KYP Lemma for Cone-preserving Systems and Its Applications to Controller Design

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***Abstract*—This paper presents a new version of the Kalman-Yakubovich-Popov (KYP) Lemma for linear systems with their states constrained in proper cones. Based on this lemma, two important applications are introduced. One is the stabilization controller design to satisfy the spectral radius performance while preserving cone invariance. The other is to obtain an H^∞ state-feedback controller such that both H^∞ performance and cone invariance are guaranteed. Moreover, to address these two problems, a practical algorithm based on the linear matrix inequality is provided. Finally, two numerical examples on a linear system defined in the second-order cone are used to illustrate the results.**

***Index Terms*—Cone invariance, H^∞ control, KYP Lemma, Linear matrix inequality, Second-order cone, Stabilization.**

I. Introduction

Positive systems, characterized by the property that their states and outputs remain in the nonnegative orthant whenever initialized in the nonnegative orthant, are widely used to describe various dynamic processes, including virus treatment [1], population variation [2], and network congestion [3]. This special property has brought new analytical methods and extensive results of stability and input-output performance analysis, which are only valid for positive systems but not for general linear systems [4]–[9]. For positive systems, the storage function in dissipative theory can be constructed in the form of a quadratic Lyapunov function (LF) with a diagonal matrix. Instead of a symmetric matrix for general linear systems, the diagonal matrix can simplify the computation of stability condition and H^∞ control scheme [6]. Another form of LF frequently used in positive systems is the linear co-positive LF, represented as the inner product of two positive vectors. By doing so, the stability analysis and input-output gain characterization can be solved using linear programming (LP) [7]–[9].

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When the invariance property of states in the nonnegative orthant is generalized to a cone, systems with such constraints are called cone-preserving systems. In addition to the positive systems mentioned above, cone-preserving systems also find applications in the rendezvous problems of multi-agent systems [10], covariance dynamics of stochastic systems [11], and chemical reaction networks [12]. Despite the rich results on positive systems, the research on cone-preserving systems is limited. The input-output performance for positive systems based on the L_1-, L_2-, L_∞-norm was proved to be determined by a static gain matrix [6], [9], [13] . Similar results were obtained in [14], where the input-output gain was characterized by the cone linear absolute-norm [15] and the cone max-norm [16], regarded as an analogous of L_1- and L_∞-norm for positive systems, respectively. However, it was found in [11] that the property of L_2-gain fails to be extended to the cone-preserving system directly. A weaker result derived in [17], presented that the property of L_2-induced gain can still hold for systems with the invariance condition in symmetric cones. In [17], the H^∞ performance was shown to be completely governed by a static matrix, and such performance also can be characterized in terms of a linear matrix inequality (LMI) with the quadratic representation of the Jordan algebra instead of the diagonal form in positive systems [6]. A similar result is derived in this paper by an alternative proof based on the Kalman-Yakubovich-Popov (KYP) Lemma.

The claimed KYP Lemma is used to build an equivalence between a frequency domain inequality and a state space matrix based LMI, which has developed into different versions for various control performances. Specifically, the KYP Lemma frequently used in H^∞ control states that the H^∞ performance can be evaluated by checking the existence of a symmetric matrix in an LMI. It turns out in [6] that this symmetric matrix in the LMI can be replaced by a diagonal matrix for positive systems. A new equivalent condition of matrix inequalities between LP and semi-definite programming was introduced in the KYP Lemma for positive systems [18]. The corresponding work on discrete-time positive systems was proposed in [19]. Therefore, a natural question arises: is there a KYP Lemma for cone-preserving systems? A special version of KYP Lemma was provided in [17], where the H^∞ performance characterized by an LMI was equivalent to a set of inequalities solved by cone programming and also equivalent to the 2-norm constraint on a static matrix. Based on the existing results, we aim to generalize the claimed KYP Lemma to cone-preserving systems. Via the KYP Lemma in this paper, two performance-based stabilization control problems are addressed: spectral radius performance and H^∞ performance.

Compared with the previous work, the main contributions of this paper are as follows. First, a general version of the KYP Lemma is provided, which gives four equivalent inequalities including the frequency domain inequality, LMI, matrix inequality with quadratic form and inequality in terms of cone programming. Compared with the result in [17], the matrix (denoted as M in this paper) presented in these four inequalities is less restrictive and can be selected in various forms according to the control targets, while in [17], this matrix is fixed in terms of system matrices. Hence, this KYP Lemma can deal with more control problems besides the H^∞ control. Second, for second-order cones, a controller gain is obtained such that the stability with performance requirements and cone invariance are guaranteed simultaneously. Although there are plenty of works on stability analysis for cone-preserving systems [20], [21], the stabilization problem still remains open due to the complex computational issues. By the KYP Lemma proposed in [22], a necessary and sufficient condition is first derived for the analysis of the spectral radius for systems defined in symmetric cones. For second-order cone-invariant systems, a detailed algorithm is given for the stabilization with spectral radius specification. The KYP Lemma is further applied to H^∞ controller design, as the work in [17] only considered the analysis of H^∞ performance. The design of H^∞ controller is considerably more difficult for cone-preserving systems than ordinary linear systems. Different from the work in [17], a new proof to characterize the H^∞ norm is also provided.

The rest of this paper is organized as follows. After recalling some necessary definitions and mathematical preliminaries about cone-preserving systems in Section II, we present a KYP Lemma and its proof in Section III. The applications of the KYP Lemma, including the stabilization with spectral radius performance and the H^∞ control, are presented in Section IV. The corresponding LMI algorithms for solving these two problems are also provided in this section. Section V utilizes two numerical examples to illustrate the effectiveness of proposed results. Finally, Section VI concludes the paper.

Notations: $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ represent the set of real numbers, the space of vectors of n-tuples of real numbers, and the space of $n\times n$ matrices with real entries, respectively. e_{1n} denotes the first column of identity matrix $I_n \in \mathbb{R}^{n\times n}$. The spectral radius of a square matrix A is denoted as $\rho(A)$. For matrix A, A^T, A^*, $\det(A)$ and $\mathrm{Trace}(A)$ stand for its transpose, conjugate transpose, determinant, and trace, respectively. A square matrix A is Hurwitz if all its eigenvalues lie in the open left- half-plane. $\langle \xi, \zeta\rangle = \xi^T\zeta$ represents the inner product of vectors ξ and ζ. $\mathrm{diag}(\xi)$ denotes a diagonal matrix obtained by orderly putting all elements of vector ξ on the diagonal. $\mathrm{diag}(A_1, A_2, \ldots, A_n)$ denotes a block diagonal matrix with square matrices $A_1, A_2, \ldots, A_n$ on the diagonal.

II. Preliminaries

In this section, some elementary notions about proper cones and cone-preserving systems, will be introduced as follows. For a set $\mathcal{K} \subseteq \mathbb{R}^n$, $\mathcal{K}^G$ stands for the set consisting of all finite nonnegative linear combinations of the elements of $\mathcal{K}$; $\mathcal{K}$ is called a cone if $\mathcal{K} = \mathcal{K}^G$; $\mathcal{K}$ is convex if $\alpha\xi_1 + (1-\alpha)\xi_2 \in \mathcal{K}$ for any $\xi_1, \xi_2 \in \mathcal{K}$ and $\alpha \in [0,1]$; $\mathcal{K}$ is solid if its interior denoted as $\mathrm{Int}(\mathcal{K})$ is not an empty set; and $\mathcal{K}$ is pointed if $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$.

Definition 1: [16] A cone which is closed, convex, solid and pointed is called a proper cone.

For a proper cone $\mathcal{K}$, its dual is defined as the set $\mathcal{K}^* = \{\zeta \in \mathbb{R}^n : \langle\zeta,\xi\rangle \ge 0, \forall \xi \in \mathcal{K}\}$, and the interior of $\mathcal{K}^*$ is denoted as $\mathrm{Int}(\mathcal{K}^*) = \{\zeta \in \mathbb{R}^n : \langle\zeta,\xi\rangle > 0, \forall\xi \in \{\mathcal{K}\setminus 0\}\}$. In addition, $\xi_2 \preceq_{\mathcal{K}} \xi_1$ indicates $\xi_1 - \xi_2 \in \mathcal{K}$, and $\xi_2 \prec_{\mathcal{K}} \xi_1$ means $\xi_1 - \xi_2 \in \mathrm{Int}(\mathcal{K})$.

Definition 2: [16] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be $\mathcal{K}$-nonnegative if $A\mathcal{K} \subseteq \mathcal{K}$, and it is said to be $\mathcal{K}$-positive if $A\{\mathcal{K}\setminus 0\} \subseteq \mathrm{Int}(\mathcal{K})$.

Definition 3: [23] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be cross-positive on $\mathcal{K}$ if for all $y \in \mathcal{K}, z \in \mathcal{K}^*$ satisfying $\langle z, y\rangle = 0$, it holds that $\langle z, Ay\rangle \ge 0$, and it is said to be strictly cross-positive on $\mathcal{K}$ if for all $y \in \{\mathcal{K}\setminus 0\}, z \in \{\mathcal{K}^*\setminus 0\}$ satisfying $\langle z, y\rangle = 0$, it holds that $\langle z, Ay\rangle > 0$.

Lemma 1: [24] For a proper cone $\mathcal{K}$, if matrix A is cross-positive on $\mathcal{K}$, then the following statements are equivalent:

(1) A is Hurwitz.
(2) There exists a vector $\xi \succ_{\mathcal{K}} 0$ such that $A\xi \prec_{\mathcal{K}} 0$.
(3) There exists a vector $\zeta \succ_{\mathcal{K}^*} 0$ such that $A^T\zeta \prec_{\mathcal{K}^*} 0$.
(4) $-A^{-1}$ is $\mathcal{K}$-nonnegative.

Below, some basic properties about second-order cones and Jordan algebra are presented. Matrix Q_n, used to characterize second-order cones, is defined as $Q_n = 2e_{1n}e_{1n}^T - I_n$.

Definition 4: [22] An n-dimensional second-order cone $\mathcal{I}_n$ is defined as $\mathcal{I}_n = \{x \in \mathbb{R}^n : x^TQ_nx \ge 0, x^Te_{1n} \ge 0\}$, and its interior is $\mathrm{Int}(\mathcal{I}_n) = \{x \in \mathbb{R}^n : x^TQ_nx > 0, x^Te_{1n} > 0\}$.

Based on Schur complement, a nonzero vector $s \succeq_{\mathcal{I}_n} 0$ ($s \succ_{\mathcal{I}_n} 0$, respectively) if and only if $e_{1n}s^T + se_{1n}^T - e_{1n}^TsQ_n \ge 0$ ($e_{1n}s^T + se_{1n}^T - e_{1n}^TsQ_n > 0$, respectively), which is also called the arrow-shaped representation of s [22]. Note that second-order cones are self-dual, namely $\mathcal{I}_n^* = \mathcal{I}_n$. The following two lemmas build the connection between some properties of second-order cone and matrix inequalities.

Lemma 2: [22], [23] Consider a second-order cone $\mathcal{I}_n$, then the following statements are equivalent:

(1) Matrix A is strictly cross-positive on $\mathcal{I}_n$.
(2) There exists $\alpha \ge 0$ such that $(A + \alpha I_n)$ is $\mathcal{I}_n$-positive.
(3) There exists $\lambda \in \mathbb{R}$ such that $A^TQ_n + Q_nA + \lambda Q_n > 0$.
(4) There exist $\alpha \ge 0$ and $\eta \ge 0$ such that $(A + \alpha I_n)^Te_{1n} \in \mathrm{Int}(\mathcal{I}_n)$ and $(A + \alpha I_n)^TQ_n(A + \alpha I_n) > \eta Q_n$.

Lemma 3: For second-order cones $\mathcal{I}_n$ and $\mathcal{I}_m$, $A\mathcal{I}_n \subseteq \mathcal{I}_m$ holds if and only if there exists a scalar $\eta \ge 0$ such that $A^Te_{1m} \in \mathcal{I}_n$ and $A^TQ_mA \ge \eta Q_n$.

When $m = n$, Lemma 3 corresponds to Lemma 2.9 in [22]. If $n \ne m$, it can be derived by applying Lemma 2.9 in [22] through making A square by patching zeros.

Definition 5: [17], [25] A Euclidean Jordan algebra $(\mathfrak{J}, \circ)$ is a finite-dimensional vector space over $\mathbb{R}$ endowed with a multiplication $\circ$. $(x, y) \to x\circ y$ is a bilinear mapping satisfying the following conditions:

(1) $x \circ y = y \circ x$, $\forall x, y \in \mathfrak{J}$.

(2) $x^2 \circ (y \circ x) = (x^2 \circ y) \circ x$, $\forall x, y \in \mathfrak{J}$, where $x^2 = x \circ x$.
(3) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$, $\forall x, y, z \in \mathfrak{J}$.

The following properties of the Euclidean Jordan algebra and symmetric cones can be found in [26].

For all $x \in \mathfrak{J}$, there exists a spectral decomposition: $x = \sum_{i=1}^{r} \lambda_i f_i$, where λ_i are real numbers, r is said to be the rank of $\mathfrak{J}$, and the set $\{f_1, f_2 \dots f_r\}$ is said to be a Jordan frame with each f_j being primitive idempotent, $f_i \circ f_j = 0$ for $i \neq j$ and $\sum_{i=1}^{r} f_i = \mathbf{e}$. $\mathbf{e}$ is called the identity element of $\mathfrak{J}$ with $x \circ \mathbf{e} = x$. The inverse and square root of $x \in \mathfrak{J}$ are denoted as $x^{-1} = \sum_{i=1}^{r} \lambda_i^{-1} f_i$ and $x^{\frac{1}{2}} = \sum_{i=1}^{r} \lambda_i^{\frac{1}{2}} f_i$, respectively. It indicates that x^{-1} is valid if and only if $\lambda_i \neq 0$, and $x^{\frac{1}{2}}$ is valid if and only if $\lambda_i \geq 0$. For all $x \in \mathfrak{J}$, there exist a self-adjoint operator L satisfying $x \circ y = L(x)y$, $\forall y \in \mathfrak{J}$, and a quadratic representation of x as $P(x) := 2L(x)^2 - L(x^2)$.

A proper cone $\mathcal{K}_{\mathfrak{J}}$ is called a symmetric cone if it is self-dual and homogenous, i.e., the automorphism group of $\mathcal{K}_{\mathfrak{J}}$ acts transitively on $\text{Int}(\mathcal{K}_{\mathfrak{J}})$. Based on the Euclidean Jordan algebra, a symmetric cone can be written as $\mathcal{K}_{\mathfrak{J}} = \{x \circ x \mid x \in \mathfrak{J}\}$. For more details, one can refer to [26]. Hence, for a symmetric cone $\mathcal{K}_{\mathfrak{J}}$, if $x \in \mathcal{K}_{\mathfrak{J}}$, we have $\lambda_i \geq 0$ and

$$P(x)^2 = P(x^2), \qquad P(x^{\frac{1}{2}}) = P(x)^{\frac{1}{2}}, \tag{1}$$

and if $x \in \text{Int}(\mathcal{K}_{\mathfrak{J}})$, we have $\lambda_i > 0$, and

$$\begin{aligned} &P(x^{-1}) = P(x)^{-1}, \quad P(x) = P(x)^T > 0,\\ &P(x)\mathcal{K}_{\mathfrak{J}} \subseteq \mathcal{K}_{\mathfrak{J}}, \qquad P(x)\text{Int}(\mathcal{K}_{\mathfrak{J}}) \subseteq \text{Int}(\mathcal{K}_{\mathfrak{J}}). \end{aligned} \tag{2}$$

In addition, for all $x, y \in \text{Int}(\mathcal{K}_{\mathfrak{J}})$, we have

$$z = P(y^{-1/2})(P(y^{1/2})x)^{1/2} \in \text{Int}(\mathcal{K}_{\mathfrak{J}}), \quad x = P(z)y. \tag{3}$$

Note that a symmetric cone is a self-dual cone but not vice versa.

III. KYP Lemma

Consider the following linear system:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{4}$$

which has the property that $x(t) \in \mathcal{K}_x \subset \mathbb{R}^n$ for all $t > 0$ under the condition of $x(0) \in \mathcal{K}_x$ and $u(t) \in \mathcal{K}_u \subset \mathbb{R}^m$, where $x(t)$, $u(t)$ are the state vector and the input vector, respectively, and $\mathcal{K}_x$, $\mathcal{K}_u$ are given proper cones. In this case, system (4) is called a cone-preserving system, which can be also characterized by the following lemma.

Lemma 4: [14] System (4) is a cone-preserving system with respect to $(\mathcal{K}_x, \mathcal{K}_u)$ if and only if A is cross-positive on $\mathcal{K}_x$ and $B\mathcal{K}_u \subseteq \mathcal{K}_x$.

The following theorem presents the KYP Lemma for cone-preserving systems.

Theorem 1: Consider two symmetric cones $\mathcal{K}_x \subset \mathbb{R}^n$ and $\mathcal{K}_u \subset \mathbb{R}^m$. Let A be cross-positive on $\mathcal{K}_x$ and Hurwitz, B satisfy $B\mathcal{K}_u \subseteq \mathcal{K}_x$. Suppose that symmetric matrix $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$ has the property of M_{11} being $\mathcal{K}_x$-nonnegative, $M_{12}\mathcal{K}_u \subseteq \mathcal{K}_x$ and M_{22} being cross-positive on $\mathcal{K}_u$. Then the following statements are equivalent:

1) For all $\omega \geq 0$,
$$\begin{bmatrix} (j\omega I_n - A)^{-1}B \\ I_m \end{bmatrix}^* M \begin{bmatrix} (j\omega I_n - A)^{-1}B \\ I_m \end{bmatrix} < 0.$$
2)
$$\begin{bmatrix} -A^{-1}B \\ I_m \end{bmatrix}^T M \begin{bmatrix} -A^{-1}B \\ I_m \end{bmatrix} < 0.$$
3) There exist vectors $\theta_x \succ_{\mathcal{K}_x} 0$, $p \succ_{\mathcal{K}_x} 0$ and $\theta_u \succ_{\mathcal{K}_u} 0$ with $A\theta_x + B\theta_u \prec_{\mathcal{K}_x} 0$ such that
$$M \begin{bmatrix} \theta_x \\ \theta_u \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} p \prec_{\mathcal{K}_x \times \mathcal{K}_u} 0.$$
4) There exists $w \succ_{\mathcal{K}_x} 0$ such that
$$M + \begin{bmatrix} A^T P(w) + P(w)A & P(w)B \\ B^T P(w) & 0 \end{bmatrix} < 0.$$

Proof: 1)$\Rightarrow$ 2): The inequality in 2) is directly derived from 1) by letting $\omega = 0$.

2) $\Rightarrow$ 3): Based on the definition of self-dual cone [16], one can obtain $Z^T\mathcal{K}_x \subseteq \mathcal{K}_u$ for any matrix Z satisfying $Z\mathcal{K}_u \subseteq \mathcal{K}_x$. Since A is cross-positive on $\mathcal{K}_x$ and Hurwitz, based on Lemma 1, it holds that $-A^{-1}B\mathcal{K}_u \subseteq \mathcal{K}_x$. Denote $\Gamma = \begin{bmatrix} (-A^{-1}B)^T & I_m \end{bmatrix} M \begin{bmatrix} (-A^{-1}B)^T & I_m \end{bmatrix}^T$. Therefore, $\forall \phi_1, \phi_2 \succeq_{\mathcal{K}_u} 0$ satisfying $\langle \phi_1, \phi_2 \rangle = 0$, we have $\phi_1^T \Gamma \phi_2 \geq 0$, which means that Γ is cross-positive on $\mathcal{K}_u$. Then based on Lemma 1, there must exist $\theta_u \succ_{\mathcal{K}_u} 0$ such that $q := \Gamma\theta_u \prec_{\mathcal{K}_u} 0$, and vectors $\xi \succ_{\mathcal{K}_x} 0$, $\zeta \succ_{\mathcal{K}_x} 0$ such that $A\xi \prec_{\mathcal{K}_x} 0$ and $A^T\zeta \prec_{\mathcal{K}_x} 0$. Define $\theta_x := -A^{-1}B\theta_u + k_1\xi$, and $p := \begin{bmatrix} -A^{-T} & 0 \end{bmatrix} M \begin{bmatrix} \theta_x^T & \theta_u^T \end{bmatrix}^T + k_2\zeta$, where k_1 and k_2 are positive real numbers. One can get that $\theta_x \succ_{\mathcal{K}_x} 0$, $p \succ_{\mathcal{K}_x} 0$, and $A\theta_x + B\theta_u = k_1 A\xi \prec_{\mathcal{K}_x} 0$. When k_1 and k_2 are sufficiently small, we have

$$M \begin{bmatrix} \theta_x \\ \theta_u \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} p = \begin{bmatrix} k_2 A^T\zeta \\ q + \eta \end{bmatrix} \prec_{\mathcal{K}_x \times \mathcal{K}_u} 0,$$

where $\eta = k_1(-A^{-1}B)^T M_{11}\xi + k_1 M_{12}^T \xi + k_2 B^T \zeta$.

3) $\Rightarrow$ 4): With the properties of $P(\cdot)$ in (2), (3), it holds that $w := P\left(\theta_x^{-\frac{1}{2}}\right)\left(P(\theta_x^{\frac{1}{2}})p\right)^{\frac{1}{2}} \succ_{\mathcal{K}_x} 0$, $P(w)\theta_x = p$ and $P(w)(A\theta_x + B\theta_u) \prec_{\mathcal{K}_x} 0$. Denote

$$\Xi = M + \begin{bmatrix} A^T P(w) + P(w)A & P(w)B \\ B^T P(w) & 0 \end{bmatrix}.$$

Denote $S = \text{diag}(P(w)^{-\frac{1}{2}}, I_m)$, according to the definition of symmetric cones [26] and properties of $P(\cdot)$ in (1) and (2), then for all $\varphi, \phi \succeq_{\mathcal{K}_x \times \mathcal{K}_u} 0$ satisfying $\langle \varphi, \phi \rangle = 0$, we have

$$\begin{aligned} \varphi^T S^T \Xi S \phi = &\varphi^T \begin{bmatrix} X_1 + X_1^T & P(w)^{\frac{1}{2}} B \\ B^T P(w)^{\frac{1}{2}} & 0 \end{bmatrix} \phi \\ &+ \varphi^T \begin{bmatrix} P(w)^{-\frac{1}{2}} M_{11} P(w)^{-\frac{1}{2}} & P(w)^{-\frac{1}{2}} M_{12} \\ M_{12}^T P(w)^{-\frac{1}{2}} & M_{22} \end{bmatrix} \phi \geq 0, \end{aligned}$$

where $X_1 = P(w)^{\frac{1}{2}} A P(w)^{-\frac{1}{2}}$, which means $S^T \Xi S$ is cross-positive on $\mathcal{K}_x \times \mathcal{K}_u$. Since

$$\Xi \begin{bmatrix} \theta_x \\ \theta_u \end{bmatrix} = M \begin{bmatrix} \theta_x \\ \theta_u \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} p + \begin{bmatrix} P(w)(A\theta_x + B\theta_u) \\ 0 \end{bmatrix} \prec_{\mathcal{K}_x \times \mathcal{K}_u} 0,$$

it follows that $S^T \Xi S s \prec_{\mathcal{K}_x \times \mathcal{K}_u} 0$, where $s = \begin{bmatrix}(P(w)^{\frac{1}{2}}\theta_x)^T & \theta_u^T\end{bmatrix}^T \succ_{\mathcal{K}_x \times \mathcal{K}_u} 0$. From Lemma 1, it holds that $S^T \Xi S < 0$, which implies $\Xi < 0$.

4)⇒ 1): One can derive it by recalling the proof of the KYP Lemma for general linear systems in [27]. ■

Remark 1: Note that the equivalence of conditions 3) and 2) is also valid when proper cones $\mathcal{K}_x$ and $\mathcal{K}_u$ are self-dual cones rather than symmetric cones. It could be derived as follows.

3) ⇒ 2): Since $A\theta_x + B\theta_u \prec_{\mathcal{K}_x} 0$ and $-A^{-1}$ is $\mathcal{K}_x$-nonnegative, we have $\epsilon := \theta_x + A^{-1}B\theta_u \succeq_{\mathcal{K}_x} 0$. Note that

$$\begin{aligned}&\begin{bmatrix}-A^{-1}B\\ I_m\end{bmatrix}^T \left(M\begin{bmatrix}\theta_x\\ \theta_u\end{bmatrix} + \begin{bmatrix}A^T\\ B^T\end{bmatrix}p\right)\\ &= (-A^{-1}B)^T(M_{11}\theta_x + M_{12}\theta_u) + M_{12}^T\theta_x + M_{22}\theta_u \prec_{\mathcal{K}_u} 0.\end{aligned}$$

With $\Gamma = \begin{bmatrix}(-A^{-1}B)^T & I_m\end{bmatrix} M \begin{bmatrix}(-A^{-1}B)^T & I_m\end{bmatrix}^T$, we have $\Gamma\theta_u \prec_{\mathcal{K}_u} 0$. Since matrix Γ is cross-positive on $\mathcal{K}_u$ (as shown in the proof of 2)⇒ 3)), we can obtain $\Gamma < 0$. ■

Remark 2: Note that when $\mathcal{K}_x$ and $\mathcal{K}_u$ are nonnegative orthants, the obtained KYP Lemma is applicable to positive systems. The quadratic representation in nonnegative orthant $P(w) = \text{diag}(w)^2$ is a positive diagonal matrix, which can be found in Theorem 1 of [18].

IV. Applications of KYP Lemma

In the light of the KYP Lemma derived in Section III, the state-feedback stabilization problems with various performance objectives are addressed in this section. The spectral radius and H^∞ performances are analyzed under the assumption of symmetric cone invariance. Systems with positivity and second-order cone invariance are two typical subclasses of symmetric cone-invariant systems, which play important roles in the control of cone-preserving systems. Since the former one has been extensively studied [6]–[9], in this section we will focus on the stabilization problem for second-order cone-preserving systems, which can be applied to twist systems and augmented transportation networks with additional catch-all buffers [25].

Consider the following linear system:

$$\mathcal{G}_K: \quad \begin{cases}\dot{x}(t) = A_K x(t) + Bu(t)\\ y(t) = C_K x(t) + Du(t)\end{cases}, \tag{5}$$

where $A_K = A + B_1K, C_K = C + D_1K$, and K is the controller gain to be designed. Based on Lemma 4 and Definition 1, system $\mathcal{G}_K$ is a second-order cone-preserving system with respect to (I_n, I_{m_u}, I_{m_y}) if and only if A_K is cross-positive on $\mathcal{I}_n$, $B\mathcal{I}_{m_u} \subseteq \mathcal{I}_n$, $C_K\mathcal{I}_n \subseteq \mathcal{I}_{m_y}$, and $D\mathcal{I}_{m_u} \subseteq \mathcal{I}_{m_y}$. For simplicity, assume that $m_u = m_y = m$, i.e., $\mathcal{I}_{m_u} = \mathcal{I}_{m_y} = \mathcal{I}_m$. Note that when $m_u \neq m_y$, we can patch zeros to make $m_u = m_y$. Assume that $B\mathcal{I}_m \subseteq \mathcal{I}_n$ and D is $\mathcal{I}_m$-nonnegative, and denote the transfer function of system $\mathcal{G}_K$ as G_K.

A. Stabilization with spectral radius performance

In this subsection, for system $\mathcal{G}_K$, the KYP Lemma is applied to designing a controller gain to ensure stability, spectral radius performance and second-order cone invariance. Based on Lemma 4, the constraints of cone invariance and stability can be guaranteed by finding a controller gain K such that A_K is Hurwitz and cross-positive on $\mathcal{I}_n$, and $C_K\mathcal{I}_n \subseteq \mathcal{I}_m$. However, as discussed in [22], it is hard to characterize the cross positivity by formulas. One viable way to solve this problem is to let A_K be strictly cross-positive on $\mathcal{I}_n$, and then resort to Lemma 2 to enforce it. Therefore, the stabilization control problem can be stated as follows.

Problem 1 (Spectral Radius Design): For system $\mathcal{G}_K$, design a controller gain K such that the second-order cone invariance, stability, and spectral radius performance can be guaranteed, that is,

P1(A): A_K is strictly cross-positive on $\mathcal{I}_n$, and $C_K\mathcal{I}_n \subseteq \mathcal{I}_m$;
P1(B): A_K is Hurwitz;
P1(C): $\rho(G_K(j\omega)) < \delta, \delta > 0, \forall\omega$.

Remark 3: Note that for a cone-preserving system $\mathcal{G}_K$, $\rho(G_K(j\omega))$ attains its maximum value at $\omega = 0$, which is called the DC-dominant property [11]. Hence, P1(C) holds if and only if $\rho(G_K(0)) = \rho(-C_K A_K^{-1} B + D) < \delta$ holds.

We first characterize the spectral radius performance by matrix inequalities using the following lemma.

Lemma 5: For a $\mathcal{K}_{\mathfrak{J}}$- nonnegative matrix Y, where $\mathcal{K}_{\mathfrak{J}}$ is a symmetric cone, $\rho(Y) < \delta$ holds if and only if there exists a vector $\theta \succ_{\mathcal{K}_{\mathfrak{J}}} 0$ such that $Y^T P(\theta) Y - \delta^2 P(\theta) < 0$.

Proof: Sufficiency. Since $P(\theta)^{-\frac{1}{2}} > 0$ holds, one can get that $Y^T P(\theta) Y - \delta^2 P(\theta) < 0$ if and only if $P(\theta)^{-\frac{1}{2}}\left(Y^T P(\theta) Y - \delta^2 P(\theta)\right) P(\theta)^{-\frac{1}{2}} = P(\theta)^{-\frac{1}{2}} Y^T P(\theta)^{\frac{1}{2}} P(\theta)^{\frac{1}{2}} Y P(\theta)^{-\frac{1}{2}} - \delta^2 I < 0$ holds. It means that $\delta > ||P(\theta)^{\frac{1}{2}} Y P(\theta)^{-\frac{1}{2}}||_2 \geq \rho(P(\theta)^{\frac{1}{2}} Y P(\theta)^{-\frac{1}{2}}) = \rho(Y)$.

Necessity. Assume $\rho(Y) < \delta$, then $Y - \delta I$ is Hurwitz. Based on Definition 3, $Y - \delta I$ is cross-positive on $\mathcal{K}_{\mathfrak{J}}$. From Lemma 1, there exist $\xi \succ_{\mathcal{K}_{\mathfrak{J}}} 0$, $\zeta \succ_{\mathcal{K}_{\mathfrak{J}}} 0$ such that $(Y - \delta I)\xi \prec_{\mathcal{K}_{\mathfrak{J}}} 0$ and $(Y^T - \delta I)\zeta \prec_{\mathcal{K}_{\mathfrak{J}}} 0$ hold, respectively. With $\theta := P\left(\xi^{-\frac{1}{2}}\right)\left(P(\xi^{\frac{1}{2}})\zeta\right)^{\frac{1}{2}}$, $\zeta = P(\theta)\xi$ holds based on formulas (3). It is obvious that $Y_P := P(\theta)^{-\frac{1}{2}}\left(Y^T P(\theta) Y - \delta^2 P(\theta)\right) P(\theta)^{-\frac{1}{2}}$ is cross-positive on $\mathcal{K}_{\mathfrak{J}}$. Further, with $\eta := P(\theta)^{\frac{1}{2}}\xi \succ_{\mathcal{K}_{\mathfrak{J}}} 0$, we have $Y_P\eta = P(\theta)^{-\frac{1}{2}} Y^T P(\theta)(Y - \delta I)\xi + \delta P(\theta)^{-\frac{1}{2}}(Y^T - \delta I)\zeta \prec_{\mathcal{K}_{\mathfrak{J}}} 0$, which means $Y_P < 0$. By congruence transformation, it holds that $Y^T P(\theta) Y - \delta^2 P(\theta) < 0$. ■

Theorem 2: Consider a cone-preserving system $\mathcal{G}_K$ with respect to $(\mathcal{K}_x, \mathcal{K}_m, \mathcal{K}_m)$, where $\mathcal{K}_x, \mathcal{K}_m$ are symmetric cones, and A_K is Hurwitz. Introduce $\theta \succ_{\mathcal{K}_m} 0$ and define M as

$$\begin{bmatrix} C_K^T P(\theta) C_K & C_K^T P(\theta) D P(\theta)^{-\frac{1}{2}} \\ P(\theta)^{-\frac{1}{2}} D^T P(\theta) C_K & P(\theta)^{-\frac{1}{2}} D^T P(\theta) D P(\theta)^{-\frac{1}{2}} - \delta^2 I_m \end{bmatrix}.$$

Then for a given positive real number δ, the following statements are equivalent:

1) $\rho(G_K(0)) < \delta$.
2) There exist $\theta_x \succ_{\mathcal{K}_x} 0$, $p \succ_{\mathcal{K}_x} 0$, $\theta_u \succ_{\mathcal{K}_m} 0$ with $A_K\theta_x + BP(\theta)^{-\frac{1}{2}}\theta_u \prec_{\mathcal{K}_x} 0$ such that

$$M\begin{bmatrix}\theta_x\\ \theta_u\end{bmatrix} + \begin{bmatrix}A_K^T\\ P(\theta)^{-\frac{1}{2}}B^T\end{bmatrix} p \prec_{\mathcal{K}_x \times \mathcal{K}_m} 0.$$

3) There exists $w \succ_{\mathcal{K}_x} 0$ such that

$$M+\begin{bmatrix} P(w)A_K + A_K^T P(w) & P(w)BP(\theta)^{-\frac{1}{2}} \\ P(\theta)^{-\frac{1}{2}}B^T P(w) & 0 \end{bmatrix} < 0. \quad (6)$$

4) There exist $v \succ_{\mathcal{K}_x} 0$, $\xi \succ_{\mathcal{K}_m} 0$, and $L \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} \mathcal{A}+\mathcal{A}^T & B\Theta & \mathcal{C}^T \\ \Theta B^T & -\delta^2\Theta & \Theta D^T \\ \mathcal{C} & D\Theta & -\Theta \end{bmatrix} < 0, \quad (7)$$

where $\mathcal{A} = AW + B_1 L$, $\mathcal{C} = CW + D_1 L$, $W = P(v)$, $\Theta = P(\xi)$, $K = LW^{-1}$.

5) $\rho(G_K(jw)) < \delta, \forall\omega$.

Proof: First, based on Lemma 5, condition 1) holds if and only if there exists $\theta \succ_{\mathcal{K}_m} 0$ satisfying $G_K(0)^T P(\theta) G_K(0) - \delta^2 P(\theta) < 0$, which is rewritten as

$$\mathcal{Y} := \begin{bmatrix}(-A_K^{-1}B)^T & I_m\end{bmatrix} \overline{M} \begin{bmatrix}(-A_K^{-1}B)^T & I_m\end{bmatrix}^T < 0,$$

where $\overline{M} = \mathrm{diag}(I_n, P(\theta)^{\frac{1}{2}}) M \mathrm{diag}(I_n, P(\theta)^{\frac{1}{2}})$. Note that $P(\theta)^{-\frac{1}{2}} > 0$, then $\mathcal{Y} < 0$ holds if and only if

$$P(\theta)^{-\frac{1}{2}} \mathcal{Y} P(\theta)^{-\frac{1}{2}}$$
$$= \begin{bmatrix} -A_K^{-1}BP(\theta)^{-\frac{1}{2}} \\ I_m \end{bmatrix}^T M \begin{bmatrix} -A_K^{-1}BP(\theta)^{-\frac{1}{2}} \\ I_m \end{bmatrix} < 0.$$

It is obvious that M satisfies all properties required in Theorem 1. Therefore, according to Theorem 1, we can conclude that statements 1), 2) and 3) are equivalent. Pre- and post-multiplying inequality (6) by $\mathrm{diag}(I_n, P(\theta)^{\frac{1}{2}})$, one can get

$$\overline{M} + \begin{bmatrix} P(w)A_K + A_K^T P(w) & P(w)B \\ B^T P(w) & 0 \end{bmatrix} < 0. \quad (8)$$

Based on Schur complement, inequality (8) can be rewritten as

$$\begin{bmatrix} P(w)A_K + A_K^T P(w) & P(w)B & C_K^T \\ B^T P(w) & -\delta^2 P(\theta) & D^T \\ C_K & D & -P(\theta)^{-1} \end{bmatrix} < 0. \quad (9)$$

With $v := w^{-1}$, $\xi := \theta^{-1}$, one has $W = P(w)^{-1}$ and $\Theta = P(\theta)^{-1}$. By pre- and post- multiplying (9) by $\mathrm{diag}(W, \Theta, I_m)$, we obtain (7), and this establishes equivalence between statements 3) and 4). Finally, based on Remark 3, the equivalence between 1) and 5) is derived, which completes the proof. ■

Remark 4: When $\delta = 1$, condition 1) in Theorem 2 can be used to determine the stability for a positive feedback system, referred to [11].

With the characterization of spectral radius performance in Theorem 2, Problem 1 can be solved as follows.

Proposition 1: A necessary and sufficient condition for the existence of controller gain K satisfying conditions P1(A)–P1(C) is that there exist vectors $v \succ_{\mathcal{I}_n} 0$, $\xi \succ_{\mathcal{I}_m} 0$, $\iota \succ_{\mathcal{I}_n} 0$, matrix $L \in \mathbb{R}^{m\times n}$ and scalars $\eta_1 \geq 0, \eta_2 \geq 0, \alpha \geq 0$ such that inequality (7) and

$$(\mathcal{A}+\alpha W)Q_n(\mathcal{A}+\alpha W)^T - \eta_1 Q_n > 0, \quad (10a)$$
$$e_{1n}s_1^T + s_1 e_{1n}^T - e_{1n}^T s_1 Q_n > 0, \quad (10b)$$
$$\mathcal{C}^T Q_m \mathcal{C} - \eta_2 Q_n \geq 0, \quad (10c)$$
$$e_{1n}s_2^T + s_2 e_{1n}^T - e_{1n}^T s_2 Q_n \geq 0, \quad (10d)$$
$$e_{1n}s_3^T + s_3 e_{1n}^T - e_{1n}^T s_3 Q_n < 0, \quad (10e)$$

hold, where $\mathcal{A} = AW + B_1 L$, $\mathcal{C} = CW + D_1 L$, $W = P(v)$, $\Theta = P(\xi)$, $s_1 = (\mathcal{A}+\alpha W)e_{1n}$, $s_2 = \mathcal{C}^T e_{1m}$, $s_3 = \mathcal{A}\iota$, $K = LW^{-1}$.

Proof: Since $\mathcal{I}_n$ is self-dual, A_K is strictly cross-positive on $\mathcal{I}_n$ if and only if A_K^T is strictly cross-positive on $\mathcal{I}_n$. Based on Lemma 2 and formulas in (2), A_K^T is strictly cross-positive on $\mathcal{I}_n$ if and only if there exists $\alpha \geq 0$ such that $W(A_K + \alpha I_n)^T$ is $\mathcal{I}_n$-positive, which is equivalent to the condition that there exist scalars $\alpha \geq 0$ and $\eta_1 \geq 0$ such that (10a) and $(\mathcal{A}+\alpha W)e_{1n} \in \mathrm{Int}(\mathcal{I}_n)$ hold, where the later is equivalent to (10b) by its arrow-shaped representation in [22]. Similarly, based on Lemma 3, the second condition in P1(A) is equivalent to (10c) and (10d). From Lemma 1, P1(B) holds if and only if there exists $s \succ_{\mathcal{I}_n} 0$ satisfying $A_K s \prec_{\mathcal{I}_n} 0$ under the condition of P1(A). With $W^{-1}s = \iota \succ_{\mathcal{I}_n} 0$, we have $\mathcal{A}\iota \prec_{\mathcal{I}_n} 0$, which is equivalent to (10e). Finally, P1(C) is equivalent to (7) based on Theorem 2, which completes the proof. ■

Note that inequalities (10a)–(10e) in Proposition 1 are nonlinear with respect to W, L, ι and α, which can be improved by an alternative corollary inspired by [22].

Corollary 1: If there exist vectors $v \succ_{\mathcal{I}_n} 0, \xi \succ_{\mathcal{I}_m} 0$, matrix $L \in \mathbb{R}^{m\times n}$ and scalars $\eta_1 \geq 0, \eta_2 \geq 0, \alpha > \alpha_1 > 0$ such that inequalities (7), (10d) and

$$\begin{bmatrix} \Psi_1 - \eta_1 Q_n & \mathcal{A}+\alpha W \\ (\mathcal{A}+\alpha W)^T & I_n \end{bmatrix} > 0, \quad (11a)$$
$$(\mathcal{A}+\alpha_1 W)e_{1n} = 0, \quad (11b)$$
$$\begin{bmatrix} \Psi_2 - \eta_2 Q_n & \mathcal{C}^T \\ \mathcal{C} & I_m \end{bmatrix} \geq 0, \quad (11c)$$

hold, where $\mathcal{A} = AW + B_1 L$, $\mathcal{C} = CW + D_1 L$, $\Psi_1 = 2(\alpha-\alpha_1)^2 W e_{1n} e_{1n}^T W$, $\Psi_2 = 2\mathcal{C}^T e_{1m} e_{1m}^T \mathcal{C}$, $W = P(v)$, $\Theta = P(\xi)$, then there exists a controller gain $K = LW^{-1}$ such that system $\mathcal{G}_K$ is cone-preserving and stable.

Proof: If (11b) holds, we have $\mathcal{A}e_{1n} = -\alpha_1 W e_{1n}$, $s_1 = (\alpha-\alpha_1)We_{1n}$ and $s_3 = -\alpha_1 W e_{1n} \prec_{\mathcal{I}_n} 0$ in Proposition 1 such that (10b) and (10e) hold. Then by Schur complement, (11a) is equivalent to (10a), and (11c) is equivalent to (10c). The rest of the proof is similar to that in Proposition 1. ■

Recall that, in second-order cones, $P(v)$ and $P(\xi)$ are given as $W = P(v) = vv^T - \frac{1}{2}(v^T Q_n v)Q_n$, $\Theta = P(\xi) = \xi\xi^T - \frac{1}{2}(\xi^T Q_m \xi)Q_m$, respectively [28]. Define $R := vv^T$ and $T := \xi\xi^T$. Without loss of generality, assume that $v^T e_{1n} = \xi^T e_{1m} = 1$. Then it derives that $W = R - Q_n + \frac{1}{2}\mathrm{Trace}(R)Q_n$, $\Theta = T - Q_m + \frac{1}{2}\mathrm{Trace}(T)Q_m$.

Remark 5: Note that it is difficult to solve the inequalities in Corollary 1 since R, T, Ψ_1, Ψ_2 are quadratic with respect to unknown vectors, and $(\alpha-\alpha_1)^2$ is nonlinear with respect to α and α_1. To address these nonlinearities, a practical way to find a feasible state-feedback gain through LMI can be presented in Algorithm 1, where the dominant eigenvalue of A_K is first set as $-\alpha_1$, and α is then chosen based on $\alpha > \alpha_1$. For more details about Algorithm 1, one can refer to [22].

Algorithm 1 Stabilization with Cone Invariance and Spectral Radius Performance

Step 1: Initialize the prescribed tolerance: ϵ, the maximum iteration: k_{max}, and the upper bounds of α_1 and α: $\bar{\alpha}_1$, $\bar{\alpha}$ ($\bar{\alpha} > \bar{\alpha}_1$), respectively. Set $k = 1$.
Step 2: Randomly generate $\alpha_1^{(k)}$ between 0 and $\bar{\alpha}_1$, and $\alpha^{(k)}$ between $\alpha_1^{(k)}$ and $\bar{\alpha}$ without repetition.
Step 3: Minimize $f = \mathrm{Trace}(R)+\mathrm{Trace}(T)+\mathrm{Trace}(\Psi_1)+\mathrm{Trace}(\Psi_2)$ with respect to $\eta_1 \geq 0$, $\eta_2 \geq 0$, $v \succ_{\mathcal{I}_n} 0$ with $v^T e_{1n} = 1$, $\xi \succ_{\mathcal{I}_m} 0$ with $\xi^T e_{1m} = 1$, $R \geq 0$, $T \geq 0$, $\Psi_1 \geq 0$, $\Psi_2 \geq 0$, inequalities (7), (10d), (11c), and

$$\begin{bmatrix} \Psi_1 - \eta_1 Q_n & \mathcal{A} + \alpha^{(k)} W \\ (\mathcal{A} + \alpha^{(k)} W)^T & I_n \end{bmatrix} > 0, \quad (12a)$$

$$\begin{bmatrix} \Psi_1 & \sqrt{2}(\alpha^{(k)} - \alpha_1^{(k)}) W e_{1n} \\ \sqrt{2}(\alpha^{(k)} - \alpha_1^{(k)}) e_{1n}^T W & 1 \end{bmatrix} \geq 0, \quad (12b)$$

$$\begin{bmatrix} \Psi_2 & \sqrt{2}\mathcal{C}^T e_{1m} \\ \sqrt{2} e_{1m}^T \mathcal{C} & 1 \end{bmatrix} \geq 0, \quad (12c)$$

$$(\mathcal{A} + \alpha_1^{(k)} W) e_{1n} = 0, \quad (12d)$$

$$\begin{bmatrix} R & v \\ v^T & 1 \end{bmatrix} \geq 0, \quad (12e)$$

$$\begin{bmatrix} T & \xi \\ \xi^T & 1 \end{bmatrix} \geq 0, \quad (12f)$$

where $\mathcal{A} = AW + B_1 L$, $\mathcal{C} = CW + D_1 L$, $W = R - Q_n + \frac{1}{2}\mathrm{Trace}(R) Q_n$, $\Theta = T - Q_m + \frac{1}{2}\mathrm{Trace}(T) Q_m$.
Step 4: Check the conditions:

$$\mathrm{Trace}(\Psi_1 - 2(\alpha^{(k)} - \alpha_1^{(k)})^2 W e_{1n} e_{1n}^T W) < \epsilon, \quad (13a)$$

$$\mathrm{Trace}(\Psi_2 - 2\mathcal{C}^T e_{1m} e_{1m}^T \mathcal{C}) < \epsilon, \quad (13b)$$

$$\mathrm{Trace}(R - vv^T) < \epsilon, \quad (13c)$$

$$\mathrm{Trace}(T - \xi\xi^T) < \epsilon. \quad (13d)$$

If they all hold, go to Step 5; otherwise, go to Step 6.
Step 5: Compute the controller gain $K = LW^{-1}$, and stop.
Step 6: Set $k = k + 1$. If $k \geq k_{max}$, input new bounds $\bar{\alpha}_1$ and $\bar{\alpha}$, and go to Step 1; else, go to Step 2.

B. H^∞ controller design

In this subsection, the property that the H^∞ norm of cone-preserving system with respect to symmetric cones is determined by the 2-norm of a static gain matrix is provided, which is used to solve the H^∞ controller design problem described below for the case of second-order cones.

Problem 2 (H^∞ Design): For system $\mathcal{G}_K$, design a controller gain K such that the second-order cone invariance, stability, and H^∞ performance can be guaranteed, that is,

P2(A): A_K is strictly cross-positive on $\mathcal{I}_n$, and $C_K \mathcal{I}_n \subseteq \mathcal{I}_m$;
P2(B): A_K is Hurwitz;
P2(C): $\|G_K\|_\infty < \gamma$.

The H^∞ norm for cone-preserving systems is first characterized by the following theorem.

Theorem 3: Suppose system $\mathcal{G}_K$ is a cone-preserving system with respect to $(\mathcal{K}_x, \mathcal{K}_m, \mathcal{K}_m)$, where $\mathcal{K}_x$, $\mathcal{K}_m$ are symmetric cones, and A_K is Hurwitz. Define M as

$$\frac{1}{\gamma}\begin{bmatrix} C_K^T C_K & C_K^T D \\ D^T C_K & D^T D - \gamma^2 I_m \end{bmatrix}.$$

Then for a given positive real number γ, the following statements are equivalent:

1) $\|G_K(0)\|_2 < \gamma$.
2) There exist $\theta_x \succ_{\mathcal{K}_x} 0$, $p \succ_{\mathcal{K}_x} 0$, $\theta_u \succ_{\mathcal{K}_m} 0$ with $A_K \theta_x + B\theta_u \prec_{\mathcal{K}_x} 0$ such that
$$M \begin{bmatrix} \theta_x \\ \theta_u \end{bmatrix} + \begin{bmatrix} A_K^T \\ B^T \end{bmatrix} p \prec_{\mathcal{K}_x \times \mathcal{K}_m} 0.$$
3) There exists $w \succ_{\mathcal{K}_x} 0$ such that
$$M + \begin{bmatrix} P(w)A_K + A_K^T P(w) & P(w)B \\ B^T P(w) & 0 \end{bmatrix} < 0.$$
4) There exist $v \succ_{\mathcal{K}_x} 0$ and $L \in \mathbb{R}^{m\times n}$ such that
$$\begin{bmatrix} \mathcal{A} + \mathcal{A}^T & B & \mathcal{C}^T \\ B^T & -\gamma I_m & D^T \\ \mathcal{C} & D & -\gamma I_m \end{bmatrix} < 0, \quad (14)$$
where $\mathcal{A} = AW + B_1 L$, $\mathcal{C} = CW + D_1 L$, $W = P(v)$, $K = LW^{-1}$.
5) $\|G_K\|_\infty < \gamma$.

Proof: Assume that condition 1) holds. Since $\gamma > 0$, it holds that $\frac{1}{\gamma}(G_K(0)^T G_K(0) - \gamma^2 I_m) < 0$, that is, $\left[(-A_K^{-1}B)^T \quad I_m\right] M \left[(-A_K^{-1}B)^T \quad I_m\right]^T < 0$. Since M satisfies all properties required in Theorem 1, based on Theorem 1, we can obtain that 2) and 3) hold and the equivalence of 2) and 3). Assume that the inequality in 3) holds, then based on Schur complement, we can obtain

$$\begin{bmatrix} P(w)A_K + A_K^T P(w) & P(w)B & C_K^T \\ B^T P(w) & -\gamma I_m & D^T \\ C_K & D & -\gamma I_m \end{bmatrix} < 0, \quad (15)$$

which indicates that 5) holds. By introducing $v = w^{-1}$ and pre- and post- multiplying (15) by $\mathrm{diag}(W, I_m, I_m)$, it holds that 4) is equivalent to (15). Finally, 5) directly implies 1) by the definition of H^∞ norm, which completes the proof. ■

Remark 6: A similar result of H^∞ norm (an equivalence between 1)and 5)) has been given in [17] by cone programming, while in this paper, the result is directly derived from the KYP Lemma under the case of special M.

Similar to Proposition 1 and Corollary 1, one can obtain the following results to design a controller gain K to sufficiently solve Problem 2 and the proofs are omitted here for simplicity.

Proposition 2: A necessary and sufficient condition for the existence of controller gain K satisfying conditions P2(A)–P2(C) is that there exist vectors $v \succ_{\mathcal{I}_n} 0$, $\iota \succ_{\mathcal{I}_n} 0$, matrix $L \in \mathbb{R}^{m\times n}$, and scalars $\eta_1 \geq 0, \eta_2 \geq 0, \alpha \geq 0$ such that inequalities (10a)–(10e) and (14) hold.

Corollary 2: If there exist vector $v \succ_{\mathcal{I}_n} 0$, matrix $L \in \mathbb{R}^{m\times n}$ and scalars $\eta_1 \geq 0, \eta_2 \geq 0, \alpha > \alpha_1 > 0$ such that inequalities ((11a)–(11c), (10d) and (14) hold, then there exists a controller gain $K = LW^{-1}$ such that conditions P2(A)–P2(C) are satisfied.

Remark 7: The algorithm to obtain the controller gain via the LMI is similar to Algorithm 1 except Steps 3 and 4 which are replaced by the following statements, respectively.

Step 3: Minimize $f = \mathrm{Trace}(R + \Psi_1 + \Psi_2)$ with respect to $\eta_1 \geq 0$, $\eta_2 \geq 0$, $v \succ_{\mathcal{I}_n} 0$ with $v^T e_{1n} = 1$, $R \geq 0$, $\Psi_1 \geq 0$, $\Psi_2 \geq 0$, inequalities (14), (10d), (11c), and (12a)–(12e).

Step 4: Check the conditions (13a)–(13c). If they all hold, go to Step 5. Otherwise, go to Step 6.

V. Numerical Examples

Consider system $\mathcal{G}_K$ with parameters given as follows:

$$A = \begin{bmatrix} 0.0550 & -0.4349 & -0.4408 \\ 0.2769 & 0.3513 & -0.3964 \\ 0.7218 & -0.6907 & -2.6823 \end{bmatrix}, B_1 = \begin{bmatrix} 0.8147 & 0.9134 \\ 0.9058 & 0.6324 \\ 0.1270 & 0.0975 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.8547 & 0.4134 \\ 0.3058 & 0.6324 \\ 0.1270 & 0.0975 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3350 & 0.0352 & 0.0170 \\ -0.0077 & 0.0477 & -0.0425 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.0110 & -0.0052 \\ 0.0227 & 0.0110 \end{bmatrix}, D = \begin{bmatrix} 0.3100 & -0.0517 \\ 0.1771 & 0.0603 \end{bmatrix}.$$

We consider input constrained in second-order cone, and the goal is to obtain state-feedback controller which will preserve system state and output in second-order cones. Note that $B\mathcal{I}_2 \subseteq \mathcal{I}_3$ and $D\mathcal{I}_2 \subseteq \mathcal{I}_2$. For simplicity, system $\mathcal{G}_K$ with $K = 0$ is denoted as $\mathcal{G}$, and correspondingly, denote its transfer function as G. Based on Remark 5, the upper bounds of α_1 and α are selected as $\bar{\alpha}_1 = 2$, and $\bar{\alpha} = 12$, respectively.

A. Example 1

Based on Algorithm 1, for given $\delta = 1$, one can get that

$$K = \begin{bmatrix} 6.2178 & -12.9533 & -6.1256 \\ -6.2856 & 10.6132 & 4.4395 \end{bmatrix},$$

and $\rho(G_K(0)) = 0.5304 < 1$. It can be shown that A_K is Hurwitz and strictly cross-positive on $\mathcal{I}_3$, and $C_K\mathcal{I}_3 \subseteq \mathcal{I}_2$ based on Lemmas 2 and 3, respectively. Hence system $\mathcal{G}_K$ is stable and cone -preserving, which can be also illustrated by Fig. 1 and Fig. 2. It is obvious that, with the control gain K, both the state trajectory $x(t)$ and output trajectory $y(t)$ are restricted in second-order cones, but this property can not be guaranteed in the case of $K = 0$ by observing the trajectories of $x(t)$, $y(t)$ eventually run out of second-order cones.

B. Example 2

For given $\gamma = 0.8$, one can get

$$K = \begin{bmatrix} 9.2228 & -17.5606 & -5.8048 \\ -8.6839 & 13.5277 & 3.9148 \end{bmatrix}.$$

From Lemmas 2 and 3, one can verify that A_K is strictly cross-positive on $\mathcal{I}_3$ and $C_K\mathcal{I}_3 \subseteq \mathcal{I}_2$. Based on Lemma 4, with such H^∞ controller gain K, the cone invariance for system $\mathcal{G}_K$ is guaranteed, which is also illustrated by Fig. 3 and Fig. 4. In addition, Fig. 3 and Fig. 4 also show the stability of $\mathcal{G}_K$. From Theorem 3, it holds that $\|G_K\|_\infty = \|G_K(0)\|_2 = 0.5188 < \gamma$. Fig. 5 describes the 2-norm of transfer functions for systems $\mathcal{G}$ and $\mathcal{G}_K$, respectively. It illustrates that the obtained controller K can realize the given H^∞ performance.

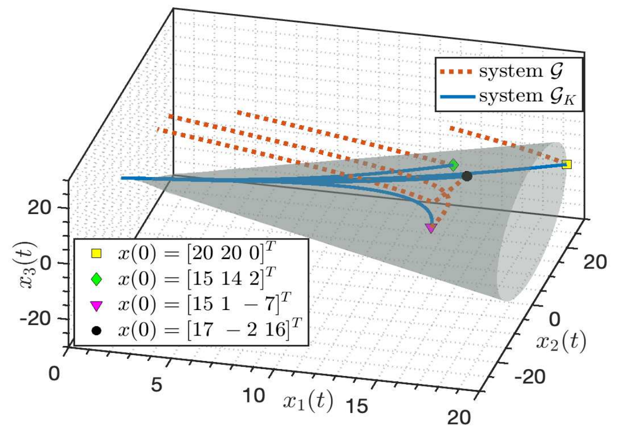

Fig. 1: State trajectories $x(t)$ under the various initial conditions for systems $\mathcal{G}$ and $\mathcal{G}_K$ for a three-dimensional second-order cone.

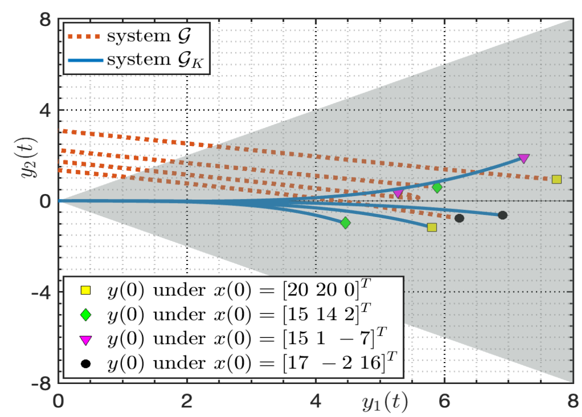

Fig. 2: Output trajectories $y(t)$ under the various initial conditions for systems $\mathcal{G}$ and $\mathcal{G}_K$ for a two-dimensional second-order cone.

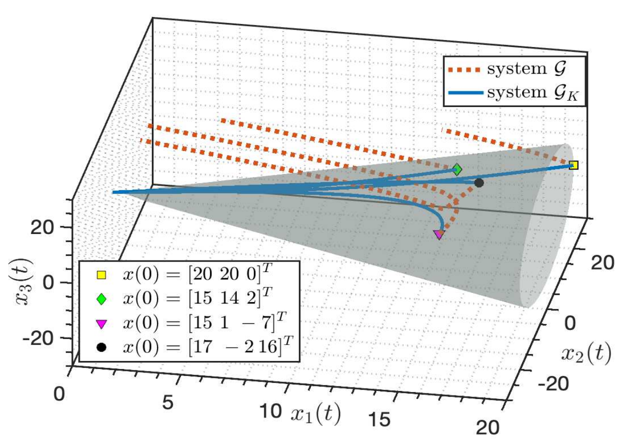

Fig. 3: State trajectories $x(t)$ under the various initial conditions for systems $\mathcal{G}$ and $\mathcal{G}_K$ for a three-dimensional second-order cone.

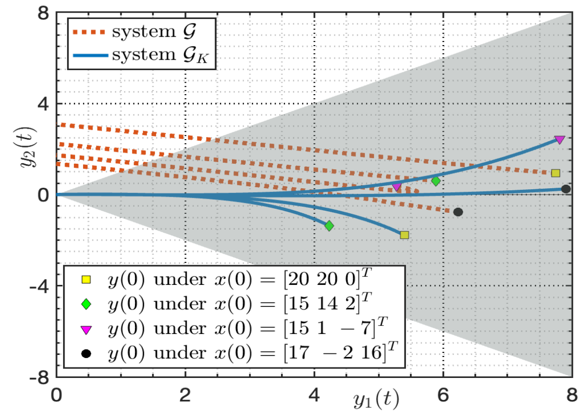

Fig. 4: Output trajectories $y(t)$ under the various initial conditions for systems $\mathcal{G}$ and $\mathcal{G}_K$ for a two-dimensional second-order cone.

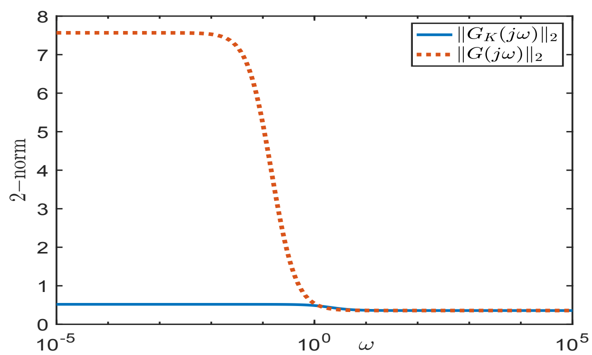

Fig. 5: 2-norm of transfer functions G and G_K.

VI. Conclusions

In this paper, we have investigated the KYP Lemma for cone-preserving systems, where the inequalities in terms of cone programming and LMI have been presented. The derived KYP Lemma was used to design a state-feedback controller, introducing the conditions for spectral radius and H^∞ performance of the closed-loop system. Additionally, the obtained controller synthesis methods guarantee second-order cone invariance and stability, and have the form of an iterative LMI-based algorithm. It is worth noting that the results obtained in this paper, which focus on symmetric cones, can be readily adapted to handle general ellipsoidal cones. This is because every ellipsoidal cone can be transformed into a second-order cone via nonsingular linear transformation. More applications and less restrictive assumptions on KYP Lemma are foreseen. This involves developing versions of the KYP Lemma with non-strict inequalities and incorporating cone invariance for a wider range of proper cones beyond symmetric cones.

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