mathrm{soc}(\pi)$ be the socle (i.e. maximal semisimple submodule) of π and let $\mathrm{cosoc}(\pi)$ be the cosocle (i.e. maximal semisimple quotient) of π.

Since we are working on representations over $\mathbb{C}$, we shall not distinguish cuspidal representations and supercuspidal representations.

2.2 More Notations for Segments and Multisegments

For $\mathfrak{m} \in \text{Mult}$, two segments Δ_1 and Δ_2 in $\mathfrak{m}$ are said to be *linked* if $\Delta_1 \cup \Delta_2$ is still a segment and $\Delta_1 \not\subset \Delta_2$ and $\Delta_2 \not\subset \Delta_1$. We write $\Delta_1 < \Delta_2$ if Δ_1 and Δ_2 are linked and $a(\Delta_1) < a(\Delta_2)$. Note $\Delta_1 < \Delta_2$ automatically implies $b(\Delta_1) < b(\Delta_2)$.

A multisegment $\mathfrak{m}$ is said to be *generic* if any two segments in $\mathfrak{m}$ are not linked.

As in [53], for $\mathfrak{m}, \mathfrak{n} \in \text{Mult}_n$, we say that $\mathfrak{m}$ is obtained by an intersection-union process if there are two linked segments Δ_1, Δ_2 in $\mathfrak{n}$ such that

$$\mathfrak{m} = \begin{cases} \mathfrak{n} - \{\Delta_1, \Delta_2\} + \{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\} & \text{if } \Delta_1 \cap \Delta_2 \neq \emptyset \\ \mathfrak{n} - \{\Delta_1, \Delta_2\} + \{\Delta_1 \cup \Delta_2\} & \text{if } \Delta_1 \cap \Delta_2 = \emptyset \end{cases}$$

Here $+$ and $-$ represent the union and substraction as multisets.

Write $\mathfrak{m} <_Z \mathfrak{n}$ if $\mathfrak{m}$ can be obtained by a sequence of intersection-union operations from $\mathfrak{n}$, and write $\mathfrak{m} \leq_Z \mathfrak{n}$ if $\mathfrak{m} = \mathfrak{n}$ or $\mathfrak{m} <_Z \mathfrak{n}$.

2.3 Langlands and Zelevinsky Classification

For a segment $\Delta = [c, d]_\rho$, let $\langle \Delta \rangle$ (resp. $\text{St}(\Delta)$) be the unique simple submodule (resp. quotient) of

$$\nu_\rho^c \rho \times \ldots \times \nu_\rho^d \rho.$$

For any multisegment $\mathfrak{m}$, write the segments in $\mathfrak{m}$ as $\Delta_1, \ldots, \Delta_r$. We label the segments in $\mathfrak{m}$ such that

$$b(\Delta_1) \not< b(\Delta_2) \not< \ldots \not< b(\Delta_r).$$

Let

$$\zeta(\mathfrak{m}) = \langle \Delta_1 \rangle \times \ldots \langle \Delta_r \rangle, \quad \lambda(\mathfrak{m}) = \text{St}(\Delta_1) \times \ldots \times \text{St}(\Delta_r).$$

It is shown in [53] that there is a unique simple submodule in $\zeta(\mathfrak{m})$, which will be denoted $\langle \mathfrak{m} \rangle$. It is independent of a choice of a labeling above. There is a one-to-one correspondence

$$\text{Mult}_n \longleftrightarrow \text{Irr}(G_n), \quad \mathfrak{m} \mapsto \langle \mathfrak{m} \rangle.$$

The Zelevinsky classification is due to Zelevinsky [53] when $D = F$ and due to Mínguez-Sécherre [41, 42] when D is general.

On the other hand, $\lambda(\mathfrak{m})$ has unique simple quotient, denoted by $\text{St}(\mathfrak{m})$. This gives another one-to-one correspondence

$$\text{Mult}_n \longleftrightarrow \text{Irr}(G_n), \quad \mathfrak{m} \mapsto \text{St}(\mathfrak{m}).$$

The above correspondence in the form due to Langlands is known for $D = F$ in [53] and the general case follows from the local Jacquet-Langlands correspondence due to Deligne-Kazhdan-Vignéras [20] for the zero characteristic and Badulescu [4] for positive characteristics. Such classification also has significance in the unitary dual problem, see work of Tadić, Sécherre, Badulescu-Henniart-Lemaire-Sécherre [8, 47, 51].

2.4 Parabolic Inductions and Jacquet Functors

For non-negative integers $n_1, \ldots, n_r$ with $n_1 + \ldots + n_r = n$, let $P_{n_1,\ldots,n_r}$ be the parabolic subgroup containing the subgroup $G_{n_1} \times \ldots \times G_{n_r}$ as block diagonal matrices and all upper triangular matrices. We shall call $P_{n_1,\ldots,n_r}$ to be a standard parabolic subgroup. (Note that when n_i is zero, G_{n_i} is regarded as the trivial group and we may simply drop the term. We include such case for the convenience of notations later.) Let $N_{n_1,\ldots,n_r}$ be the unipotent radical of $P_{n_1,\ldots,n_r}$, and let $N^-_{n_1,\ldots,n_r}$ be the unipotent radical of the parabolic subgroup opposite to $P_{n_1,\ldots,n_r}$.

For $\pi_1 \in \mathrm{Alg}(G_{n_1})$ and $\pi_2 \in \mathrm{Alg}(G_{n_2})$, define $\pi_1 \times \pi_2$ to be

$$\pi_1 \times \pi_2 = \mathrm{Ind}_{P_{n_1,n_2}}^{G_{n_1+n_2}} (\pi_1 \boxtimes \pi_2),$$

the space of smooth functions from $G_{n_1+n_2}$ to $\pi_1 \boxtimes \pi_2$ satisfying

$$f(pg) = \delta(p)^{1/2} p.f(g),$$

where δ is the modular character of P_{n_1,n_2}. The G_n-action on $\pi_1 \times \pi_2$ is the right translation on those functions i.e. for $f \in \pi_1 \times \pi_2$,

$$(g.f)(g') = f(g'g).$$

Here we consider $\pi_1 \boxtimes \pi_2$ as a P_{n_1,n_2}-representation by the inflation. We shall simply call $\pi_1 \times \pi_2$ to be a *product*. The product is indeed an associative operation and so there is no ambiguity in defining $\pi_1 \times \ldots \times \pi_r$.

For a parabolic subgroup P of G_n with the Levi decomposition LN, define the Jacquet functor, as a L-representation:

$$\pi_N = \delta_P^{-1/2} \cdot \frac{\pi}{\langle n.x - x : x \in \pi, n \in N \rangle},$$

where δ_P is the modular character of P.

Both parabolic inductions and Jacquet functors are exact functors. For $n_1 + \ldots + n_r = n$, the parabolic induction $\pi \mapsto \mathrm{Ind}_{P_{n_1,\ldots,n_r}}^{G_n} \pi$ has the Jacquet functor $\pi \mapsto \pi_{N_{n_1,\ldots,n_r}}$ as its left adjoint functor, and has the opposite Jacquet functor $\pi \mapsto \pi_{N^-_{n_1,\ldots,n_r}}$ as its right adjoint functor.

Following [34], for $\pi \in \mathrm{Irr}$, π is said to be $\square$-*irreducible* if $\pi \times \pi$ is irreducible. Let $\mathrm{Irr}^{\square}$ be the set of $\square$-irreducible representations. In the content of quantum affine algebras, it is called real modules, see e.g. work of Hernandez-Leclerc and Kang-Kashiwara-Kim-Oh [27, 30]. A particular class of $\square$-irreducible representations is those $\mathrm{St}(\mathfrak{m})$ for a generic multisegment $\mathfrak{m} \in \mathrm{Mult}$.

2.5 Geometric Lemma

For $i \leq n$, we sometimes abbreviate N_i for $N_{n-i,i} \subset G_n$.

Let $\pi_1 \in \mathrm{Alg}(G_{n_1})$ and let $\pi_2 \in \mathrm{Alg}(G_{n_2})$. Let $n = n_1 + n_2$. The geometric lemma [10] gives that $(\pi_1 \times \pi_2)_{N_i}$ admits a filtration, whose successive subquotients are of the form:

$$\mathrm{Ind}_{P_{n_1-i_1,n_2-i_2} \times P_{i_1,i_2}}^{G_{n-i} \times G_i} ((\pi_1)_{N_{i_1}} \boxtimes (\pi_2)_{N_{i_2}})^{\phi},$$

where $i_1 + i_2 = i$ and ϕ sends a $G_{n_1-i_1} \times G_{i_1} \times G_{n_2-i_2} \times G_{i_2}$-representation to a $G_{n_1-i_1} \times G_{n_2-i_2} \times G_{i_1} \times G_{i_2}$-representation via the map

$$(g_1, g_2, g_3, g_4) \mapsto (g_1, g_3, g_2, g_4).$$

Moreover, the bottom layer in the filtration of $(\pi_1 \times \pi_2)_{N_i}$ is when $i_1 = \min\{n_1, i\}$ and the top layer in the filtration of $(\pi_1 \times \pi_2)_{N_i}$ is when $i_2 = \min\{n_2, i\}$.

2.6 Jacquet Functors on Segment and Steinberg Representations

Let $[a, b]_\rho \in \mathrm{Seg}$. Let $k = \deg(\rho)$. It follows from [53, Propositions 3.4 and 9.5] and [51, Proposition 3.1] that

$$\langle [a, b]_\rho \rangle_{N_{ik}} \cong \langle [a, b-i]_\rho \rangle \boxtimes \langle [b-i+1, b]_\rho \rangle \tag{2.1}$$

and

$$\mathrm{St}([a, b]_\rho)_{N_{ik}} \cong \mathrm{St}([a+i, b]_\rho) \boxtimes \mathrm{St}([a, a+i-1]_\rho), \tag{2.2}$$

and the Jacquet modules $\langle [a, b]_\rho \rangle_{N_j}$ and $\mathrm{St}([a, b]_\rho)_{N_j}$ are zero if k does not divide j.

We sometimes use the formulas implicitly in computing layers involving the geometric lemma.

3 Some Generalities of the Product Functor

3.1 Product Functor

Let $\mathcal{A}$ be a full Serre subcategory of $\mathrm{Alg}(G_n)$. For a fixed irreducible representation π of G_k, we define the product functor

$$\times_{\pi,\mathcal{A}} : \mathcal{A} \to \mathrm{Alg}(G_{n+k})$$

as

$$\times_{\pi,\mathcal{A}}(\tau) = \pi \times \tau$$

and, for a morphism f from τ to τ' in $\mathcal{A}$ and $F \in \pi \times \tau$ (under the realization in Section 2.4),

$$(\times_{\pi,\mathcal{A}}(f)(F))(g) = (\mathrm{Id}_\pi \boxtimes f)(F(g)), \quad \text{for any } g \in G_{n+k}.$$

Note that we do not assume $\times_{\pi,\mathcal{A}}$ preserves simple objects at this point.

Proposition 3.1 *The functor $\times_{\pi,\mathcal{A}}$ is exact and faithful.*

Proof Exactness follows from that the parabolic induction is an exact functor. The faithfulness then follows from that the functor sends a non-zero object to a non-zero object. □

3.2 Smooth Dual Functor

In this section, we specify to $D = F$. Let $\theta : G_n \to G_n$ given by $\theta(g) = g^{-t}$, the transpose inverse of g. This induces a covariant auto-equivalence for $\mathrm{Alg}(G_n)$, still denoted by θ. For $D = F$, it is a classical result of Gelfand-Kazhdan that $\theta(\pi) \cong \pi^\vee$ for any $\pi \in \mathrm{Irr}$.

Definition 3.2 A full Serre subcategory $\mathcal{A}$ of $\mathrm{Alg}_f(G_n)$ is said to be $\sim$ *-closed* if for any object C in $\mathcal{A}$, $\widetilde{C}$ is still in $\mathcal{A}$.

One main example is the category $\mathrm{Alg}_\pi(G_n)$, as shown in Theorem 4.1 later. Define$^\sim = \theta \circ {}^\vee$ and so it is also a contravariant functor.

Proposition 3.3 *Let $D = F$. Let $\mathcal{A}$ be a $\sim$-closed full subcategory of* $\mathrm{Alg}_f(G_n)$. *Let $\pi \in \mathrm{Irr}(G_k)$. Define the right product functor*

$$\times^{\pi,\mathcal{A}} : \mathcal{A} \to \mathrm{Alg}(G_{n+k}), \quad \times^{\pi,\mathcal{A}}(\pi') = \pi' \times \pi.$$

Then $\times_{\pi,\mathcal{A}}$ is fully-faithful if and only if $\times^{\pi,\mathcal{A}}$ is fully-faithful.

Proof We only prove the if direction, and the only if direction can be proved similarly. We have the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{G_{n+k}}(\pi \times \pi_1', \pi \times \pi_2') &\cong \mathrm{Hom}_{G_{n+k}}(\widetilde{\pi \times \pi_2'}, \widetilde{\pi \times \pi_1'}) \\
&\cong \mathrm{Hom}_{G_{n+k}}(\widetilde{\pi}_2' \times \widetilde{\pi}, \widetilde{\pi}_1' \times \widetilde{\pi}) \\
&\cong \mathrm{Hom}_{G_{n+k}}(\widetilde{\pi}_2' \times \pi, \widetilde{\pi}_1' \times \pi) \\
&\cong \mathrm{Hom}_{G_n}(\widetilde{\pi}_2', \widetilde{\pi}_1') \\
&\cong \mathrm{Hom}_{G_n}(\pi_1', \pi_2')
\end{aligned}$$

The first and last isomorphisms follow from taking the duals (and the representations are admissible). The second isomorphism follows from the compatibility between taking duals and parabolic inductions [43, Page 173]. The third isomorphism follows from a result of Gelfand-Kazhdan [22, Theorem 2] (see [9, Theorem 7.3]). The fourth isomorphism follows from the if direction. □

3.3 Cohomological Dual Functor

Let $\mathcal{H}(G_n)$ be the space of compactly supported smooth $\mathbb{C}$-valued functions on G_n, viewed as a G_n-representation with the action given by: for $f \in \mathcal{H}(G_n)$, $(g.f)(g') =$

$f(g'g)$. Let $\mathfrak{R}$ be a Bernstein component of $\mathrm{Alg}(G_n)$ and let d be the homological dimension of $\mathfrak{R}$. Given a finitely-generated G_n-module π in $\mathfrak{R}$, define

$$\mathcal{D}(\pi) = \mathrm{Ext}^d_{G_n}(\pi, \mathcal{H}(G_n))$$

viewed as a G_n-module. As shown in [6, Page 102] and [48, Page 132], $\mathcal{D}$ is a contravariant exact functor. With the property that $\mathcal{D}^2 = \mathrm{Id}$, $\mathcal{D}$ is a fully-faithful functor. The functor also sends a simple object to a simple object and agrees with the Aubert-Zelevinsky dual [2, 53] in the Grothendieck group level, see the work of Schneider-Stuhler [48, Proposition IV.5.2] and Bernstein-Bezrukavnikov-Kazhdan [7, Section 3.2]. Explicit algorithm for computing $\mathcal{D}(\pi)$ for $\pi \in \mathrm{Irr}$ is given by Mœglin-Waldspurger [43].

We first recall the following result of Bernstein:

Theorem 3.4 [6, Theorem 31(4)] *We fix Bernstein components $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of $\mathrm{Alg}(G_{n_1})$ and $\mathrm{Alg}(G_{n_2})$ respectively. Let π_1 and π_2 be finitely-generated objects in $\mathfrak{R}_1$ and $\mathfrak{R}_2$ respectively. Let $\mathfrak{R}$ be the unique Bernstein component containing the object $\pi_1 \times \pi_2$. Then*

$$\mathcal{D}(\pi_1 \times \pi_2) \cong \mathcal{D}(\pi_2) \times \mathcal{D}(\pi_1).$$

We remark that the switch in the terms on the RHS comes from switching the induction between a standard parabolic subgroup and its opposite one.

Corollary 3.5 *Let $\mathfrak{R}$ be a Bernstein component of $\mathrm{Alg}(G_n)$ and let $\mathfrak{R}_f$ be the full subcategory of $\mathfrak{R}$ of all objects of finite lengths. Let $\mathcal{A}$ be a full Serre subcategory of $\mathfrak{R}_f$. Let $\pi \in \mathrm{Irr}$. This gives a full subcategory $\mathcal{D}(\mathcal{A})$ whose objects are $\mathcal{D}(\pi)$ for objects π in $\mathcal{A}$ and morphisms $\mathcal{D}(f)$ for morphisms f in $\mathcal{A}$. Then $\times_{\pi,\mathcal{A}}$ is a fully-faithful functor if and only if $\times^{\mathcal{D}(\pi),\mathcal{D}(\mathcal{A})}$ is a fully-faithful functor. Here $\times^{\mathcal{D}(\pi),\mathcal{D}(\mathcal{A})}$ is defined in Proposition 3.3.*

Proof This follows from Theorem 3.4 and that $\mathcal{D}$ is a fully-faithful contravariant functor. □

4 Product with A Segment Representation and Intersection-union Process

Recall that for $\pi \in \mathrm{Irr}$, $\mathcal{M}_\pi$ is the set of all multisegments $\mathfrak{n}$ such that for any segment Δ in $\mathfrak{n}$, $\langle\Delta\rangle \times \pi$ is irreducible.

We say that $\pi \in \mathrm{Alg}_f$ is *SI* or *socle irreducible* if $\mathrm{soc}(\pi)$ is irreducible and occurs with multiplicity one in the Jordan-Hölder sequence of π.

Theorem 4.1 *Let $\pi \in \mathrm{Irr}(G_n)$. Let $\mathfrak{m} \in \mathcal{M}_\pi$. For any $\mathfrak{n} \in \mathrm{Mult}$ with $\mathfrak{n} \leq_Z \mathfrak{m}$, $\mathfrak{n} \in \mathcal{M}_\pi$.*

Proof For $\pi_1, \pi_2 \in \mathrm{Alg}_f$, let R_{π_1,π_2} be the normalized non-zero intertwining operator from $\pi_1 \times \pi_2$ to $\pi_2 \times \pi_1$ (see [34, Section 2]). By the transitivity of $\leq_Z$, we reduce to the case that $\mathfrak{m} \in \mathcal{M}_\pi$ is of two linked segments. Now, fix an arbitrary $\pi \in \mathrm{Irr}$, and let $\Delta_1, \Delta_2 \in \mathrm{Seg}$ such that $\langle \Delta_1 \rangle \times \pi$ and $\langle \Delta_2 \rangle \times \pi$ are irreducible. Then $(R_{\langle \Delta_2 \rangle \times \pi} \times \mathrm{Id}_{\langle \Delta_1 \rangle}) \circ (\mathrm{Id}_{\langle \Delta_2 \rangle} \times R_{\langle \Delta_1 \rangle \times \pi})$ sends $\langle \Delta_2 \rangle \times \langle \Delta_1 \rangle \times \pi$ to $\pi \times \langle \Delta_2 \rangle \times \langle \Delta_1 \rangle$, and is an isomorphism:

$$\langle \Delta_2 \rangle \times \langle \Delta_1 \rangle \times \pi \cong \pi \times \langle \Delta_2 \rangle \times \langle \Delta_1 \rangle. \tag{4.3}$$

By switching the labelling if necessary, we also have:

$$\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \pi \cong \pi \times \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle. \tag{4.4}$$

Again, by switching labelling if necessary, we may and shall assume that $\langle \Delta_1 + \Delta_2 \rangle$ is in the quotient of $\langle \Delta_2 \rangle \times \langle \Delta_1 \rangle$.

Let $\tau = \langle \{\Delta_1, \Delta_2\} \rangle \times \pi$. Let

$$\tau_1 := \mathrm{soc}(\langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle \times \pi), \quad \tau_2 := \mathrm{cosoc}(\langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle \times \pi).$$

Here $\Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2$ is equal to the multisegment $\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}$ if $\Delta_1 \cap \Delta_2$ is non-empty and is equal to the multisegment $\{\Delta_1 \cup \Delta_2\}$ if $\Delta_1 \cap \Delta_2$ is empty.

Suppose $\langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle \times \pi$ is not irreducible to arrive a contradiction. Then, we must have $\tau_1 \not\cong \tau_2$, which follows from $\square$-irreducibility of $\langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle$ ([53, Theorem 9.7] for $D = F$, see [51, Lemma 2.5] and [33, Lemma 6.17] for general) and the SI property of $\langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle$ [34, Lemma 2.8]. Thus, we must have either $\tau_1 \not\cong \tau$ or $\tau_2 \not\cong \tau$.

We now consider two cases separately:

(1) $\tau_1 \not\cong \tau$. Then τ_1 appears in the submodule of $\langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle \times \pi$ and so appears in the submodule of $\langle \Delta_2 \rangle \times \langle \Delta_1 \rangle \times \pi$. Using Eq. 4.3, we also have that τ_1 is a submodule of $\pi \times \langle \Delta_2 \rangle \times \langle \Delta_1 \rangle$. Since $\tau_1 \not\cong \tau$, τ_1 must come from the submodule of $\pi \times \langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle$. This shows that

$$\tau_1 \cong \mathrm{soc}(\pi \times \langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle) \cong \mathrm{soc}(\langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle \times \pi).$$

In other words, the socle and cosocle of $\pi \times \langle \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \rangle$ coincides by [34, Corollary 2.4]. By [34, Lemma 2.8], we must then have that $\pi \times \langle \Delta_1 \cap \Delta_2 + \Delta_1 \cup \Delta_2 \rangle$ is irreducible.

(2) $\tau_2 \not\cong \tau$. The proof is similar to the previous case, but we consider quotients rather than submodules and use Eq. 4.4 rather than Eq. 4.3.

$\square$

4.1 Some Explicit Criteria for A Multisegment in $\mathcal{M}_\pi$

A segment $\Delta = [a, b]_\rho$ is said to be *juxtaposed* to another segment $\Delta' = [a', b']_\rho$ if either $b + 1 = a'$ or $b' + 1 = a$.

Remark 4.2 (1) Let $\pi \cong \mathrm{St}(\mathfrak{m})$ for $\mathfrak{m} \in \mathrm{Mult}$ with all segments in $\mathfrak{m}$ mutually unlinked. Let $\mathfrak{m} \in \mathrm{Mult}$ such that $\pi \cong \mathrm{St}(\mathfrak{m})$. Then $\mathfrak{n} \in \mathcal{M}_\pi$ if and only if any segment in $\mathfrak{n}$ is not juxtaposed to any segment in $\mathfrak{m}$. (See [5, THÉORÈME 0.1])

(2) Let π be a Speh representation with the corresponding multisegment $\mathfrak{m}$. We label the segments in $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_r\}$ satisfying $\Delta_1 < \ldots < \Delta_r$. Then $\mathfrak{n} \in \mathcal{M}_\pi$ if and only if any segment Δ in $\mathfrak{n}$ satisfies $\Delta \not< \Delta_1$ and $\Delta_r \not< \Delta$. (See [33, Lemma 6.5])

5 Indecomposability Under Product Functor: Non-isomorphic Cases

5.1 Some Results on Irreducibility

In this section, we recall some results on the irreducibility of parabolic inductions. Most results are from or deduced from [33].

Lemma 5.1 [33, Lemma 3.9] *Let* $\Delta \in \mathrm{Seg}$ *and let* $\mathfrak{m} \in \mathrm{Mult}$. *Then* $\langle \Delta \rangle \times \langle \mathfrak{m} \rangle$ *is irreducible if and only if* $\langle \Delta \rangle \times \langle \mathfrak{m} \rangle \cong \langle \mathfrak{m} \rangle \times \langle \Delta \rangle$.

Proposition 5.2 [33, Proposition 6.1] *Let* $\mathfrak{m} \in \mathrm{Mult}$ *and let* $\pi = \langle \mathfrak{m} \rangle$. *Let* $\mathfrak{p} \in \mathcal{M}_\pi$. *Then*

(1) $\langle \mathfrak{p} \rangle \times \pi$ *is irreducible; and*
(2) $\zeta(\mathfrak{p}) \times \pi \hookrightarrow \zeta(\mathfrak{p} + \mathfrak{m})$*; and*
(3) $\pi \times \zeta(\mathfrak{p}) \hookrightarrow \zeta(\mathfrak{p} + \mathfrak{m})$.

Lemma 5.3 *Let* $\mathfrak{m}, \mathfrak{p} \in \mathrm{Mult}$. *Then* $\mathfrak{p} \in \mathcal{M}_{\langle \mathfrak{m} \rangle}$ *if and only if* $\mathfrak{p}^\vee \in \mathcal{M}_{\langle \mathfrak{m}^\vee \rangle}$.

Proof This follows from definitions. □

5.2 Indecomposability

We remark that an analogous result holds for other connected reductive groups with replacing the Zelevinsky classification by the Langlands classification (also see [12]). For the Langlands classification version, we remark that there is also an analogous statement for branching laws [14], with the generic case conjectured by D. Prasad [46] and proved in [19] by Savin and the author.

Lemma 5.4 *Let* $\mathfrak{m}, \mathfrak{n} \in \mathrm{Mult}_n$. *Suppose* $\mathfrak{n} \neq \mathfrak{m}$. *Then*

$$\mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{m}^\vee)^\vee, \zeta(\mathfrak{n})) = 0$$

for all i.

Proof We shall prove by an induction on the sum of the numbers of segments in $\mathfrak{m}$ and $\mathfrak{n}$. When both $\mathfrak{m}$ and $\mathfrak{n}$ are empty sets, there is nothing to prove.

Let $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_r\}$ with

$$b(\Delta_1) \not< \ldots \not< b(\Delta_r).$$

Similarly, let $\mathfrak{n} = \left\{\Delta'_1, \ldots, \Delta'_s\right\}$ with

$$b(\Delta'_1) \not< \ldots \not< b(\Delta'_s).$$

If no segment in $\mathfrak{n}$ satisfies $b(\Delta'_i) \geq b(\Delta_1)$, then a cuspidal support argument gives that $\mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{m}^\vee)^\vee, \zeta(\mathfrak{n})) = 0$ for all i. Furthermore, if we have Δ'_i such that $b(\Delta'_i) > b(\Delta_1)$, then a cuspidal support argument again gives that

$$\mathrm{Ext}^k_{G_n}(\zeta(\mathfrak{m}^\vee)^\vee, \zeta(\mathfrak{n})) = 0.$$

Set $\rho = b(\Delta_1)$. Thus, now we consider that $b(\Delta'_i) \cong \rho$ and there is no segment Δ'_j in $\mathfrak{n}$ satisfying $b(\Delta'_j) > \rho$. Now, by relabelling if necessary (using Lemma 5.1), we may assume that Δ_1 is a shortest segment in $\mathfrak{m}$ with $b(\Delta_1) \cong \rho$, and similarly assume that Δ'_1 is a shortest segment in $\mathfrak{n}$ with $b(\Delta'_1) \cong \rho$. We now consider the following three cases:

- Suppose $\Delta'_1 \subsetneq \Delta_1$. Then, Frobenius reciprocity gives that:

$$\begin{aligned}(*)\quad &\mathrm{Ext}^i_{G_n}(\langle\Delta_r\rangle \times \ldots \times \langle\Delta_1\rangle, \langle\Delta_1\rangle \boxtimes \zeta(\mathfrak{n} - \Delta'_1)) \\ \cong\ &\mathrm{Ext}^i_{G_n}((\langle\Delta_r\rangle \times \ldots \times \langle\Delta_1\rangle)_{N_{n-l_{abs}(\Delta'_1)}}, \langle\Delta'_1\rangle \boxtimes \zeta(\mathfrak{n} - \Delta'_1)).\end{aligned}$$

  Now one analyzes the layers from the geometric lemma on the term $(\langle\Delta_r\rangle \times \ldots \times \langle\Delta_1\rangle)_{N_{n-l_{abs}(\Delta'_1)}}$ (also see Section 2.6). One sees that no layer has the same cuspidal support as $\langle\Delta'_1\rangle \boxtimes \zeta(\mathfrak{n} - \Delta'_1)$, and so this gives such desired Ext-vanishing by the standard argument on an action of the Bernstein center.
- Suppose $\Delta_1 \subsetneq \Delta'_1$. Then, one uses that

$$\mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{m}^\vee)^\vee, \zeta(\mathfrak{n})) \cong \mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{n})^\vee, \zeta(\mathfrak{m}^\vee)).$$

  Now, we write $\zeta(\mathfrak{m}^\vee) \cong \zeta(\mathfrak{m}^\vee - \Delta_1^\vee) \times \langle\Delta_1^\vee\rangle$. One applies Frobenius reciprocity to give that:

$$\begin{aligned}\mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{n})^\vee, \zeta(\mathfrak{m}^\vee)) \cong\ &\mathrm{Ext}^i_{G_{n-l_{abs}(\Delta_1)} \times G_{l_{abs}(\Delta_1)}}((\langle{\Delta'_1}^\vee\rangle \times \ldots \\ &\times \langle{\Delta'_s}^\vee\rangle)_{N_{l_{abs}(\Delta_1)}}, \langle{\Delta_r}^\vee\rangle \times \ldots \times \langle{\Delta_2}^\vee\rangle \boxtimes \langle{\Delta_1}^\vee\rangle).\end{aligned}$$

  Now again analysing layers in the geometric lemma on $(\langle{\Delta'_1}^\vee\rangle \times \ldots \times \langle{\Delta'_s}^\vee\rangle)_{N_{l_{abs}(\Delta_1)}}$ (see Section 2.6 again), one can compare cuspidal supports to give Ext-vanishing.

- Suppose $\Delta_1 = \Delta_1'$. Then we apply the Frobenius reciprocity as (*). Then, again we compute the layers from the geometric lemma on the term

$$(\langle \Delta_r \rangle \times \ldots \times \langle \Delta_1 \rangle)_{N_{n-l_{abs}(\Delta_1')}}.$$

Then, a cuspidal support consideration on the $G_{l_{abs}(\Delta_1)}$ factor in $G_{l_{abs}(\Delta_1)} \times G_{n-l_{abs}(\Delta_1)}$ gives that only possible layers contributing a non-zero Ext-group take the form:

$$\langle \Delta_1 \rangle \boxtimes (\langle \Delta_r \rangle \times \ldots \times \langle \Delta_2 \rangle).$$

Let $G' = G_{l_{abs}(\Delta_1)} \times G_{n-l_{abs}(\Delta_1)}$. Now, by the Künneth formula,

$$\begin{aligned} &\mathrm{Ext}^i_{G'}(\langle \Delta_1 \rangle \boxtimes (\langle \Delta_r \rangle \times \ldots \times \langle \Delta_2 \rangle), \langle \Delta_1' \rangle \boxtimes \zeta(\mathfrak{n} - \Delta_1')) \\ = &\bigoplus_{k+l=i} \mathrm{Ext}^k_{G'}(\langle \Delta_1 \rangle, \langle \Delta_1' \rangle) \boxtimes \mathrm{Ext}^l_{G'}(\langle \Delta_r \rangle \times \ldots \times \langle \Delta_2 \rangle, \zeta(\mathfrak{n} - \Delta_1')) \end{aligned}$$

The latter term is zero by the induction, and so such layer will also give vanishing Ext-groups. Now, since all layers in the geometric lemma give vanishing Ext-groups, we again have that $\mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{m}^\vee)^\vee, \zeta(\mathfrak{n})) = 0$ for all i.

□

Lemma 5.5 *Let* $\mathfrak{m}, \mathfrak{n} \in \mathrm{Mult}_n$. *Suppose* $\mathfrak{n} \not\leq_Z \mathfrak{m}$. *Then*

$$\mathrm{Ext}^i_{G_n}(\langle \mathfrak{m} \rangle, \zeta(\mathfrak{n})) = 0$$

for all i.

Proof The basic case is that when all the segments in $\mathfrak{m}$ are unlinked. In such case, either $\zeta(\mathfrak{n})$ does not have the same cuspidal support as $\langle \mathfrak{m} \rangle$; or $\mathfrak{n}$ is not generic. That case then follows from Lemma 5.4.

We now consider that some segments in $\mathfrak{m}$ are unlinked. Then it admits a short exact sequence:

$$0 \to \omega \to \zeta(\mathfrak{m}^\vee)^\vee \to \langle \mathfrak{m} \rangle \to 0,$$

where ω is the kernel of the surjection. Then, the Zelevinsky theory [53, Theorem 7.1] implies that any simple composition factor of ω has the associated multisegment $\mathfrak{m}'$ with $\mathfrak{m}' \leq_Z \mathfrak{m}$. Thus we still have $\mathfrak{m}' \not\leq_Z \mathfrak{n}$. Inductively on $\leq_Z$ (the basic case explained above), we have that:

$$\mathrm{Ext}^i_{G_n}(\omega, \zeta(\mathfrak{n})) = 0$$

for all i. Thus a long exact sequence argument gives that

$$\mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{m}^\vee)^\vee, \zeta(\mathfrak{n})) \cong \mathrm{Ext}^i_{G_n}(\langle \mathfrak{m} \rangle, \zeta(\mathfrak{n})).$$

Now the former one is zero by Lemma 5.4 and so the latter one is also zero. □

For $\pi_1, \pi_2 \in \mathrm{Irr}$, we write $\pi_1 \leq_Z \pi_2$ if $\mathfrak{m}_1 \leq_Z \mathfrak{m}_2$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the unique multisegments such that $\pi_1 \cong \langle \mathfrak{m}_1 \rangle$ and $\pi_2 \cong \langle \mathfrak{m}_2 \rangle$.

Lemma 5.6 *Let λ be a representation of G_n of length 2. Suppose λ is indecomposable and the two simple composition factors of λ are not isomorphic. Then either*

(1) $\lambda \hookrightarrow \zeta(\mathfrak{p})$ for some multisegment $\mathfrak{p}$; or
(2) $\lambda^\vee \hookrightarrow \zeta(\mathfrak{p})$ for some multisegment $\mathfrak{p}$.

Proof Let π be the simple quotient of λ and let π' be the simple submodule of λ. We consider the following three cases:

- Case 1: $\pi <_Z \pi'$. Let $\mathfrak{p}$ be the multisegment such that $\pi' \cong \langle \mathfrak{p} \rangle$. We have the following short exact sequence:

$$0 \to \pi' \to \lambda \to \pi \to 0.$$

  Then applying $\mathrm{Hom}_{G_n}(., \zeta(\mathfrak{p}))$, we have the following long exact sequence:

$$0 \to \mathrm{Hom}_{G_n}(\pi, \zeta(\mathfrak{p})) \to \mathrm{Hom}_{G_n}(\lambda, \zeta(\mathfrak{p})) \to \mathrm{Hom}_{G_n}(\pi', \zeta(\mathfrak{p})) \to \mathrm{Ext}^1_{G_n}(\pi, \zeta(\mathfrak{p})).$$

  By Lemma 5.5, the first and last terms are zero, and the third term has one-dimensional. Thus the unique map from λ to $\zeta(\mathfrak{p})$ is still non-zero when restricting to π'. Since π' is the unique simple module, the map must then be an embedding.
- Case 2: $\pi' <_Z \pi$. In such case, we consider $\lambda^\vee$, which has simple submodule $\pi'^\vee$ and simple quotient $\pi^\vee$. We still have that $\pi'^\vee <_Z \pi^\vee$. Now the argument in Case 1 gives the embedding $\lambda^\vee \hookrightarrow \zeta(\mathfrak{p})$ for some multisegment $\mathfrak{p}$.
- Case 3: π' and π are not $\leq_Z$-comparable. It suffices to prove that

$$\mathrm{Ext}^1_{G_n}(\pi, \pi') = 0$$

  i.e. such indecomposable λ does not happen. To this end, let $\mathfrak{p}$ and $\mathfrak{p}'$ be the multisegments such that $\pi \cong \langle \mathfrak{p} \rangle$ and $\pi' \cong \langle \mathfrak{p}' \rangle$. We consider the following short exact sequences:

$$0 \to \langle \mathfrak{p}' \rangle \to \zeta(\mathfrak{p}') \to \omega \to 0,$$

  where ω is the cokernel of the first injection. Then, a long exact sequence argument with Lemma 5.5 gives

$$\mathrm{Hom}_{G_n}(\langle \mathfrak{p} \rangle, \omega) \cong \mathrm{Ext}^1_{G_n}(\langle \mathfrak{p} \rangle, \langle \mathfrak{p}' \rangle).$$

  The former one is zero since any simple composition factor ω' in ω also satisfies $\langle \mathfrak{p} \rangle \not\leq_Z \omega'$. Thus the latter Ext is also zero. □

Define

$$\mathrm{Alg}_\pi(G_n)$$

to be the full subcategory of $\mathrm{Alg}_f(G_n)$ of objects, all of whose simple composition factors are isomorphic to $\langle \mathfrak{m} \rangle$ for some $\mathfrak{m} \in \mathcal{M}_\pi$. In other words, $\mathrm{Alg}_\pi(G_n)$ is the full Serre subcategory generated by simple objects of the form $\langle \mathfrak{m} \rangle$ for $\mathfrak{m}$ in $\mathcal{M}_\pi$.

Proposition 5.7 *Let λ be a representation of G_n of length 2. Suppose λ is indecomposable. Suppose the two simple composition factors of λ are not isomorphic and both are in $\mathrm{Alg}_{\pi}(G_n)$. Then $\pi \times \lambda$ is still an indecomposable representation of length 2.*

Proof By Proposition 5.2(1), we have that $\pi \times \lambda$ has length 2. To show the indecomposability, it suffices to show that $\pi \times \lambda$ has either unique simple quotient or unique simple submodule. Let π_1 be the simple quotient of λ and let π_2 be the simple submodule of λ. Let $\mathfrak{m}$ be the multisegment associated to π.

According to the proof of Lemma 5.6, we must have one of the following two cases:

- Case (1): $\pi_1 <_Z \pi_2$. In such case, there exists an embedding, by Lemma 5.6,

$$\lambda \hookrightarrow \zeta(\mathfrak{p})$$

  for some multisegment $\mathfrak{p}$. Thus we also have an embedding:

$$\pi \times \lambda \hookrightarrow \pi \times \zeta(\mathfrak{p}).$$

  But the latter module embeds to $\zeta(\mathfrak{m}+\mathfrak{p})$ by Proposition 5.2(3), which has a unique submodule. Thus, $\pi \times \lambda$ also has unique submodule and so is indecomposable.
- Case (2): $\pi_2 <_Z \pi_1$. It suffices to show that

$$(\pi \times \lambda)^{\vee} = \pi^{\vee} \times \lambda^{\vee}$$

  has unique simple submodule. We have the embedding, by (the proof of) Lemma 5.6 again:

$$\lambda^{\vee} \hookrightarrow \zeta(\mathfrak{q}) \tag{5.5}$$

  for some $\mathfrak{q} \in \mathrm{Mult}$. We now consider the following embeddings:

$$\pi^{\vee} \times \lambda^{\vee} \hookrightarrow \pi^{\vee} \times \zeta(\mathfrak{q}) \hookrightarrow \zeta(\mathfrak{q} + \mathfrak{m}^{\vee}),$$

  where the first embedding follows from Eq. 5.5 and the second embedding follows from Proposition 5.2 and Lemma 5.3. □

6 Some Results Involving the Geometric Lemma

6.1 A Counting Problem

In order to give a favour of using the geometric lemma below, let us first consider the following lemma involving some counting arguments. We first define some notions.

For $\mathfrak{m} \in \mathrm{Mult}$ and $\Delta \in \mathrm{Seg}$, let

$$\mathfrak{m}_{\Delta} = \{\overbrace{\Delta, \ldots, \Delta}^{k \text{ times}}\},$$

where k is the multiplicity of Δ in $\mathfrak{m}$. In particular, $\mathfrak{m}_\Delta$ is a submultisegment of $\mathfrak{m}$. For example, if $\mathfrak{m} = \{[1], [1, 2], [1, 2], [2, 3], [2, 3], [4]\}$, then $\mathfrak{m}_{[1,2]} = \{[1, 2], [1, 2]\}$ and $\mathfrak{m}_{[4]} = \{[4]\}$.

For two segments Δ, Δ', we write $\Delta <_b \Delta'$ if either one of the following conditions holds:

- $b(\Delta) < b(\Delta')$; or
- $b(\Delta) \cong b(\Delta')$ and $a(\Delta) \leq a(\Delta')$.

We write $\Delta \leq_b \Delta'$ if $\Delta = \Delta'$ or $\Delta <_b \Delta'$. This defines a partial ordering on Seg.

Lemma 6.1 *Let* $\mathfrak{m} \in$ Mult. *We write* $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_r\}$. *Let* $\Delta = [a, b]_\rho$ *be a* $\leq_b$*-maximal element in* $\mathfrak{m}$. *For each segment* $\Delta_i = [a_i, b_i]_{\rho_i}$, *we write:* $\Delta_i^+ = [a_i, c_i]_{\rho_i}$ *and* $\Delta_i^- = [c_i + 1, b_i]_{\rho_i}$ *for some* $a_i - 1 \leq c_i \leq b_i$. *By abuse of notations, we write* $A = \cup_i \Delta_i^+$ *as a multiset of cuspidal representations. Let k be the number of segments in* $\mathfrak{m}_\Delta$. *If*

$$A = \cup_{j=1}^k \Delta$$

as multisets, then $c_i = b_i$ *if* $\Delta_i = \Delta$ *and* $c_i = a_i - 1$ *if* $\Delta_i \neq \Delta$.

Proof Note that there are k copies of $b(\Delta)$ in $\cup_{j=1}^k \Delta$. Hence, we must also have k copies of $b(\Delta)$ in A. Thus, we must have k-copies of Δ_i in $\mathfrak{m}_{b=b(\Delta)}$ such that $\Delta_i^+ = \Delta_i$. We write such k segments as $\Delta_{i_1}, \ldots, \Delta_{i_k}$. Now, recall that Δ is $\leq_b$-maximal from our choice, and so if one of $\Delta_{i_j} \neq \Delta$, then Δ_{i_j} contains the cuspidal representation $\nu_\rho^{-1} a(\Delta)$. Thus it is impossible. Hence, all $\Delta_{i_j} = \Delta$. Then a simple count gives that $A = \cup_{j=1}^k \Delta$. □

We now study some applications on the above Lemma 6.1. For notational simplicity, for $\mathfrak{m} \in$ Mult, we set

$$\widetilde{\zeta}(\mathfrak{m}) = \zeta(\mathfrak{m}^\vee)^\vee.$$

This coincides with the notion $\widetilde{\zeta(\mathfrak{m})}$ in Section 3.2 when $D = F$.

Lemma 6.2 *Let* $\mathfrak{m}_1, \mathfrak{m}_2 \in$ Mult. *Let* $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$. *Let* Δ *be a* $\leq_b$*-maximal element in* $\mathfrak{m}$. *Let* $n_1 = l_{abs}(\mathfrak{m}_1)$, $n_2 = l_{abs}(\mathfrak{m}_2)$, $i_1 = n_1 - l_{abs}((\mathfrak{m}_1)_\Delta)$ *and let* $i_2 = n_2 - l_{abs}((\mathfrak{m}_2)_\Delta)$. *Let* $i = i_1 + i_2$. *We now consider the filtration for* $(\widetilde{\zeta}(\mathfrak{m}_1) \times \widetilde{\zeta}(\mathfrak{m}_2))_{N_i}$ *from the geometric lemma in Section 2.5. The only layer from that filtration, which has the same cuspidal support as* $\langle \mathfrak{m}_\Delta \rangle \boxtimes \langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle$, *takes the form*

$$(*) \quad \mathrm{Ind}_P^{G'} (\widetilde{\zeta}(\mathfrak{m}_1)_{N_{i_1}} \boxtimes \widetilde{\zeta}(\mathfrak{m}_2)_{N_{i_2}})^\phi,$$

where $G' = G_{n_1+n_2-i_1-i_2} \times G_{i_1+i_2}$ *and* $\phi : G_{n_1-i_1} \times G_{i_1} \times G_{n_2-i_2} \times G_{i_2} \to G_{n_1-i_1} \times G_{n_2-i_2} \times G_{i_1} \times G_{i_2}$.

Moreover, the component in $\widetilde{\zeta}(\mathfrak{m}_1)_{N_{i_1}}$ *and* $\widetilde{\zeta}(\mathfrak{m}_2)_{N_{i_2}}$ *in (*) contributing to the factor* $\langle \mathfrak{m}_\Delta \rangle \boxtimes \langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle$ *can be refined to*

$$\langle (\mathfrak{m}_1)_\Delta \rangle \boxtimes \widetilde{\zeta}(\mathfrak{m}_1 - (\mathfrak{m}_1)_\Delta), \quad \langle (\mathfrak{m}_2)_\Delta \rangle \boxtimes \widetilde{\zeta}(\mathfrak{m}_1 - (\mathfrak{m}_2)_\Delta \rangle.$$

Proof The problem on the layer can be transferred to the counting problem by using the Jacquet functor computations in Section 2.6. Then the lemma follows from Lemma 6.1. □

6.2 A Direct Summand Computation

Lemma 6.3 *Let* $\mathfrak{m} \in \mathrm{Mult}$. *Let* Δ *be a* $\leq_b$*-maximal element. Let* $i = l_{abs}(\mathfrak{m}) - l_{abs}(\mathfrak{m}_\Delta)$. *Then the direct summand in* $\langle \mathfrak{m} \rangle_{N_i}$ *with same cuspidal support as* $\langle \mathfrak{m}_\Delta \rangle \boxtimes \langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle$ *is actually isomorphic to* $\langle \mathfrak{m}_\Delta \rangle \boxtimes \langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle$.

Proof Let $\pi = \langle \mathfrak{m} \rangle$. Then, from standard results of the Zelevinsky classification [53],

$$\pi \hookrightarrow \langle \mathfrak{m}_\Delta \rangle \times \langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle.$$

This implies

$$\pi_{N_i} \hookrightarrow (\langle \mathfrak{m}_\Delta \rangle \times \langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle)_{N_i}.$$

Thus, any simple composition factor in π_{N_i} appears as a composition factor in:

$$(\tau_1 \times \tau_2) \boxtimes (\tau_3 \times \tau_4)$$

such that $\tau_1 \boxtimes \tau_3$ is a simple composition factor in $\langle \mathfrak{m}_\Delta \rangle_{N_{i'}}$ and $\tau_2 \boxtimes \tau_4$ is a simple composition factor in $\langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle_{N_{i''}}$, where $i' + i'' = i$.

Let $k = |\mathfrak{m}_\Delta|$ and write $\Delta = [a, b]_\rho$. Then $\mathrm{csupp}(\tau_1) \cup \mathrm{csupp}(\tau_3)$ (union as a multiset) has k number of $b(\Delta)$. Now we suppose some $b(\Delta)$ come from $\mathrm{csupp}(\tau_3)$ to obtain a contradiction. In such case, $\tau_3 \boxtimes \tau_4$ also appears in $\zeta(\mathfrak{m} - \mathfrak{m}_\Delta)_{N_{i''}}$. But the latter term can be computed from the geometric lemma again. One sees that if $\mathrm{csupp}(\tau_3)$ contains $b(\Delta)$, then it contains all the cuspidal representation in a segment $\Delta' \in \mathfrak{m}_{b=b(\Delta)} - \mathfrak{m}_\Delta$. Since Δ is $\leq_b$-maximal, Δ' contains $\nu_\rho \cdot a(\Delta)$. This gives a contradiction that $\mathrm{csupp}(\tau_1) \cup \mathrm{csupp}(\tau_3) = \mathrm{csupp}(\langle \mathfrak{m}_\Delta \rangle)$.

We have concluded that all $b(\Delta)$ in $\mathrm{csupp}(\langle \mathfrak{m}_\Delta \rangle)$ arises from $\mathrm{csupp}(\tau_1)$. Then, we must have that $\tau_1 = \langle \mathfrak{m}_\Delta \rangle$ and $i' = i$ and $i'' = 0$. This shows that the only layer in the geometric lemma giving the desired module is $\langle \mathfrak{m}_\Delta \rangle \boxtimes \langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle$. This shows the lemma. □

6.3 A Refined Computation

Lemma 6.4 *We use the notations in Lemma 6.2. We consider the filtration for* $(\langle \mathfrak{m}_1 \rangle \times \langle \mathfrak{m}_2 \rangle)_{N_i}$ *from the geometric lemma. The only layer from that filtration, which has the same cuspidal support as* $\langle \mathfrak{m}_\Delta \rangle \boxtimes \langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle$, *takes the form*

$$(**) \quad \mathrm{Ind}_P^{G'} (\langle \mathfrak{m}_1 \rangle_{N_{i_1}} \boxtimes \langle \mathfrak{m}_2 \rangle_{N_{i_2}})^\phi.$$

Moreover, the component in $\langle \mathfrak{m}_1 \rangle_{N_{i_1}}$ *and* $\langle \mathfrak{m}_2 \rangle_{N_{i_2}}$ *in (**) contributing to the factor* $\langle \mathfrak{m}_\Delta \rangle \boxtimes \langle \mathfrak{m} - \mathfrak{m}_\Delta \rangle$ *can be refined to*

$$\langle (\mathfrak{m}_1)_\Delta \rangle \boxtimes \langle \mathfrak{m}_1 - (\mathfrak{m}_1)_\Delta \rangle, \quad \langle (\mathfrak{m}_2)_\Delta \rangle \boxtimes \langle \mathfrak{m}_1 - (\mathfrak{m}_2)_\Delta \rangle.$$

Proof Note that the geometric lemma is functorial and so the first assertion follows from the corresponding one in Lemma 6.2. The second assertion then follows from Lemma 6.3. □

In the following applications, we shall need two modifications. One is to use Lemma 6.4 repeatedly while another one is to replace $\langle \mathfrak{m}_2 \rangle$ with an indecomposable module of length 2. We shall avoid notation complications to give such precise statements and the meaning will become clearer when one sees the required statements in the following proofs.

7 Constructing Self-extensions

For π in $\mathrm{Irr}(G_n)$, we first show that self-extensions of π can be constructed via self-extensions of its associated Zelevinsky standard module. Then we study self-extensions of Zelevinsky standard modules and show it can be reduced to a tempered case via a categorical equivalence in Corollary 7.5.

7.1 Constructing Extensions from $\zeta(\mathfrak{m})$

Let $\mathfrak{m} \in \mathrm{Mult}$. Let $\pi = \langle \mathfrak{m} \rangle$. In this subsection, we explain how to construct extensions between two copies of $\langle \mathfrak{m} \rangle$ from extensions of two copies of $\zeta(\mathfrak{m})$. One may compare with the study in [12, Section 3]. We first show that one can do that by showing Lemma 7.1 and then reinterpret the result via the Yoneda extension lemma.

Lemma 7.1 *Let* $\mathfrak{m} \in \mathrm{Mult}_n$. *Then we have a natural embedding*

$$\mathrm{Ext}^1_{G_n}(\langle \mathfrak{m} \rangle, \langle \mathfrak{m} \rangle) \hookrightarrow \mathrm{Ext}^1_{G_n}(\langle \mathfrak{m} \rangle, \zeta(\mathfrak{m})) \cong \mathrm{Ext}^1_{G_n}(\zeta(\mathfrak{m}), \zeta(\mathfrak{m})).$$

Proof We have

$$0 \to \langle \mathfrak{m} \rangle \hookrightarrow \zeta(\mathfrak{m}) \to K \to 0,$$

where K is the cokernel of the first embedding.

Now, by Lemma 5.4, we have that, for all i,

$$\mathrm{Ext}^i_{G_n}(K, \zeta(\mathfrak{m})) = 0.$$

Thus a standard long exact sequence gives that

$$\mathrm{Ext}^i_{G_n}(\langle \mathfrak{m} \rangle, \zeta(\mathfrak{m})) \cong \mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{m}), \zeta(\mathfrak{m})). \tag{7.6}$$

Long exact sequence now gives that

$$0 = \mathrm{Hom}_{G_n}(\langle \mathfrak{m} \rangle, K) \to \mathrm{Ext}^1_{G_n}(\langle \mathfrak{m} \rangle, \langle \mathfrak{m} \rangle) \to \mathrm{Ext}^1_{G_n}(\langle \mathfrak{m} \rangle, \zeta(\mathfrak{m}))$$

Thus, combining with the above isomorphism,

$$0 \to \mathrm{Ext}^1_{G_n}(\langle \mathfrak{m} \rangle, \langle \mathfrak{m} \rangle) \hookrightarrow \mathrm{Ext}^1_{G_n}(\zeta(\mathfrak{m}), \zeta(\mathfrak{m}))$$

□

We remark that the injection in Lemma 7.1 is not an isomorphism in general. For example, if one takes $\mathfrak{m} = \{[0], [1]\}$, then $\langle \mathfrak{m} \rangle \cong \mathrm{St}([0, 1])$ and so $\dim \mathrm{Ext}^1_{G_2}(\langle \mathfrak{m} \rangle, \langle \mathfrak{m} \rangle) = 1$, but $\dim \mathrm{Ext}^1_{G_2}(\zeta(\mathfrak{m}), \zeta(\mathfrak{m})) = 2$.

We now explain Lemma 7.1 in module language via the Yoneda extension interpretation ([38, Ch III Theorem 9.1], also see [38, Section 6, Pages 71 and 83]). We can interpret an element in $\mathrm{Ext}^1_{G_n}(\langle \mathfrak{m} \rangle, \langle \mathfrak{m} \rangle)$ as a short exact sequence:

$$0 \to \langle \mathfrak{m} \rangle \to \pi \to \langle \mathfrak{m} \rangle \to 0.$$

By using Lemma 7.1, there exist short exact sequences such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \langle \mathfrak{m} \rangle & \longrightarrow & \pi & \longrightarrow & \langle \mathfrak{m} \rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \zeta(\mathfrak{m}) & \longrightarrow & \pi' & \longrightarrow & \langle \mathfrak{m} \rangle & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \zeta(\mathfrak{m}) & \longrightarrow & \pi'' & \longrightarrow & \zeta(\mathfrak{m}) & \longrightarrow & 0
\end{array}$$

Since the leftmost and rightmost vertical maps are injections, the middle vertical maps are also injective. In other words, we obtain:

Lemma 7.2 *Let π be an indecomposable representation of G_n of length two with both irreducible composition factors isomorphic to $\langle \mathfrak{m} \rangle$ for some $\mathfrak{m} \in \mathrm{Mult}_n$. Then there exists an indecomposable representation π'' of G_n which*

- *admits a short exact sequence:*

$$0 \to \zeta(\mathfrak{m}) \to \pi'' \to \zeta(\mathfrak{m}) \to 0$$

  and
- *π embeds to π''.*

7.2 Extensions Between Two $\zeta(\mathfrak{m})$

Let $\Delta_1, \ldots, \Delta_r$ be all the distinct segments such that $\mathfrak{m}_{\Delta_i} \neq \emptyset$ and label in the way that $\Delta_1 \not\leq_b \Delta_2 \not\leq_b \ldots \not\leq_b \Delta_r$. Let $n_i = l_{abs}(\mathfrak{m}_{\Delta_i})$ for $i = 1, \ldots, r$. Denote by $G(\mathfrak{m})$ the group $G_{n_1} \times \ldots \times G_{n_r}$. Let, as a $G(\mathfrak{m})$-representation,

$$[\mathfrak{m}] = \langle \mathfrak{m}_{\Delta_1} \rangle \boxtimes \ldots \boxtimes \langle \mathfrak{m}_{\Delta_r} \rangle.$$

Lemma 7.3 *For* $\mathfrak{m} \in \mathrm{Mult}_n$*, and for any* i*,*

$$\mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{m}), \zeta(\mathfrak{m})) \cong \mathrm{Ext}^i_{G(\mathfrak{m})}([\mathfrak{m}], [\mathfrak{m}]).$$

Proof Let Δ be a $\leq_b$-maximal element such that $\mathfrak{m}_\Delta \neq \emptyset$. Then we write

$$\zeta(\mathfrak{m}) = \langle \mathfrak{m}_\Delta \rangle \times \zeta(\mathfrak{m} - \mathfrak{m}_\Delta).$$

Let $n_1 = l_{abs}(\mathfrak{m}_{b=\rho'})$ and let $n = l_{abs}(\mathfrak{m})$. Now,

$$\mathrm{Ext}^i_{G_n}(\zeta(\mathfrak{m}), \zeta(\mathfrak{m})) \cong \mathrm{Ext}^i_{G_{n_1} \times G_{n-n_1}}(\langle \mathfrak{m}_\Delta \rangle \boxtimes \zeta(\mathfrak{m} - \mathfrak{m}_\Delta), \langle \mathfrak{m}_\Delta \rangle \boxtimes \zeta(\mathfrak{m} - \mathfrak{m}_\Delta)),$$

which follows from first applying Frobenius reciprocity and then using the geometric lemma and Lemma 6.1 to single out the only layer that has the same cuspidal support as $\zeta(\mathfrak{m}_\Delta) \boxtimes \zeta(\mathfrak{m} - \mathfrak{m}_\Delta)$. We now repeat similar process for $\zeta(\mathfrak{m} - \mathfrak{m}_\Delta)$. □

We now focus on $i = 1$ in Lemma 7.3 to study first extensions. We now have the following:

Proposition 7.4 *Let* $\pi_1, \pi_2 \in \mathrm{Alg}_f(G_n)$*. Suppose each of* π_1 *and* π_2 *admits a filtration with successive subquotients isomorphic to* $\zeta(\mathfrak{m})$*. Let* $M = G(\mathfrak{m})$ *and let* P *be the standard parabolic subgroup containing* M*. Then*

(1) for each $i = 1, 2$*, there exists an admissible* M*-representation* τ_i *which admits a filtration with successive subquotients isomorphic to* $[\mathfrak{m}]$ *such that* $\pi_i \cong \mathrm{Ind}_P^{G_n} \tau_i$*, and*

(2) $\pi_1 \cong \pi_2$ *if and only if* $\tau_1 \cong \tau_2$*.*

Proof Let $n = l_{abs}(\mathfrak{m})$. Let $P = MN$ be the Levi decomposition. We first consider $(\pi_i)_N$. Let τ_i be the projection of $(\pi_i)_N$ to the component that has the same cuspidal support as $[\mathfrak{m}]$. By repeated use of Lemma 6.2 (under the situation that $\mathfrak{m}_2$ in Lemma 6.2 is empty), we have that τ_i admits a filtration whose successive subquotients are isomorphic to $[\mathfrak{m}]$. Then, applying Frobenius reciprocity, we have

$$\mathrm{Hom}_{G(\mathfrak{m})}((\pi_i)_N, \tau_i) \cong \mathrm{Hom}_{G_n}(\pi_i, \mathrm{Ind}_P^{G_n} \tau_i)$$

and so we obtain a map f in $\mathrm{Hom}_{G_n}(\pi_i, \mathrm{Ind}_P^{G_n} \tau_i)$ corresponding to the surjection

$$(\pi_i)_N \twoheadrightarrow \tau_i.$$

Claim: f is an isomorphism.
Proof of the claim: If f is not an isomorphism, then by counting the number of composition factors, we must have an embedding $\langle \mathfrak{m} \rangle \overset{\iota}{\hookrightarrow} \pi_i$ such that $f \circ \iota = 0$. However, via the functoriality of Forbenius reciprocity, we also have a composition of maps

$$[\mathfrak{m}] \hookrightarrow \langle \mathfrak{m} \rangle_N \hookrightarrow (\pi_i)_N \twoheadrightarrow \tau$$

is zero. However, this is not possible since the multiplicity of $[\mathfrak{m}]$ in $(\pi_i)_N$ agrees with that in τ via the construction above.

Now the claim gives that $\pi_i \cong \mathrm{Ind}_P^{G_n} \tau_i$ and this proves (1).

We now prove (2). The if direction is clear. For the only if direction, let f : $\mathrm{Ind}_P^{G_n} \tau_1 \to \mathrm{Ind}_P^{G_n} \tau_2$ be the isomorphism. Then the corresponding map, denoted $\widetilde{f}$, under Frobenius reciprocity is given by: $\widetilde{f}(x) = f(x)(1)$, where 1 is the evaluation at the identity by viewing $f(x)$ as a function in $\mathrm{Ind}_P^{G_n} \tau_2$; and x is any representative in $(\pi_1)_N$. Since f is an isomorphism, the map $\widetilde{f}$ is surjective. Thus the multiplicity of $[\mathfrak{m}]$ in τ_1 must be at least that in τ_2. Similarly, we can obtain that the multiplicity of $[\mathfrak{m}]$ in τ_2 must be at least that in τ_1. Since the two multiplicities agree, $\widetilde{f}$ restricted to τ_1 in $(\pi_1)_N$ must be an isomorphism. □

Corollary 7.5 *Let* $\mathfrak{m} \in \mathrm{Mult}_n$. *Let* $\mathcal{C}_1$ *be the full subcategory of* $\mathrm{Alg}_f(G_n)$ *whose objects admit a finite filtration with successive subquotients isomorphic to* $\zeta(\mathfrak{m})$. *Let* $\mathcal{C}_2$ *be the full subcategory of* $\mathrm{Alg}_f(G(\mathfrak{m}))$ *whose objects admit a finite filtration with successive subquotients isomorphic to* $[\mathfrak{m}]$. *There is a categorical equivalence between* $\mathcal{C}_1$ *and* $\mathcal{C}_2$. *Here* $\mathrm{Alg}_f(G(\mathfrak{m}))$ *is the category of smooth representations of finite length of* $G(\mathfrak{m})$.

Proof Using Proposition 7.4 (and its notations), one can write $\pi_i = \mathrm{Ind}_P^{G_n} \tau_i$ for some τ_i in $\mathcal{C}_1$. It remains to see that the induction functor in the previous proposition also defines an isomorphism on the morphism spaces. The induced map

$$\mathrm{Hom}_{G(\mathfrak{m})}(\tau_1, \tau_2) \to \mathrm{Hom}_{G_n}(\pi_1, \pi_2)$$

is injective since the parabolic induction sends any non-zero objects to non-zero objects. Now it follows from Frobenius reciprocity,

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \cong \mathrm{Hom}_{G(\mathfrak{m})}((\pi_1)_N, \tau_2) \cong \mathrm{Hom}_{G(\mathfrak{m})}(\tau_1, \tau_2).$$

The last isomorphism follows from the proof of Proposition 7.4 that τ_i is the component of $(\pi_i)_N$ that has the same cuspidal support as τ_i. Now the finite-dimensionality of the Hom spaces implies that the injection is also an isomorphism, as desired. □

Corollary 7.6 *We use the notations in Corollary 7.5. Let* τ *be an object in* $\mathcal{C}_2$ *and let* π *be the corresponding representation under the equivalence in Corollary 7.5. Then*

$$\dim \mathrm{Hom}_{G_n}(\langle \mathfrak{m} \rangle, \pi) = \dim \mathrm{Hom}_{G(\mathfrak{m})}([\mathfrak{m}], \tau).$$

Proof Let $k = \dim \mathrm{Hom}_{G(\mathfrak{m})}([\mathfrak{m}], \tau)$. By the equivalence of categories, we have an embedding:

$$\overbrace{\zeta(\mathfrak{m}) \oplus \ldots \oplus \zeta(\mathfrak{m})}^{k \text{ times}} \hookrightarrow \pi = \mathrm{Ind}_P^{G_n} \tau.$$

Hence, $\dim \mathrm{Hom}_{G_n}(\langle \mathfrak{m} \rangle, \pi) \geq k$.

Let $l = \dim \mathrm{Hom}_{G_n}(\langle \mathfrak{m} \rangle, \pi)$. Suppose $l > k$. Then, we have an embedding:

$$\overbrace{\langle \mathfrak{m} \rangle \oplus \ldots \oplus \langle \mathfrak{m} \rangle}^{l \text{ times}} \hookrightarrow \pi = \mathrm{Ind}_P^{G_n} \tau.$$

Now since the Jacquet functor is exact, we have that:

$$\overbrace{[\mathfrak{m}] \oplus \ldots \oplus [\mathfrak{m}]}^{l \text{ times}} \hookrightarrow \tau.$$

This then gives a contradiction. Hence, we must have that $l = k$. □

8 Big Derivatives

In this section, we introduce the notion of big derivatives and describe some basic properties.

8.1 Big Derivatives

Definition 8.1 Let $\sigma \in \mathrm{Irr}(G_i)$. For $\pi \in \mathrm{Alg}_f(G_n)$, define

$$\mathbb{D}_\sigma(\pi) = \mathrm{Hom}_{G_i}(\sigma, \pi_{N_{n-i}^-}),$$

where $\pi_{N_{n-i}^-}$ is regarded as a G_i-representation via embedding G_i to the first factor of $G_i \times G_{n-i}$. We regard $\mathbb{D}_\sigma(\pi)$ as a G_{n-i}-representation via:

$$(g.f)(x) = (I_i, g).f(x).$$

By applying the element $\begin{pmatrix} & I_{n-i} \\ I_i & \end{pmatrix}$, one can switch the $G_i \times G_{n-i}$-representation $\pi_{N_{n-i}^-}$ to $G_{n-i} \times G_i$-representation π_{N_i}. This gives the following isomorphism:

$$\mathbb{D}_\sigma(\pi) \cong \mathrm{Hom}_{G_i}(\sigma, \pi_{N_i}). \tag{8.7}$$

We shall use the identification frequently in Section 11.

We similarly define the left version as:

$$\mathbb{D}'_\sigma(\pi) = \mathrm{Hom}_{G_i}(\sigma, \pi_{N_i^-}),$$

where $\pi_{N_i^-}$ is regarded as a $G_{n-i} \times G_i$-representation via the embedding $G_i \hookrightarrow G_{n-i} \times G_i$ into the second factor of $G_{n-i} \times G_i$. We remark that $\mathbb{D}_\sigma$ and $\mathbb{D}'_\sigma$ are left-exact, but not exact.

We only prove results for $\mathbb{D}$ and results for $\mathbb{D}'$ can be formulated and proved similarly (c.f. Section 3.2). When π is $\square$-irreducible, $\mathbb{D}_\pi(\tau)$ is either zero or has unique simple submodule [30]. If $\mathbb{D}_\pi(\tau) \neq 0$, we shall denote by $D_\pi(\tau)$ the unique submodule.

8.2 Composition of Big Derivatives

Proposition 8.2 *Let $\sigma_1, \ldots, \sigma_r \in \mathrm{Irr}$ such that $\sigma_1 \times \ldots \times \sigma_r$ is still irreducible. Then*

$$\mathbb{D}_{\sigma_r} \circ \ldots \circ \mathbb{D}_{\sigma_1}(\pi) \cong \mathbb{D}_{\sigma_1 \times \ldots \times \sigma_r}(\pi).$$

Proof For the given condition, we have that $\sigma_1 \times \ldots \times \sigma_s$ is still irreducible for $s \leq r$. Thus it reduces to $r = 2$. Let $n_1 = \deg(\sigma_1)$ and $n_2 = \deg(\sigma_2)$. We have: for any $\tau \in \mathrm{Alg}_f(G_{n-n_1-n_2})$ and $\pi \in \mathrm{Alg}_f(G_n)$,

$$\begin{aligned}
\mathrm{Hom}_{G_{n-n_1-n_2}}(\tau, \mathbb{D}_{\sigma_2} \circ \mathbb{D}_{\sigma_1}(\pi)) \cong & \mathrm{Hom}_{G_{n_2} \times G_{n-n_1-n_2}}(\sigma_2 \boxtimes \tau, \mathbb{D}_{\sigma_1}(\pi)_{N^-_{n_2,n-n_1-n_2}}) \\
\cong & \mathrm{Hom}_{G_{n-n_1}}(\sigma_2 \times \tau, \mathbb{D}_{\sigma_1}(\pi)) \\
\cong & \mathrm{Hom}_{G_{n_1} \times G_{n-n_1}}(\sigma_1 \boxtimes (\sigma_2 \times \tau), \pi_{N^-_{n_1,n-n_1}}) \\
\cong & \mathrm{Hom}_{G_n}(\sigma_1 \times \sigma_2 \times \tau, \pi) \\
\cong & \mathrm{Hom}_{G_{n_1+n_2} \times G_{n-n_1-n_2}}((\sigma_1 \times \sigma_2) \boxtimes \tau, \pi_{N_{n_1+n_2,n-n_1-n_2}}) \\
\cong & \mathrm{Hom}_{G_{n-n_1-n_2}}(\tau, \mathbb{D}_{\sigma_1 \times \sigma_2}(\pi)),
\end{aligned}$$

where the second, forth and fifth isomorphisms follow from Frobenius reciprocity, the first, third and last ones follow from the adjointness of the functors. We remark that the forth isomorphism also uses taking parabolic inductions in stages. Now the natural isomorphism between the two derivatives follows from the Yoneda lemma. □

Proposition 8.3 *Let $\sigma_1, \ldots, \sigma_r \in \mathrm{Irr}^{\square}$ such that $\sigma_1 \times \ldots \times \sigma_r$ is still $\square$-irreducible. Suppose $D_{\sigma_1 \times \ldots \times \sigma_r}(\pi) \neq 0$. Then $D_{\sigma_1 \times \ldots \times \sigma_r}(\pi) \cong D_{\sigma_1} \circ \ldots \circ D_{\sigma_r}(\pi)$.*

Proof Let $\tau = D_{\sigma_1 \times \ldots \times \sigma_r}(\pi)$. Then

$$\pi \hookrightarrow \tau \times \sigma_1 \times \ldots \times \sigma_r.$$

Let $I_{\sigma_1}(\tau)$ be the unique simple submodule of $\tau \times \sigma_1$. The unique submodule must factor through the embedding:

$$I_{\sigma_1}(\tau) \times \sigma_2 \times \ldots \times \sigma_r \hookrightarrow \tau \times \sigma_1 \times \ldots \times \sigma_r.$$

Then inductively, we have that $D_{\sigma_2} \circ \ldots \circ D_{\sigma_r}(\pi) \cong I_{\sigma_1}(\tau)$. This implies that $D_{\sigma_1} \circ D_{\sigma_2} \circ \ldots \circ D_{\sigma_r}(\pi) \cong \tau$. □

8.3 The Segment Case

We now consider a special case of the product functor and we recall the following result shown in [15]. For $\Delta \in \mathrm{Seg}$, let $\mathcal{C} = \mathcal{C}_\Delta$ be the full subcategory of $\mathrm{Alg}_f(G_n)$ whose objects π satisfy the property that for any simple composition factor τ in π and any $\sigma \in \mathrm{csupp}(\tau)$, $\sigma \in \Delta$ (c.f. [15, Section 9.1]).

Let $k = l_{abs}(\Delta)$. The product functor

$$\times_{\Delta,\mathcal{C}} : \mathcal{C}_\Delta \to \mathrm{Alg}(G_{n+k})$$

as

$$\times_{\Delta,\mathcal{C}}(\pi) = \langle \Delta \rangle \times \pi .$$

Lemma 8.4 *Let $\Delta \in \mathrm{Seg}$. Then $\times_{\Delta,\mathcal{C}}$ is a fully-faithful functor.*

Proof This is a special case of [15, Theorem 9.1]. □

Corollary 8.5 *Let $\Delta \in \mathrm{Seg}$. Let $\mathfrak{m}$ be a multisegment with all segments equal to Δ. Let $\mathfrak{n}$ be a submultisegment of $\mathfrak{m}$. Then $\mathbb{D}_{\langle \mathfrak{n} \rangle}(\langle \mathfrak{m} \rangle) = \langle \mathfrak{m} - \mathfrak{n} \rangle$.*

Proof This follows from Proposition 8.2, [15, Corollary 9.2] and Lemma 8.4. □

One may also compare the above two statements with Lemma 11.3 and Remark 11.4 below.

9 Indecomposability Under Product Functor: Isomorphic Cases

9.1 Indecomposibility

The following result is well-known (see [51, Proposition 2.3]), but we shall use some similar computations in the proof of Theorem 9.2 and so we sketch some main steps in the following proof.

Lemma 9.1 *Let $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Mult}$. Suppose $\langle \mathfrak{m}_1 \rangle \times \langle \mathfrak{m}_2 \rangle$ is irreducible. Then*

$$\langle \mathfrak{m}_1 \rangle \times \langle \mathfrak{m}_2 \rangle \cong \langle \mathfrak{m}_1 + \mathfrak{m}_2 \rangle .$$

Proof Let $n = l_{abs}(\mathfrak{m}_1) + l_{abs}(\mathfrak{m}_2)$. It suffices to show that

$$\mathrm{Hom}_{G_n}(\langle \mathfrak{m}_1 \rangle \times \langle \mathfrak{m}_2 \rangle, \zeta(\mathfrak{m}_1 + \mathfrak{m}_2)) \cong \mathbb{C}. \tag{9.8}$$

Let $\mathfrak{n} = \mathfrak{m}_1 + \mathfrak{m}_2$. Let $\Delta_1, \ldots, \Delta_r$ be all the segments such that $\mathfrak{n}_{\Delta_i} \neq \emptyset$. We shall label the segments such that $\Delta_1 \not\leq_b \ldots \not\leq_b \Delta_r$.

Let $s_i = l_{abs}((\mathfrak{m}_1)_{\Delta_i})$ and let $t_i = l_{abs}((\mathfrak{m}_2)_{\Delta_i})$ for $i = 1, \ldots, r$. Let $u_i = s_i + t_i$. Let

$$G' = G_{u_1} \times \ldots \times G_{u_r},$$

and let

$$G'' = (G_{s_1} \times G_{t_1}) \times \ldots \times (G_{s_r} \times G_{t_r}).$$

Let $n_1 = l_{abs}(\mathfrak{m}_1)$ and let $n_2 = l_{abs}(\mathfrak{m}_2)$. Note that

$$\zeta(\mathfrak{m}_1 + \mathfrak{m}_2) = \langle \mathfrak{n}_{\Delta_1} \rangle \times \ldots \times \langle \mathfrak{n}_{\Delta_r} \rangle.$$

We now apply the Frobenius reciprocity:

$$\mathrm{Hom}_{G_{u_1} \times G_{n-u_1}}(\langle \mathfrak{m}_1 \rangle \times \langle \mathfrak{m}_2 \rangle, \zeta(\mathfrak{n}_{\Delta_1}) \times \ldots \times \zeta(\mathfrak{n}_{\Delta_r})).$$

Then, by Lemma 6.4, a possible layer that can contribute to the non-zero Hom is

$$(*) \quad \mathrm{Ind}_P^{G_{u_1} \times G_{n-u_1}}((\langle \mathfrak{m}_1 \rangle_{N_{n_1-s_1}} \boxtimes \langle \mathfrak{m}_2 \rangle_{N_{n_2-t_1}}))^{\phi},$$

where

- the superscript ϕ is to twist the $G_{s_1} \times G_{n_1-s_1} \times G_{t_1} \times G_{n_2-t_1}$-action to $G_{s_1} \times G_{t_1} \times G_{n_1-s_1} \times G_{n_2-t_1}$ in an obvious way;
- P is the standard parabolic subgroup containing $G_{s_1} \times G_{t_1} \times G_{n_1-s_1} \times G_{n_2-t_1}$.

Indeed, this is the only possible layer by Lemma 6.4.

In such layer (*), by Lemma 6.4 again, the only direct summand that can (possibly) contribute the Hom via Frobenius reciprocity is

$$\langle (\mathfrak{m}_1)_{\Delta_1} \rangle \times \langle (\mathfrak{m}_2)_{\Delta_1} \rangle \boxtimes \langle \mathfrak{m}_1 - (\mathfrak{m}_1)_{\Delta_1} \rangle \times \langle \mathfrak{m}_2 - (\mathfrak{m}_2)_{\Delta_1} \rangle.$$

Now we inductively have that $\mathrm{Hom}_{G_{n-s_1-t_1}}(\langle \mathfrak{m}_1 - (\mathfrak{m}_1)_{\Delta_1} \rangle \times \langle \mathfrak{m}_2 - (\mathfrak{m}_2)_{\Delta_1} \rangle, \zeta(\mathfrak{n} - \mathfrak{n}_{\Delta_1})) \cong \mathbb{C}$ and so Künneth's formula gives Eq. 9.8. □

The idea of proving the following theorem is that one first enlarges to some modules close to standard modules. In particular, one uses Lemma 7.2 for a module of length 2. Then one applies Frobenius reciprocity and does some analysis as in the proof of Lemma 9.1.

Theorem 9.2 *Let $\pi_1, \pi_2 \in \mathrm{Irr}$. Suppose $\pi_1 \times \pi_2$ is irreducible. Let π be a representation of length 2 with the two simple composition factors isomorphic to π_2. Then π is indecomposable if and only if $\pi_1 \times \pi$ is indecomposable.*

Proof Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be multisegments such that

$$\pi_1 \cong \langle \mathfrak{m}_1 \rangle, \quad \pi_2 \cong \langle \mathfrak{m}_2 \rangle.$$

Since $\pi_1 \times \pi_2$ is irreducible, $\pi_1 \times \pi_2 \cong \langle \mathfrak{m}_1 + \mathfrak{m}_2 \rangle$ by Lemma 9.1.

Note that the if direction is clear. We now prove the only if direction. By Lemma 7.2 and taking the dual, there exists $\pi'' \in \mathrm{Alg}_f$ such that

- π'' admits a short exact sequence

$$0 \to \widetilde{\zeta}(\mathfrak{m}_2) \to \pi'' \to \widetilde{\zeta}(\mathfrak{m}_2) \to 0$$

 and;
- $\pi'' \twoheadrightarrow \pi$.

Let $n = l_{abs}(\mathfrak{m}_1) + l_{abs}(\mathfrak{m}_2)$. We see that $\pi_1 \times \pi''$ is indecomposable if and only if $\mathrm{Hom}_{G_n}(\pi_1 \times \pi'', \langle \mathfrak{m}_1 + \mathfrak{m}_2 \rangle) \cong \mathbb{C}$. To prove the latter one, it suffices to show that

$$\mathrm{Hom}_{G_n}(\pi_1 \times \pi'', \zeta(\mathfrak{m}_1 + \mathfrak{m}_2)) \cong \mathbb{C}.$$

Now we apply some similar strategy as the proof of Lemma 9.1. Let $\mathfrak{n} = \mathfrak{m}_1 + \mathfrak{m}_2$. Let $\Delta_1, \ldots, \Delta_r$ be all the distinct segments such that $\mathfrak{n}_{\Delta_i} \neq \emptyset$, and $\Delta_1 \not\leq_b \Delta_2 \not\leq_b \ldots \not\leq_b \Delta_r$. Let $s_i = l_{abs}((\mathfrak{m}_1)_{\Delta_i})$ and $t_i = l_{abs}((\mathfrak{m}_2)_{\Delta_i})$. Let $u_i = s_i + t_i$.

Now we write

$$\zeta(\mathfrak{n}) = \zeta(\mathfrak{n}_{\Delta_1}) \times \ldots \times \zeta(\mathfrak{n}_{\Delta_r}).$$

Let $G' = G(\mathfrak{n})$. Let P be the standard parabolic subgroup in G_n containing G' with the Levi decomposition $P = G'N$. Recall that $[\mathfrak{n}] = \langle \mathfrak{n}_{\Delta_1} \rangle \boxtimes \ldots \boxtimes \langle \mathfrak{n}_{\Delta_r} \rangle$ and so $\zeta(\mathfrak{n}) = \mathrm{Ind}_P^{G_n}[\mathfrak{n}]$. Now Frobenius reciprocity gives that:

$$(*) \quad \mathrm{Hom}_{G_n}(\pi_1 \times \pi'', \zeta(\mathfrak{n})) \cong \mathrm{Hom}_{G'}((\pi_1 \times \pi'')_N, [\mathfrak{n}]).$$

The analysis in the proof of Lemma 9.1 gives that the only possible layer contributing a non-zero Hom in (*) (via the geometric lemma on $(\pi_1 \times \pi'')_N$) is of the form:

$$\mathrm{Ind}_{\widetilde{P}}^{G'}((\pi_1)_{N'} \boxtimes (\pi'')_{N''})^{\phi},$$

where

- N' is the unipotent radical corresponding to the partition $(s_1, \ldots, s_r)$ and N'' is the unipotent radical corresponding to the partition $(t_1, \ldots, t_r)$;
- the superscript ϕ is a twist sending $G_{s_1} \times \ldots \times G_{s_r} \times G_{t_1} \times \ldots \times G_{t_r}$ to $G_{s_1} \times G_{t_1} \times \ldots \times G_{s_r} \times G_{t_r}$ (by permutating the factors in an obvious way);
- $\widetilde{P}$ is the standard parabolic subgroup in G' containing $G_{s_1} \times G_{t_1} \times \ldots \times G_{s_r} \times G_{t_r}$.

Thus, we have that:

$$\mathrm{Hom}_{G_n}(\pi_1 \times \pi'', \zeta(\mathfrak{n})) \cong \mathrm{Hom}_{G'}(\mathrm{Ind}_{\widetilde{P}}^{G'}((\pi_1)_{N'} \boxtimes (\pi'')_{N''})^{\phi}, [\mathfrak{n}]).$$

Indeed, as in Lemma 9.1, which uses Lemma 6.4 inductively, the only component in $(\mathrm{Ind}_{\widetilde{P}}^{G'}((\pi_1)_{N'} \boxtimes (\pi'')_{N''})^{\phi}$ that can contribute to the nonzero Hom is:

$$\mathrm{Ind}_{\widetilde{P}}^{G'}([\mathfrak{m}_1] \boxtimes \tau)^{\phi},$$

where τ is the direct summand in $(\pi)_{N''}$ whose simple composition factors have the same cuspidal support as $[\mathfrak{m}_2]$. Attributing to the multiple use of Lemma 6.3, we conclude that τ has length 2 and both composition factors of τ are isomorphic to $[\mathfrak{m}_2]$. Thus we further have

$$(**) \quad \mathrm{Hom}_{G_n}(\pi_1 \times \pi'', \zeta(\mathfrak{n})) \cong \mathrm{Hom}_{G'}(\mathrm{Ind}_{\widetilde{P}}^{G'}([\mathfrak{m}_1] \boxtimes \tau)^{\phi}, [\mathfrak{n}]).$$

As the functors described in the proof of Corollary 7.5, τ satisfies $\mathrm{Ind}_{P^*}^{G_{n_2}} \tau \cong \pi''$, where P^* is the standard parabolic subgroup containing $G_{t_1} \times \ldots \times G_{t_r}$, and Corollary 7.5 implies that τ is indecomposable and of length 2 with both factors isomorphic to $[\mathfrak{m}_2]$. In particular, τ has a unique simple quotient.

Now we return to compute the latter Hom of (**). Let

$$G'' = G_{s_1} \times G_{t_1} \times \ldots \times G_{s_r} \times G_{t_r}.$$

In such case, applying the second adjointness, such Hom is equal to

$$\mathrm{Hom}_{G''}(([\mathfrak{m}_1] \boxtimes \tau))^{\phi}, [\mathfrak{n}]_{N^-}),$$

where N^- is the unipotent radical in $P^t_{s_1,t_1} \times \ldots \times P^t_{s_r,t_r} \subset G_{u_1} \times \ldots \times G_{u_r}$. By using Hom-tensoring adjointness, the previous Hom turns to:

$$(***) \quad \mathrm{Hom}_{G_{t_1} \times \ldots \times G_{t_r}}(\tau, \mathrm{Hom}_{G_{s_1} \times \ldots \times G_{s_r}}([\mathfrak{m}_1], [\mathfrak{n}])),$$

where we regard $[\mathfrak{n}]$ as a $G_{s_1} \times \ldots \times G_{s_r}$-representation via the embedding:

$$(g_1, \ldots g_r) \mapsto (\begin{pmatrix} g_1 & \\ & I_{t_1} \end{pmatrix}, \ldots, \begin{pmatrix} g_r & \\ & I_{t_r} \end{pmatrix}).$$

Let $\sigma_i = \langle (\mathfrak{m}_1)_{\Delta_i} \rangle$. Finally, using Künneth formula on (***) and combining with (*), we have that:

$$\mathrm{Hom}_{G_n}(\pi_1 \times \pi'', \zeta(\mathfrak{n})) \cong \mathrm{Hom}_{G_{t_1} \times \ldots \times G_{t_r}}(\tau, \mathbb{D}_{\sigma_1}(\langle \mathfrak{n}_{\Delta_1} \rangle) \boxtimes \ldots \boxtimes \mathbb{D}_{\sigma_r}(\langle \mathfrak{n}_{\Delta_r} \rangle))$$

and so, by Corollary 8.5,

$$\mathrm{Hom}_{G_n}(\pi_1 \times \pi'', \zeta(\mathfrak{n})) \cong \mathrm{Hom}_{G_{t_1} \times \ldots \times G_{t_r}}(\tau, \langle \mathfrak{n}'_1 \rangle \boxtimes \ldots \boxtimes \langle \mathfrak{n}'_r \rangle),$$

where $\mathfrak{n}'_i = (\mathfrak{n})_{\Delta_i} - (\mathfrak{m}_1)_{\Delta_i}$. Now, as discussed above τ has only unique simple quotient and we so have that the Hom space has dimension 1, as desired. Thus, we have $\mathrm{Hom}_{G_n}(\pi_1 \times \pi'', \zeta(\mathfrak{m}_1 + \mathfrak{m}_2)) \cong \mathbb{C}$ and so $\mathrm{Hom}_{G_n}(\pi_1 \times \pi'', \langle \mathfrak{m}_1 + \mathfrak{m}_2 \rangle) \cong \mathbb{C}$. □

Remark 9.3 In general, the parabolic induction does not preserve self-extensions. For example, we consider $\pi = \langle [0] \rangle$. Let $\tau = (\pi \times \pi)_{N_1}$. Then

$$\mathrm{Ind}_{P_{1,1}}^{G_2} \tau \cong \pi \times \pi \oplus \pi \times \pi.$$

10 Fully-faithfulness of the Product Functor

10.1 A Criteria for Proving fully-faithfulness

We recall the following criteria for proving fully-faithfulness:

Lemma 10.1 [15, Lemma A.1] *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian Schurian k-categories, where k is a field. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive exact functor. Suppose the following holds:*

(1) any object of $\mathcal{A}$ has finite length;
(2) for any simple objects X and Y in $\mathcal{A}$, the induced map of F from $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ to $\mathrm{Ext}^1_{\mathcal{B}}(F(X), F(Y))$ is an injection;
(3) $F(X)$ is a simple object in $\mathcal{B}$ if X is a simple object in $\mathcal{A}$;
(4) for any simple objects X and Y in $\mathcal{A}$, $X \cong Y$ if and only if $F(X) \cong F(Y)$.

Then F is a fully-faithful functor.

The original statement of [15, Lemma A.1] is stated in a slightly different way, but the proof still applies.

We remark that elements in $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ can be interpreted as short exact sequences in Yoneda extensions [38], and the addition corresponds to the Baer sum. Then the exact functor $F : \mathcal{A} \to \mathcal{B}$ sends a short exact sequence to a short exact sequence and so induces a map from $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ to $\mathrm{Ext}^1_{\mathcal{B}}(F(X), F(Y))$ above.

If we consider the full subcategory $\mathcal{B}'$ of $\mathcal{B}$ containing all $F(X)$ for objects X in $\mathcal{A}$, then F defines an equivalence of categories from $\mathcal{A}$ to $\mathcal{B}'$ [50, Lemma 4.2.19]. We remark that $\mathcal{B}'$ may not be Serre in $\mathcal{B}$.

10.2 Product Functor

Recall that $\mathrm{Alg}_\pi(G_n)$ is defined in Section 5.2.

Theorem 10.2 *Let $\pi \in \mathrm{Irr}$. Let $k = \deg(\pi)$. Let $\mathcal{C} = \mathrm{Alg}_\pi(G_n)$. The functor $\times_{\pi,\mathcal{C}}$ defined in Section 3.1 is fully-faithful i.e.*

$$\mathrm{Hom}_{G_{n+k}}(\times_{\pi,\mathcal{C}}(\tau_1), \times_{\pi,\mathcal{C}}(\tau_2)) \cong \mathrm{Hom}_{G_n}(\tau_1, \tau_2)$$

for any $\tau_1, \tau_2 \in \mathrm{Alg}_\pi(G_n)$.

Proof It suffices to check the conditions in Lemma 10.1. For (1), it follows from the definitions. For (3), it follows from Proposition 5.2. For (4), it follows from Lemma 9.1. For (2), it follows from Proposition 5.7 and Theorem 9.2. □

Remark 10.3 Let $\pi \in \mathrm{Irr}$. Let λ be an indecomposable representation such that for any $\pi' \in \mathrm{JH}(\lambda)$, $\pi \times \pi'$ is irreducible. In general, it is not necessary that $\pi \times \lambda$ is indecomposable. For example, take $\pi = \langle[0]\rangle$ and let $\lambda = \langle[1]\rangle \times \langle[0]\rangle$. Then $\pi \times \lambda$ is a direct sum of two irreducible representations.

Corollary 10.4 *The functor $\times^{\pi,\mathcal{C}} : \mathcal{C} \to \mathrm{Alg}(G_{n+k})$ determined by $\tau \mapsto \tau \times \pi$ is fully-faithful.*

Proof When $D = F$, it follows from Theorem 10.2 and Proposition 3.3. In general, using the Zelevinsky type classification, one can prove in a similar way to Theorem 10.2. □

10.3 Dual Formulation

Theorem 10.5 *Let* $\pi \in \mathrm{Irr}$. *Let*

$$\mathcal{N}_\pi = \{\mathfrak{m} \in \mathrm{Mult} : \mathrm{St}(\Delta) \times \pi \textit{ is irreducible } \forall \Delta \in \mathfrak{m}\} .$$

Let $\mathcal{C}' := \mathrm{Alg}'_\pi(G_n)$ *be the full subcategory of* $\mathrm{Alg}_f(G_n)$ *whose objects have all simple composition factors isomorphic to* $\mathrm{St}(\mathfrak{m})$ *for some* $\mathfrak{m} \in \mathcal{N}_\pi$. *Then the product functors* $\times_{\pi,\mathcal{C}'}$ *and* $\times^{\pi,\mathcal{C}'}$ *are fully-faithful.*

Proof Note that $\mathcal{D}(\langle\Delta\rangle) = \mathrm{St}(\Delta^\vee)$ by using the formulation in [48]. This follows from Theorem 10.2, Corollary 10.4 and Corollary 3.5. □

10.4 Self-extensions

We now study the fully-faithfulness of some □-irreducible representations. One may compare with Lemma 8.4.

Proposition 10.6 *Let* $\pi \in \mathrm{Irr}(G_l)$. *Let* $\pi' \in \mathrm{Irr}$ *such that* $\pi \times \pi'$ *is irreducible. Let* $\mathrm{Alg}_{\pi',\mathrm{self}}(G_n)$ *be the full subcategory of* $\mathrm{Alg}_f(G_n)$ *whose objects have all simple composition factors isomorphic to* π'. *Then the product functor*

$$\times^s_\pi : \mathrm{Alg}_{\pi',\mathrm{self}}(G_n) \to \mathrm{Alg}(G_{n+l})$$

given by

$$\times^s_\pi(\tau) = \pi \times \tau$$

is fully-faithful.

Proof Again we check the conditions in Lemma 10.1. (1), (3) and (4) are automatic. (2) follows from Theorem 9.2. □

Remark 10.7 For $\pi \in \mathrm{Irr}^\square$, $\mathbb{D}_\pi(\pi \times \pi)$ is possibly not irreducible. For example, take $\mathfrak{m} = \{[0, 1], [1]\}$. Let $\pi = \langle\mathfrak{m}\rangle$. Via a geometric lemma consideration, one deduces that:

$$(\langle[0]\rangle \times \langle[1]\rangle \times \langle[1]\rangle \times \langle[0, 1]\rangle) \boxtimes \langle[1]\rangle$$

is a submodule of $\mathbb{D}_{[1]}(\pi \times \pi)$ (also see Example 11.6 below). Then

$$(\langle[0]\rangle \times \langle[1]\rangle \times \langle[1]\rangle) \boxtimes \pi$$

is a submodule of $\mathbb{D}_{[0,1]} \circ \mathbb{D}_{[1]}(\pi \times \pi) \cong \mathbb{D}_\pi(\pi \times \pi)$ (see Proposition 8.2).

One may further ask when $\pi \in \mathrm{Irr}^\square$ is prime in the Bernstein-Zelevinsky ring, is it true that $\mathbb{D}_\pi(\pi \times \pi) = \pi$? This holds for when π is a Speh representation by using [15], but it is not clear for the general situation.

11 Application on the SI Property for Big Derivatives

11.1 More Notations on Derivatives

Recall that the big derivative is defined in Definition 8.1. For $\Delta \in \mathrm{Seg}$, set $\mathbb{D}_\Delta = \mathbb{D}_{\mathrm{St}(\Delta)}$.

For $\pi \in \mathrm{Irr}$, let $D_\Delta(\pi)$ be the unique submodule of $\mathbb{D}_\Delta(\pi)$ if $\mathbb{D}_\Delta(\pi) \neq 0$. Let $D_\Delta(\pi) = 0$ if $\mathbb{D}_\Delta(\pi) = 0$.

For a generic multisegment $\mathfrak{m}$, we similarly set $\mathbb{D}_\mathfrak{m} = \mathbb{D}_{\mathrm{St}(\mathfrak{m})}$. For $\pi \in \mathrm{Irr}$, we similarly define $D_\mathfrak{m}(\pi)$ as the unique submodule of $\mathbb{D}_\mathfrak{m}(\pi)$ if $\mathbb{D}_\mathfrak{m}(\pi) \neq 0$ and define $D_\mathfrak{m}(\pi) = 0$ otherwise. The uniqueness of the simple submodule in $D_\Delta(\pi)$ and $\mathbb{D}_\mathfrak{m}(\pi)$ follows from [33] and [30].

Set $i = l_{abs}(\mathfrak{m})$. With a slight reformulation from above, we also have that:

$$\pi_{N_i} \twoheadrightarrow D_\mathfrak{m}(\pi) \boxtimes \mathrm{St}(\mathfrak{m}),$$

or equivalently, by [34, Corollary 2.4],

$$D_\mathfrak{m}(\pi) \boxtimes \mathrm{St}(\mathfrak{m}) \hookrightarrow \pi_{N_i},$$

or equivalently, by Frobenius reciprocity,

$$\pi \hookrightarrow D_\mathfrak{m}(\pi) \times \mathrm{St}(\mathfrak{m}).$$

11.2 η-invariants and Δ-reduced Representations

We shall first discuss more results on derivatives.

Define $\epsilon_\Delta(\pi)$ to be the largest non-negative integer k such that

$$D_\Delta^k(\pi) \neq 0.$$

Define

$$\eta_\Delta(\pi) = (\epsilon_{[a,b]_\rho}(\pi), \epsilon_{[a-1,b]_\rho}(\pi), \ldots, \epsilon_{[b,b]_\rho}(\pi)).$$

Using similar terminologies in [36, Section 7], a segment $[c, d]_\rho$ is said to be $[a, b]_\rho$-*saturated* if $d = b$ and $a \leq c$. Define $\mathfrak{mx}(\pi, \Delta)$ to be the multisegment that contains exactly the Δ-saturared segments Δ' with the multiplicity $\epsilon_{\Delta'}(\pi)$. We shall call π to be Δ-*reduced* if $\mathfrak{mx}(\pi, \Delta) = \emptyset$.

We give two useful properties related to the η-invariant, which will also be used in the appendix. Those properties are also useful in the study of the Bernstein-Zelevinsky derivatives [16, 18]:

Proposition 11.1 *(c.f.* [36, Proposition 7.3]*) Let* $\pi \in \mathrm{Irr}$ *and let* $\Delta \in \mathrm{Seg}$. *Let* $\mathfrak{p} = \mathfrak{mx}(\pi, \Delta)$. *Let* $i = l_{abs}(\mathfrak{p})$. *Then* $D_\mathfrak{p}(\pi) \boxtimes \mathrm{St}(\mathfrak{p})$ *is a direct summand in* π_{N_i}.

Proof Let $\tau = D_{\mathfrak{p}}(\pi)$. Then τ is Δ-reduced. We also have the embedding:

$$\pi \hookrightarrow \tau \times \mathrm{St}(\mathfrak{p}).$$

Then we apply the Jacquet functor $_{N_i}$ on $\tau \times \mathrm{St}(\mathfrak{p})$. We first mention two important ingredients. The first one is to use the Jacquet functor computations for Eq. 2.2. The second one is that the Δ-reduced property on τ implies that if a simple composition factor in τ_{N_j} (for some j) takes the form $\omega_1 \boxtimes \omega_2$ and satisfies that $\mathrm{csupp}(\omega_2) \subset \cup_{\Delta' \in \mathfrak{p}} \Delta'$ (as a multiset), then $b(\Delta) \notin \mathrm{csupp}(\omega_2)$.

Now, we have to see which layer in the geometric lemma can contribute to the same cuspidal support as $\tau \boxtimes \mathrm{St}(\mathfrak{p})$. But, the second point implies that, all $b(\Delta)$ must come from a factor from $\mathrm{St}(\mathfrak{p})$. However, the first point will then force that the layer in in $\tau \times \mathrm{St}(\mathfrak{p}))_{N_i}$ must come from the layer of the form $\tau \boxtimes \mathrm{St}(\mathfrak{p})$. Thus $\tau \boxtimes \mathrm{St}(\mathfrak{p})$ is a direct summand in $(\tau \times \mathrm{St}(\mathfrak{p}))_{N_i}$ and so is a direct summand in π_{N_i}. □

We shall use it to do a reduction later. When Δ is a singleton, Proposition 11.1 is also shown by Jantzen [28] and Mínguez [40] (also see [24]).

When $\mathfrak{m} \in \mathrm{Mult}$ is generic, $\mathrm{St}(\mathfrak{m}) = \lambda(\mathfrak{m})$ and $\mathrm{St}(\mathfrak{m})$ is generic when $D = F$ [53]. For generic $\mathfrak{m} \in \mathrm{Mult}$ and $\pi \in \mathrm{Irr}$, denote by $I_{\mathfrak{m}}(\pi)$ the unique simple submodule of $\pi \times \mathrm{St}(\mathfrak{m})$. (Here the uniqueness follows from [30] and [34] since $\mathrm{St}(\mathfrak{m}) \in \mathrm{Irr}^{\square}$.)

A multisegment $\mathfrak{m}$ is said to be Δ-*saturated* if all the segments in $\mathfrak{m}$ are Δ-saturated. We denote by $\mathrm{Mult}_{\Delta-sat}$ the set of all Δ-saturated multisegments.

Proposition 11.2 *(c.f.* [36, Proposition 7.3]*) Let* $\Delta \in \mathrm{Seg}$. *Let* $\tau \in \mathrm{Irr}$. *Suppose* τ *is* Δ*-reduced. Let* $\mathfrak{p} \in \mathrm{Mult}_{\Delta-sat}$. *Then,*

(1) $I_{\mathfrak{p}}(\tau)$ *appears with multiplicity one in* $\tau \times \mathrm{St}(\mathfrak{p})$*;*
(2) for π' *in* $\mathrm{JH}(\tau \times \mathrm{St}(\mathfrak{p}))$ *with* $\pi' \not\cong I_{\mathfrak{p}}(\tau)$, $\mathfrak{mx}(\pi', \Delta) \neq \mathfrak{p}$.

Proof Note that $\tau \cong D_{\mathfrak{p}} \circ I_{\mathfrak{p}}(\tau)$ by definitions. Then, we again have that:

$$I_{\mathfrak{p}}(\tau) \hookrightarrow \tau \times \mathrm{St}(\mathfrak{p}).$$

In Proposition 11.1, we showed that $\tau \boxtimes \mathrm{St}(\mathfrak{p})$ appears with multiplicity one in $(\tau \times \mathrm{St}(\mathfrak{p}))_{N_l}$, where $l = l_{abs}(\mathfrak{p})$. This implies (1). Moreover, the proof of Proposition 11.1 also shows that no other composition factor in $(\tau \times \mathrm{St}(\mathfrak{p}))_{N_l}$ takes the form $\tau' \boxtimes \mathrm{St}(\mathfrak{p})$. This implies (2). □

11.3 SI Property on the Segment Case

We introduce one more notions for convenience. For a cuspidal representation ρ, a multisegment $\mathfrak{p}$ is said to be *strongly* ρ*-saturated* if $b(\Delta) \cong \rho$ for any segment Δ in $\mathfrak{p}$.

Lemma 11.3 *Fix a cuspidal representation* ρ. *Let* $\mathfrak{p}$ *be a strongly* ρ*-saturated multisegment. Let* $\Delta \in \mathfrak{p}$. *Then*

$$\mathbb{D}_{\Delta}(\mathrm{St}(\mathfrak{p})) = D_{\Delta}(\mathrm{St}(\mathfrak{p})) = \mathrm{St}(\mathfrak{p} - \Delta).$$

Proof Let $\pi = \mathrm{St}(\Delta)$. By [53] and [51], $\mathrm{St}(\mathfrak{p}-\Delta) \in \mathcal{M}_\pi$. Note that $\mathrm{St}(\mathfrak{p}) = \mathrm{St}(\mathfrak{p}-\Delta) \times \mathrm{St}(\Delta)$.

Let $n = l_{abs}(\mathfrak{p}-\Delta)$ and let $k = l_{abs}(\Delta)$. We consider the full subcategory $\mathcal{A}'$ of $\mathrm{Alg}(G_n)$ whose objects have all composition factors isomorphic to $\mathrm{St}(\mathfrak{p}-\Delta)$, and let $\mathcal{B}'$ be the full subcategory of $\mathrm{Alg}(G_{n+k})$ whose objects have all composition factors isomorphic to $\mathrm{St}(\mathfrak{p})$. By Theorem 10.2, $\times_{\pi,\mathcal{A}'} : \mathcal{A}' \to \mathcal{B}'$ is a fully-faithful functor. Moreover, $\mathbb{D}_\Delta : \mathcal{B}' \to \mathcal{A}'$ is well-defined and is right-adjoint to $\times_{\pi,\mathcal{A}'}$. Thus,

$$\mathbb{D}_\Delta(\mathrm{St}(\mathfrak{p})) = \mathbb{D}_\Delta(\times_{\mathrm{St}(\Delta)}(\mathrm{St}(\mathfrak{p}-\Delta))) = \mathrm{St}(\mathfrak{p}-\Delta)$$

is irreducible, by [50, Lemma 4.24.3]. □

Remark 11.4 The statement of Lemma 11.3 does not hold in general if we simply replace $\mathbb{D}_\Delta$ by $\mathbb{D}'_\Delta$.

For $\sigma \in \mathrm{Irr}^\square$ and $\pi \in \mathrm{Irr}$, by [30], there exists at most one $\tau \in \mathrm{Irr}$ such that

$$\tau \boxtimes \sigma \hookrightarrow \pi_{N_{\deg(\sigma)}}.$$

If such τ exists, denote such τ by $D_\sigma(\pi)$. Otherwise, set $D_\sigma(\pi) = 0$.

Proposition 11.5 *Fix a cuspidal representation ρ. Let $\mathfrak{p}$ be a strongly ρ-saturated multisegment. Let $\sigma = \mathrm{St}(\mathfrak{p})$ and let $\pi \in \mathrm{Irr}$. Suppose $D_\sigma(\pi) \neq 0$. Then $\mathbb{D}_\sigma(\pi)$ is SI.*

Proof We write $\sigma = \mathrm{St}(\Delta)$ for some segment Δ. Let $\mathfrak{p} = \mathfrak{mx}(\pi, \Delta)$. Then we have that

$$\pi \hookrightarrow D_\mathfrak{p}(\pi) \times \mathrm{St}(\mathfrak{p}).$$

Now, one has:

$$\mathbb{D}_\Delta(\pi) \hookrightarrow \mathbb{D}_\Delta(D_\mathfrak{p}(\pi) \times \mathrm{St}(\mathfrak{p})).$$

From Definition 8.1, one has to compute a Jacquet module on $D_\mathfrak{p}(\pi) \times \mathrm{St}(\mathfrak{p})$ and the structure again can be computed from the geometric lemma. With a standard cuspidal support argument before, one boils down to have:

$$\mathbb{D}_\Delta(\pi) \hookrightarrow D_\mathfrak{p}(\pi) \times \mathbb{D}_\Delta(\mathrm{St}(\mathfrak{p})).$$

By Lemma 11.3, we have that

$$\mathbb{D}_\Delta(\pi) \hookrightarrow D_\mathfrak{p}(\pi) \times \mathrm{St}(\mathfrak{p}-\Delta).$$

Now $D_\Delta(\pi)$ is the unique submodule of $D_\mathfrak{p}(\pi) \times \mathrm{St}(\mathfrak{p}-\Delta)$ and appears with multiplicity one in $D_\mathfrak{p}(\pi) \times \mathrm{St}(\mathfrak{p}-\Delta)$ by [33] or [30]. Hence $D_\Delta(\pi)$ also appears with multiplicity one in $\mathbb{D}_\Delta(\pi)$. □

Example 11.6 For a segment Δ and $\pi \in \mathrm{Irr}$, $\mathbb{D}_\Delta(\pi)$ is not irreducible in general. For example, let $\mathfrak{m} = \{[1], [0, 1]\}$. Then $\mathbb{D}_{[1]}(\langle\mathfrak{m}\rangle)$ has length two.

One may further consider the indecomposable component τ in $\langle\mathfrak{m}\rangle_{N_1}$ which contains $\langle[0, 1]\rangle \boxtimes \langle[1]\rangle$ as the submodule. It is shown by (some variants of) [13, Corollary 2.9] (also see [16]) that τ is the direct summand with all the simple composition factors in $\langle\mathfrak{m}\rangle_{N_1}$ with the same cuspidal support as $\langle[0,1]\rangle \boxtimes \langle[1]\rangle$. Thus we have the following relation:

$$D_{[1]}(\langle\mathfrak{m}\rangle) \boxtimes \langle[1]\rangle \subsetneq \mathbb{D}_{[1]}(\langle\mathfrak{m}\rangle) \boxtimes \langle[1]\rangle \subsetneq \tau.$$

11.4 An Application

We give an application on studying how to embed some Jacquet modules into some layers arising from the geometric lemma. The study of how to do such embedding will be used in [17, 18] for studying commutations of some derivatives and integrals.

Proposition 11.7 *Let $\pi = \mathrm{St}(\mathfrak{n})$ for some generic $\mathfrak{n} \in \mathrm{Mult}_n$. Let $\Delta \in \mathrm{Seg}$ such that $D_\Delta(\pi) \neq 0$. Let τ be the unique indecomposable component with the unique submodule $D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta)$ in $\pi_{N_{l_{abs}(\Delta)}}$. Then $\mathfrak{mx}(\pi, \Delta)$ contains only one segment if and only if*

$$D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta) \cong \mathbb{D}_\Delta(\pi) \boxtimes \mathrm{St}(\Delta) \cong \tau.$$

Proof Let $\mathfrak{n}$ be the multisegment such that $\pi \cong \mathrm{St}(\mathfrak{n})$. Let $\mathfrak{m} = \mathfrak{mx}(\pi, \Delta)$. Let $i = l_{abs}(\Delta)$. If $\mathfrak{m}$ contains only one segment, then $\mathfrak{m} = \{\Delta\}$. Then $D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta)$ is a direct summand in π_{N_i} (see Proposition 11.1). Thus we have $D_\Delta(\pi) = \mathbb{D}_\Delta(\pi)$ and $\tau \cong D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta)$.

We now prove the converse direction. Since we are dealing with the generic case, we have a simple description on $\mathfrak{mx}$ as:

$$\mathfrak{mx}(\pi, \Delta) = \left\{[a(\Delta'), b(\Delta)] : \Delta' \in \mathfrak{n}, \quad a(\Delta') \leq b(\Delta)\right\}$$

Thus a geometric lemma shows that $D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta)$ appears more than one time in π_{N_i}.

Let $\omega = D_{\mathfrak{m}}(\pi)$. On the other hand, if $\mathbb{D}_\Delta(\pi) \boxtimes \mathrm{St}(\Delta) \cong \tau$, then

$$\tau \cong \mathbb{D}_\Delta(\pi) \boxtimes \mathrm{St}(\Delta) \hookrightarrow \mathbb{D}_\Delta(\omega \times \mathrm{St}(\mathfrak{m})) \boxtimes \mathrm{St}(\Delta).$$

Again, computing $\mathbb{D}_\Delta(\omega \times \mathrm{St}(\mathfrak{m}))$ involves a compution of a Jacquet module on $\omega \times \mathrm{St}(\mathfrak{m})$, which leads to analyzing on layers in the geometric lemma. Again with a standard comparison on cuspidal support, there is only one layer contributing the submodule $D_\Delta(\pi)$, that is of the form $\omega \times \mathbb{D}_\Delta(\mathrm{St}(\mathfrak{m}))$. Since that layer appears in the toppest one, we must then have that:

$$\tau \hookrightarrow (\omega \times \mathbb{D}_\Delta(\mathrm{St}(\mathfrak{m}))) \boxtimes \mathrm{St}(\Delta).$$

By Lemma 11.3, we then have:

$$\tau \hookrightarrow (\omega \times \mathrm{St}(\mathfrak{m} - \Delta)) \boxtimes \mathrm{St}(\Delta). \tag{11.9}$$

Since $\mathfrak{mx}(\omega, \Delta) = \emptyset$, Proposition 11.2 gives that $D_\Delta(\pi)$ appears with multiplicity one in $\omega \times \mathrm{St}(\mathfrak{m} - \Delta)$ and so as in τ. This contradicts to what we argued before. Hence, we cannot have $\tau \cong \mathbb{D}_\Delta(\pi) \boxtimes \mathrm{St}(\Delta)$. □

An alternate way to see Proposition 11.7 is that if $\mathfrak{mx}(\pi, \Delta)$ contains more than one segment, then τ cannot be written into the form $\omega' \boxtimes \mathrm{St}(\Delta)$ for some $\omega' \in \mathrm{Alg}_f$ since this otherwise will imply $\mathbb{D}_\Delta(\pi) \cong \omega'$ and so $\tau \cong \mathbb{D}_\Delta(\pi) \boxtimes \mathrm{St}(\Delta)$ giving a contradiction. This consequently gives:

Corollary 11.8 *Let $\pi = \mathrm{St}(\mathfrak{n})$ for some generic $\mathfrak{n} \in \mathrm{Mult}$. Let Δ be a segment such that $D_\Delta(\pi) \neq 0$. Let ω be a representation of finite length such that*

$$\pi \hookrightarrow \omega \times \mathrm{St}(\Delta).$$

Let $i = l_{abs}(\Delta)$. Let $p : \pi_{N_i} \twoheadrightarrow \omega \boxtimes \mathrm{St}(\Delta)$ be the projection arising from the geometric lemma (see Section 2.5). Let $\iota : D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta) \hookrightarrow \pi_{N_i}$ be the unique embedding. Suppose $\mathfrak{mx}(\pi, \Delta)$ contains at least two segments. Then $p \circ \iota = 0$.

Proof Let τ be the unique indecomposable module in π_{N_i} that contains $D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta)$ as submodule.

We have the following short exact sequence:

$$0 \to \kappa \to \pi_{N_i} \xrightarrow{p} \omega \boxtimes \mathrm{St}(\Delta) \to 0,$$

where κ is the kernel of the projection p. If $\tau \cap \kappa \neq 0$, then $D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta)$ must contain the unique submodule $D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta)$ by the uniqueness of simple submodule in τ. Thus it suffices to show $\tau \cap \kappa \neq 0$. Suppose not. Then,

$$\tau \hookrightarrow \omega \boxtimes \mathrm{St}(\Delta).$$

This implies that $\tau \cong \omega' \boxtimes \mathrm{St}(\Delta)$ for some submodule ω' of ω. Then $\mathbb{D}_\Delta(\pi) \boxtimes \mathrm{St}(\Delta) \cong \tau$. This contradicts Proposition 11.7. □

Appendix: SI Property of Big Derivatives for Generic Representations

It is interesting to generalize Proposition 11.5 to a larger family of big derivatives. We shall explain how to extend to generic representations in this appendix.

Lemma on Δ-reduced Representations

We generalize the idea of η-invariants in Section 11.2 to representations of finite lengths.

Definition 12.1 Let $\pi \in \mathrm{Alg}_f$. Let $\Delta \in \mathrm{Seg}$. We say that π is Δ-*reduced* if for any Δ-saturated segment $\widetilde{\Delta}$, $\mathbb{D}_{\widetilde{\Delta}}(\pi) = 0$.

The following lemma is a simple exercise using the Jacquet functors and we shall omit the details.

Lemma 12.2 *Let $\pi \in \mathrm{Alg}_f$. Then π is Δ-reduced if and only if π' is Δ-reduced for any π' in $\mathrm{JH}(\pi)$.*

Generic Case

We first have the following commutativity result:

Lemma 12.3 *Let* Δ_1, Δ_2 *be two unlinked segments. Let* $\pi \in \mathrm{Alg}_f$. *Then*

$$\mathbb{D}_{\Delta_1} \circ \mathbb{D}_{\Delta_2}(\pi) \cong \mathbb{D}_{\Delta_2} \circ \mathbb{D}_{\Delta_1}(\pi).$$

Proof Since $\mathrm{St}(\Delta_1) \times \mathrm{St}(\Delta_2) \cong \mathrm{St}(\Delta_2) \times \mathrm{St}(\Delta_1)$ is irreducible by [51, 53], the result follows from Proposition 8.2. □

Lemma 12.4 *Let* $\pi \in \mathrm{Irr}$. *Let* $\Delta \in \mathrm{Seg}$. *Suppose* π *is* Δ*-reduced. Let* Δ' *be a segment such that* $a(\Delta) \leq a(\Delta') \leq b(\Delta) \leq b(\Delta')$. *Then* $\mathbb{D}_{\Delta'}(\pi) = D_{\Delta'}(\pi) = 0$.

Proof Suppose $D_{\Delta'}(\pi) \neq 0$. Let $i = l_{abs}(\Delta')$. Then we have an embedding:

$$D_{\Delta'}(\pi) \boxtimes \mathrm{St}(\Delta') \hookrightarrow \pi_{N_i}.$$

Let $\Delta'' = [a(\Delta'), b(\Delta)]$. Let $n = \deg(\pi)$ and let $j = l_{abs}(\Delta'')$. Write $\Delta = [a, b]_\rho$. We apply the Jacquet functor $_{N_j}$ on the second factor, and so, by

$$\mathrm{St}(\Delta')_{N_j} = \mathrm{St}([\nu_\rho \cdot b(\Delta), b(\Delta')]) \boxtimes \mathrm{St}(\Delta''),$$

we have

$$D_{\Delta'}(\pi) \boxtimes \mathrm{St}([\nu_\rho \cdot b(\Delta), b(\Delta')]) \boxtimes \mathrm{St}(\Delta'') \hookrightarrow \pi_{N_{n-i,i-j,j}}.$$

By taking Jacquet functor in stages and applying Frobenius reciprocity, we have that $D_{\Delta''}(\pi) \neq 0$. This gives a contradiction. □

Recall that $\mathrm{Mult}_{\Delta-sat}$ is defined in Section 11.2.

Lemma 12.5 *Let* $\pi \in \mathrm{Alg}_f$. *Let* $\Delta \in \mathrm{Seg}$ *and let* $\mathfrak{p} \in \mathrm{Mult}_{\Delta-sat}$. *Let* Δ' *be a segment such that* $a(\Delta) \leq a(\Delta')$ *and* $b(\Delta) \leq b(\Delta')$. *Suppose* $D_{\Delta'}(I_{\mathfrak{p}}(\pi)) \neq 0$. *Suppose, for any segment* $\widetilde{\Delta}$ *with the following two properties:*

- $b(\widetilde{\Delta}) \cong b(\Delta)$; *and*
- $a(\widetilde{\Delta}) \leq b(\Delta)$,

we have $\mathbb{D}_{\widetilde{\Delta}}(\pi) = 0$. *Let* $\Delta'' = [a(\Delta'), b(\Delta)]$. *Then*

$$\mathbb{D}_{\Delta'}(\pi \times \mathrm{St}(\mathfrak{p})) \cong \mathbb{D}_{[\nu \cdot b(\Delta), b(\Delta')]}(\pi) \times \mathrm{St}(\mathfrak{p} - \Delta'').$$

Proof Let $i = l_{abs}(\Delta')$. Recall that

$$(*) \quad \mathbb{D}_{\Delta'}(\pi \times \mathrm{St}(\Delta')) \cong \mathrm{Hom}_{G_i}(\mathrm{St}(\Delta'), (\pi \times \mathrm{St}(\mathfrak{p}))_{N_i}).$$

Let $m = l_{abs}(\mathfrak{p})$. Then the layers in the geometric lemma of $(\pi \times \mathrm{St}(\mathfrak{p}))_{N_i}$ take the form: for $k + l = i$,

$$\omega_{k,l} = \mathrm{Ind}_{P_{n-k,m-l} \times P_{k,l}}^{G_{n+m-i} \times G_i}((\pi)_{N_k} \boxtimes (\mathrm{St}(\mathfrak{p}))_{N_l})^\phi,$$

where ϕ is a twist takes a $G_{n-k} \times G_k \times G_{m-l} \times G_l$-representation to a $G_{n-k} \times G_{m-l} \times G_k \times G_l$-representation.

Write $\Delta' = [a', b']_{\rho'}$. By Frobenius reciprocity, we have that:

$$\mathrm{Hom}_{G_i}(\mathrm{St}(\Delta'), \omega_{k,l}) \cong \mathrm{Hom}_{G_k \times G_l}(\tau_k \boxtimes \widetilde{\tau}_l, \widetilde{\omega}_{k,l})$$

where

$$\tau_k = \mathrm{St}([\nu_{\rho'}^{-(k-1)} b(\Delta'), b(\Delta')]), \quad \widetilde{\tau}_l = \mathrm{St}([a(\Delta'), \nu_{\rho'}^{l-1} a(\Delta')])$$

and

$$\widetilde{\omega}_{k,l} = \mathrm{Ind}_{P_{n-k,m-l} \times P_{k,l}}^{G_{n+m-i} \times G_i}((\pi)_{N_k} \boxtimes (\mathrm{St}(\mathfrak{p}))_{N_l})^{\phi}.$$

Thus we have that:

$$(**) \quad \mathrm{Hom}_{G_i}(\mathrm{St}(\Delta'), \omega_{k,l}) \cong \mathbb{D}_{\tau_k}(\pi) \times \mathbb{D}_{\widetilde{\tau}_l}(\mathrm{St}(\mathfrak{p})).$$

Let $l^* = l_{abs}([a(\Delta'), b(\Delta)])$ and $k^* = l_{abs}(\Delta') - l^*$. Note that if $l \neq l^*$, then either

$$\mathbb{D}_{\tau_l}(\pi) = 0 \quad \text{or} \quad \mathbb{D}_{\widetilde{\tau}_k}(\mathrm{St}(\mathfrak{p})) = 0,$$

which follows by either the assumption in the lemma or a comparison of the cuspidal support.

Now combining (*), (**) and the claim, we have that

$$\mathbb{D}_{\Delta'}(\pi \times \mathrm{St}(\Delta')) \cong \mathbb{D}_{\tau_{k^*}}(\pi) \times \mathbb{D}_{\widetilde{\tau}_{l^*}}(\mathrm{St}(\mathfrak{p}))$$

Now the lemma follows from Lemma 11.3. □

Lemma 12.6 *Let* $\pi \in \mathrm{Alg}_f$. *Let* $\mathfrak{p} \in \mathrm{Mult}_{\Delta-sat}$. *Let* Δ' *be a segment satisfying one of the following conditions:*

(1) $a(\Delta') < a(\Delta)$ *and* $b(\Delta) \leq b(\Delta')$*; or*
(2) $b(\Delta') < a(\Delta)$*; or*
(3) $b(\Delta) < a(\Delta')$.

Then

$$\mathbb{D}_{\Delta'}(\pi \times \mathrm{St}(\mathfrak{p})) = \mathbb{D}_{\Delta'}(\pi) \times \mathrm{St}(\mathfrak{p}).$$

Proof This again follows by using the geometric lemma and to notice the only layer for contributing the big derivative. We omit the details. □

We now generalize Proposition 11.5 to arbitrary generic representations. In order to use induction, one uses Section 11.2 and some properties in Proposition 11.2. For two segments Δ, Δ', we write $\Delta \leq_a \Delta'$ if either one of the conditions hold:

- $a(\Delta) < a(\Delta')$; or
- $a(\Delta) \cong a(\Delta')$ and $b(\Delta) \leq b(\Delta')$.

Theorem 12.7 *Let* $\mathfrak{m} \in \mathrm{Mult}$ *be generic. Let* $\sigma = \mathrm{St}(\mathfrak{m})$. *Then, for any* $\pi \in \mathrm{Irr}$ *with* $D_\sigma(\pi) \neq 0$, $\mathbb{D}_\sigma(\pi)$ *is SI.*

Proof We shall prove by an induction on the number of segments in $\mathfrak{m}$. When there is only one segment in $\mathfrak{m}$, it follows from Proposition 11.5.

We consider the set

$$\mathcal{B} = \{b(\Delta) : \Delta \in \mathfrak{m}\} .$$

Then we choose a minimal element ρ in $\mathcal{B}$ with respect to $\leq$ (see the ordering in Section 2.1). Among those strongly ρ-saturated segments, we choose a $\leq_a$-maximal segment Δ^* (equivalently the shortest one among those).

Let

$$\mathfrak{n} = \left\{\widetilde{\Delta} : a(\widetilde{\Delta}) \not\cong a(\Delta^*)\right\} \cup \left\{\widetilde{\Delta} \setminus \Delta^* : a(\widetilde{\Delta}) \cong a(\Delta^*), \widetilde{\Delta} \in \mathfrak{m}\right\}$$

Here $\widetilde{\Delta} \setminus \Delta^*$ is the set-theoretic subtraction i.e. for writing $\Delta^* = [a^*, b^*]_{\rho'}$ and $\widetilde{\Delta} = [a^*, \widetilde{b}]_{\rho'}$,

$$\widetilde{\Delta} \setminus \Delta^* = [b^* + 1, \widetilde{b}]_{\rho'}.$$

Claim 1: $\mathfrak{n}$ is generic.
Proof of Claim 1: It follows form the choice of Δ^* that there is no segment $\widetilde{\Delta}$ in $\mathfrak{m}$ such that $\widetilde{\Delta} \subsetneq \Delta^*$. Then it is direct to check from the genericity of $\mathfrak{m}$ that $\mathfrak{n}$ is also generic.

We shall use Claim 1 later. We now consider some other multisegments:

$$\mathfrak{o} = \left\{\widetilde{\Delta} \in \mathfrak{m} : a(\widetilde{\Delta}) \cong a(\Delta)\right\}, \quad \mathfrak{o}' = \left\{\widetilde{\Delta} \setminus \Delta : \widetilde{\Delta} \in \mathfrak{o}\right\},$$

$$\mathfrak{p} = \mathfrak{m} - \mathfrak{o}.$$

Let $\mathfrak{t} = \mathfrak{mx}(\mathfrak{m}, \Delta^*)$ and let $\tau = D_{\mathfrak{t}}(\pi)$. let k be the number of segments in $\mathfrak{t}$. We now prove another claim:
Claim 2: $\mathbb{D}_{\mathfrak{o}}(\tau \times \mathrm{St}(\mathfrak{t})) \cong \mathbb{D}_{\mathfrak{o}'}(\tau) \times \mathrm{St}(\mathfrak{t} - k \cdot \Delta)$.
Proof of Claim 2: We shall prove inductively on the number of segments. When there is only one segment in $\mathfrak{o}$, it follows from Lemmas 12.4 and 12.5. We suppose there are more than one segments. In such case, we pick a longest segment $\overline{\Delta}$ in $\mathfrak{o}$ and hence $b(\widetilde{\Delta}) \leq b(\overline{\Delta})$ for any $\widetilde{\Delta} \in \mathfrak{o}$. This also implies

$$\widetilde{\Delta} \setminus \Delta \subset \overline{\Delta} \tag{12.10}$$

for any $\widetilde{\Delta} \in \mathfrak{o}$.

Now induction gives that

$$\mathbb{D}_{\mathfrak{o}-\overline{\Delta}}(\tau) \times \mathrm{St}(\mathfrak{t})) \cong \mathbb{D}_{\mathfrak{o}''}(\tau) \times \mathrm{St}(\mathfrak{t} - (k-1) \cdot \Delta),$$

where $\mathfrak{o} = \left\{\widetilde{\Delta} \setminus \Delta\right\}_{\widetilde{\Delta} \in \mathfrak{o}-\overline{\Delta}}$. Now the claim will follow from Lemma 12.5 if we can verify that

$$\mathbb{D}_{\overline{\Delta}}(\mathbb{D}_{\mathfrak{o}-\overline{\Delta}}(\tau)) = 0.$$

To this end, we use (12.10) and so we can apply Lemma 12.3 (several times) to obtain

$$\mathbb{D}_{\overline{\Delta}}(\mathbb{D}_{\mathfrak{o}-\overline{\Delta}}(\tau) = \mathbb{D}_{\mathfrak{o}-\overline{\Delta}} \circ \mathbb{D}_{\overline{\Delta}}(\tau) = 0.$$

Now now prove another claim:

Claim 3: $\mathbb{D}_{\mathfrak{p}} \circ \mathbb{D}_{\mathfrak{o}}(\tau \times \mathrm{St}(\mathfrak{t})) \cong \mathbb{D}_{\mathfrak{o}'+\mathfrak{p}}(\tau) \times \mathrm{St}(\mathfrak{t} - k \cdot \Delta)$.

Proof: It follows from Claim 2 that we only have to prove:

$$\mathbb{D}_{\mathfrak{p}}(\mathbb{D}_{\mathfrak{o}'}(\tau) \times \mathrm{St}(\mathfrak{t} - k \cdot \Delta)) \cong \mathbb{D}_{\mathfrak{o}'+\mathfrak{p}}(\tau) \times \mathrm{St}(\mathfrak{t} - k \cdot \Delta).$$

This follows by using Lemma 12.6 several times.

Now to prove $D_\sigma(\pi)$ appears with multiplicity one in the Jordan-Hölder series of $\mathbb{D}_\sigma(\pi)$, it suffices to show that $D_\sigma(\pi)$ appears with multiplicity one in $\mathbb{D}_\sigma(D_{\mathfrak{p}}(\pi) \times \mathrm{St}(\mathfrak{p}))$. Now Claim 2 reduces to prove that (**) $\mathbb{D}_{\mathfrak{o}'+\mathfrak{p}}(\tau) \times \mathrm{St}(\mathfrak{t} - k \cdot \Delta)$.

Before proving (**), we have one more claim:

Claim 4: $\mathbb{D}_{\mathfrak{o}'}(\tau)$ and $\mathbb{D}_{\mathfrak{o}'+\mathfrak{p}}(\tau)$ are Δ-reduced.

Proof of Claim 4: With a similar argument to proving Claim 2, we have:

$$\mathbb{D}_{\mathfrak{t}-k\dot{\Delta}+\mathfrak{o}}(\tau \times \mathrm{St}(\mathfrak{t})) \cong \mathbb{D}_{\mathfrak{o}'}(\tau)$$

We see that the LHS is Δ-reduced (since $\mathbb{D}_{\mathfrak{t}-k\dot{\Delta}+\Delta'+\mathfrak{o}}(\tau \times \mathrm{St}(\mathfrak{t})) = 0$ for a Δ-saturated segment Δ') and so is the RHS. Now it follows from Lemma 12.3 that we also have $\mathfrak{mx}(\mathbb{D}_{\mathfrak{p}} \circ \mathbb{D}_{\mathfrak{o}'}(\tau), \Delta) = \emptyset$.

Let $\mathfrak{t}' = \mathfrak{t} - k \cdot \Delta$.

Claim 1 shows that $\mathbb{D}_{\mathfrak{o}'+\mathfrak{p}}(\tau)$ is SI by induction. With Claim 4, the SI property of $\mathbb{D}_{\mathfrak{o}'+\mathfrak{p}}(\tau) \times \mathrm{St}(\mathfrak{t}')$ now follows from [36, Lemma 7.1] and so we also have the SI property for $\mathbb{D}_{\mathfrak{o}+\mathfrak{p}}(\tau \times \mathrm{St}(\mathfrak{t}))$ by Claim 3. Since $\pi \hookrightarrow \tau \times \mathrm{St}(\mathfrak{t})$, we now also have the SI property of $\mathbb{D}_\sigma(\pi)$. □

Remark 12.8 We remark that [30, Corollary 3.7] shows that there is a unique simple submodule of $\mathbb{D}_\sigma(\pi)$ for $\sigma \in \mathrm{Irr}^{\square}$ and $\pi \in \mathrm{Irr}$. Indeed, for the special case of generic representations, it also follows from [13, Proposition 2.5] (also see [16]), using some inputs from branching laws.

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