Letter

Equivalence between face nonsignaling correlations, full nonlocality, all-versus-nothing proofs, and pseudotelepathy

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(Received 20 October 2023; accepted 16 October 2024; published 4 November 2024)

We show that a quantum correlation p is in a face of the nonsignaling polytope with no local points if and only if p has nonlocal content 1, if and only if p allows for a Greenberger-Horne-Zeilinger-like proof, and if and only if p provides a perfect strategy for a nonlocal game. That is, face nonsignaling (FNS) correlations, full nonlocality (FN), all-versus-nothing (AVN) proofs, and pseudotelepathy (PT) are equivalent. This shows that different resources behind a wide variety of fundamental results are in fact the same resource. We demonstrate that quantum correlations with FNS = FN = AVN = PT do not need to maximally violate a tight Bell inequality. We introduce a method for identifying quantum FNS = FN = AVN = PT correlations and use it to prove quantum mechanics does not allow for FNS = FN = AVN = PT neither in the (3, 3; 3, 2) nor in the (3, 2; 3, 4) Bell scenarios. This solves an open problem that, because of the FNS = FN = AVN = PT equivalence, has implications in several fields.

DOI: 10.1103/PhysRevResearch.6.L042035

Introduction. Bell nonlocality, i.e., the violation of Bell inequalities [1], is one of the most fundamental predictions of quantum mechanics (QM). Quantum nonlocal correlations have a wide scope of applications, ranging from secure communication [2] and randomness amplification [3] to reduction of communication complexity [4] and self-testing of quantum devices [5]. However, not all Bell nonlocal correlations are equally powerful. Specific tasks require specific types of correlations. Hereafter, we will focus on four types that have attracted interest for different reasons.

Face nonsignaling correlations. Consider a bipartite Bell scenario $(|X|, |A|; |Y|, |B|)$, where $x \in X$ and $y \in Y$ are Alice's and Bob's measurement settings, respectively, and $a \in A$ and $b \in B$ are Alice's and Bob's measurement outcomes. A Bell nonlocal correlation $p(a, b|x, y)$ in this scenario is a point outside the set of local correlations (the local polytope) and inside the set of correlations satisfying nonsignaling (NS) (the NS polytope) [6]. Using the results in [7,8], it can be proven that neither QM [9] nor any theory that assigns probabilities to sharp observables can attain a nonlocal *vertex* of a NS polytope. Still, quantum Bell nonlocal correlations can be in a face of the NS polytope. There are two possibilities [10]: either the Bell nonlocal correlation is in a face that contains local points [10], see Fig. 1(a), or it is *in a face that does not contain local points*, see Fig. 1(b). The Bell nonlocal correlations of the second type are called face nonsignaling (FNS). Why are FNS correlations so important? Some reasons are the following:

(i) Most quantum Bell nonlocal correlations can be classically simulated either by relaxing the assumption of measurement independence [11] (and admitting that some measurement settings may depend on hidden variables) or by relaxing the assumption of parameter independence [11] (and admitting that some outcomes may depend on spacelike separated settings). However, there are quantum Bell nonlocal correlations that *cannot* be classically simulated unless both assumptions are totally removed [12]. It can be proven that these Bell nonlocal correlations *must* be FNS or arbitrarily close to it [12].

(ii) FNS correlations are fundamental for identifying the principle that bounds quantum correlations [13], since a way to obtain the quantum bounds is by noticing that the studied scenario may be naturally linked to a larger scenario in which QM allows for FNS correlations that imply bounds on the smaller scenario [14,15].

(iii) FNS correlations opened up the possibility of perfect randomness from seeds with arbitrarily weak randomness [16].

Full nonlocality. One of the most widely used measures of Bell nonlocality is the nonlocal content [22]. Given a NS correlation $p(a, b|x, y)$, consider all possible decompositions

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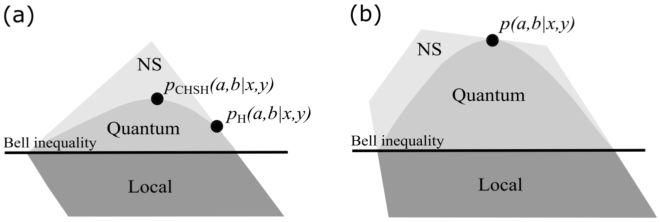

FIG. 1. (a) In the (2, 2; 2, 2) Bell scenario, $p_{\rm CHSH}(a, b|x, y)$ is the quantum correlations that maximally violates the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [17]. $p_{\rm CHSH}(a, b|x, y)$ is far from the faces of the NS polytope. $p_{\rm H}(a, b|x, y)$ is the correlation that corresponds to the optimal proof of Bell nonlocality of Hardy [18]. $p_{\rm H}(a, b|x, y)$ in a face of the NS polytope that contains a local point [10]. (b) In the (3, 4; 3, 4) Bell scenario, there is a correlation $p(a, b|x, y)$ [19,20] that is in a face of the NS polytope that has no local points [21].

of the form

$$p(a, b|x, y) = q_{\rm L} p_{\rm L}(a, b|x, y) + (1 - q_{\rm L}) p_{\rm NL}(a, b|x, y), \quad (1)$$

in terms of local correlations $p_{\rm L}(a, b|x, y)$ and Bell nonlocal NS correlations $p_{\rm NL}(a, b|x, y)$, with respective weights $q_{\rm L}$ and $1 - q_{\rm L}$, with $0 \leqslant q_{\rm L} \leqslant 1$. The local content or local fraction $q_{\rm L}^{\rm max}$ of $p(a, b|x, y)$ is the maximum local weight over all decompositions of the form (1). That is, $q_{\rm L}^{\rm max} \doteq \max_{\{p_{\rm L}, p_{\rm NL}\}} q_{\rm L}$. The nonlocal content is $q_{\rm NL}^{\rm min} \doteq 1 - q_{\rm L}^{\rm max}$. The correlation is local if and only if $q_{\rm NL}^{\rm min} = 0$. The correlation is said to have *full nonlocality* (FN) [21] or *strong nonlocality* [23] if $q_{\rm NL}^{\rm min} = 1$. For example, the maximum quantum violation of the CHSH Bell inequality [17] has $q_{\rm NL}^{\rm min} = \sqrt{2} - 1 \approx 0.414$.

FN is *necessary* for some quantum information tasks. For example: (I) Improving the number of classical messages that can be sent without error through a single use of a classical channel [24], (II) device-independent quantum key distribution (DI-QKD) based on perfect correlations [25], (III) DI-QKD based on the magic square game [26], and (IV) some types of DI-QKD based on parallel repetition [27,28].

All-versus-nothing proofs. The Einstein-Podolsky-Rosen [29] argument suggesting the possibility of completing QM with local hidden variables (LHVs) was based on *perfect* correlations that allows the parties to predict *with certainty* the outcome of the measurement of a distant party. For this reason, the proofs of Bell's theorem of impossibility of LHVs that only use perfect correlations have a special status in foundations of QM. Examples of such proofs are: (i') the proofs of Stairs [30] and Heywood and Redhead [31] based on the Kochen-Specker theorem [32], (ii') the proof of Greenberger, Horne, and Zeilinger (GHZ) with four parties [33], (iii') Mermin's simplification to three parties [34–37], (iv') the "all-versus-nothing" (AVN) proof with two parties [19–21,38–40], and (v') the proofs of Bell nonlocality based on stabilizers of graph states [41]. The name AVN was coined by Mermin [42] to designate those proofs in which the conflict between QM and LHVs is evident by looking only at predictions with certainty.

In the bipartite case, AVN proofs can be characterized (see Appendix A within the Supplemental Material, SM [43]) as follows. A table of zeros for the $(|X|, |A|; |Y|, |B|)$ Bell scenario is a matrix with $|X| \times |A|$ rows and $|Y| \times |B|$ columns containing either zeros or empty entries. A zero in the entry $(a, b|x, y)$ indicates that the probability of $(a, b|x, y)$ is zero. An AVN proof consists of a quantum correlation that produces a table of zeros which cannot be realized by any LHV model. Given $S = S_A \cup S_B$, with $S_A = \{(a|x)\}_{x\in X, a\in A}$ and $S_B = \{(b|y)\}_{y\in Y, b\in B}$, a table of zeros is not realizable by any LHV if, for every assignment $f : S \to \{0, 1\}$ satisfying $\sum_a f(a|x) = 1, \forall x \in X$, and $\sum_b f(b|y) = 1, \forall y \in Y$, there is a pair $\{(a|x), (b|y)\}$ for which $f(a|x) = f(b|y) = 1$ and $p(a, b|x, y) = 0$.

Pseudotelepathy. There is a form of Bell nonlocality that plays a fundamental role in quantum computation and quantum information. It is related to a specific type of nonlocal games. A bipartite nonlocal game [44–47] is a 4-tuple $G = (X \times Y, A \times B, \pi, W)$, where X (Y) is the input set of the first player, Alice (the second player, Bob), A (B) is the corresponding set of outputs, $\pi(X \times Y)$ is the distribution of inputs, and $W(X \times Y \times A \times B) \in \{0, 1\}$ is the winning condition, i.e., the condition that inputs and outputs should satisfy to win the game. Consequently, the winning probability of the game is given by

$$\omega(G) = \sum_{x,y,a,b} \pi(x, y) p(a, b|x, y) W(a, b, x, y). \quad (2)$$

The game G admits a *perfect strategy* or *pseudotelepathy* (PT) [46,47] if there is a correlation $p(a, b|x, y)$ that allows Alice and Bob to win every round of G. That is, if $W(a, b, x, y) = 1$ for all $(a, b|x, y)$ such that $p(a, b|x, y) \neq 0$. Therefore, a quantum strategy offers PT whenever the quantum winning probability is $\omega_{\rm Q}(G) = 1$, while using any classical strategy (that does not involve communication between Alice and Bob) the winning probability is $\omega_{\rm C}(G) < 1$.

PT is crucial in: (I') the proof of the quantum computational advantage for shallow circuits [48], (II') the proof of $\text{MIP}^* = \text{RE}$ [49], (III') device-independent randomness generation in a network with untrusted users, since PT allows to certify more local randomness (randomness known to one party but not to the other) and in a more robust way than standard Bell nonlocality [50], (IV') multiple access channels: if two senders that aim to transmit individual messages to a single receiver have PT, then they can transmit information at the maximal possible rate [51].

Equivalence. Our first result is the following.

Theorem 1. The following statements are equivalent:

(i) A quantum correlation p is face nonsignaling.

(ii) p has full nonlocality.

(iii) p allows for an all-versus-nothing proof.

(iv) p allows for a perfect (or pseudotelepathy) strategy.

Proof. The equivalence between (i) and (ii) follows from the fact that the nonlocal content $q_{\rm NL}^{\rm min}$ measures Bell nonlocality relative to the local and NS polytopes so that $q_{\rm NL}^{\rm min}$ takes the value 1 if and only if p is in a face that has no local points [since, otherwise, $q_{\rm L}^{\rm max}(p) \neq 0$; for example, for $p_{\rm H}$ in Fig. 1(a), $q_{\rm NL}^{\rm min} = 5\sqrt{5} - 11 \approx 0.18$] (see Appendix B within the SM [43]).

The equivalence between (iii) and (iv) follows from the observation that a quantum correlation p yields quantum winning probability 1 for the game G in which the winning condition is achieving (in each context asked by the referee) all the zeros in the table of zeros of p, while the classical winning probability is strictly smaller than 1, if and only if the table of zeros of p cannot be realized by any LHV variable model (see Appendix C within the SM [43]).

The equivalence between (ii) [and (i)] and (iv) [and (iii)] can be proven as follows. To prove that (iv) implies (ii), let us observe that, by (iv), there is a game G for which there is a quantum strategy (correlation) p that provides a winning probability $\omega^{(p)}(G) = 1 = \omega_{\rm NS}(G)$, while $\omega_{\rm C}(G) < 1$, where $\omega_{\rm NS}(G)$ is the winning probability allowed by NS correlations. Let us now consider any convex decomposition of p of the form (1). Then, by the linearity of the winning probability in Eq. (2),

$$\omega^{(p)}(G) = q_{\rm L}\omega^{(p_{\rm L})}(G) + (1 - q_{\rm L})\omega^{(p_{\rm NL})}(G), \qquad (3)$$

where $\omega^{(p_{\rm L})}(G)$ and $\omega^{(p_{\rm NL})}(G)$ are the winning probabilities using the local correlation $p_{\rm L}$ and the NS correlation $p_{\rm NL}$, respectively. Since $\omega^{(p)}(G) = 1$ and $0 \leqslant \omega(G) \leqslant 1$, then $\omega^{(p_{\rm L})}(G) = 1$ whenever $0 < q_{\rm L}$. This contradicts the assumption that $\omega_{\rm C}(G) < 1$. Therefore, $q_{\rm L} = 0$ in any convex decomposition of p of the form (1). That is, $q_{\rm L}^{\max} = \max_{\{p_{\rm L},p_{\rm NL}\}} q_{\rm L} = 0$.

To prove that (ii) implies (iv), let us observe that, by (ii), $q_{\rm L}^{\max}(p) = 0$. As shown in [52], the local content can be computed by the following linear program:

$$\begin{aligned} q_{\rm L}^{\max}(p) = \max \quad & \sum_i q_i \\ \text{s.t.} \quad & \sum_i q_i P_i^{\rm L} \leqslant p \\ & q_i \geqslant 0\, \forall i. \end{aligned} \qquad (4)$$

Here, $P_i^{\rm L}$ correspond to vertices of the local polytope and $\sum_i q_i P_i^{\rm L} \leqslant p$ must be interpreted term by term. The dual of this linear program can be written as follows:

$$\begin{aligned} \min_I \quad & \operatorname{tr}(I^T p) \\ \text{s.t.} \quad & \operatorname{tr}\big(I^T P_i^{\rm L}\big) \geqslant 1\, \forall i \\ & I \geqslant 0. \end{aligned} \qquad (5)$$

Again, $I \geqslant 0$ must be interpreted term by term, and i runs over all vertices of the local polytope. By the strong duality theorem of linear programming [53], the dual and primal optima are equal when one of the two problems has an optimal solution [we have that $\min_I \operatorname{tr}(I^T p) = q_{\rm L}^{\max}(p) = 0$]. In other words, I defines a Bell expression $\operatorname{tr}(I^T p)$ whose minimum value in QM is the algebraic minimum 0 achieved by p, and whose minimum local value is $\geqslant 1$. Moreover, in order to achieve the algebraic minimum, the Bell expression $I(a, b, x, y)$ has to have coefficients equal to zero for every $p(a, b|x, y) > 0$. This allows us to reformulate the Bell inequality for $I(a, b, x, y)$ as a nonlocal game G with a PT strategy. The winning condition of G is

$$W(a, b, x, y) = \begin{cases} 1, & \text{if } I(a, b, x, y) = 0 \\ 0, & \text{otherwise.} \end{cases} \qquad (6)$$

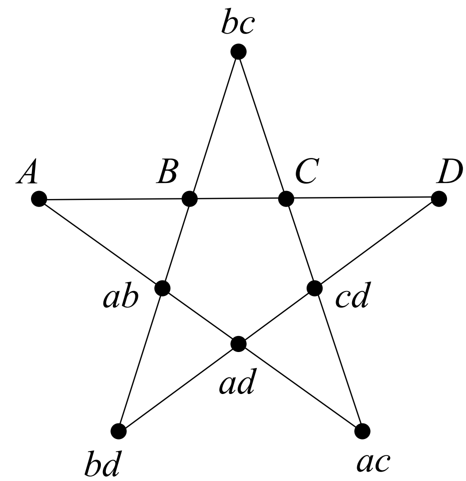

FIG. 2. Pentagram used in the game to demonstrate that games with PT strategies do not necessarily define tight Bell inequalities.

That is, by taking the complement of I, we obtain the game G with $\omega_{\rm C}(G) < 1$ and for which p provides $\omega^{(p)}(G) = 1$ ■.

FNS = FN = AVN = PT and Bell inequalities. Gisin, Méthot, and Scarani [54] made the observation that all known quantum PT strategies correspond to maximum violations of tight Bell inequalities. They also raised the question of whether this is always the case. Our Theorem 1 shows that this is in fact an important question, as it does not only concern PT. Our second result answers the question in its more general version.

Observation 1. Not all quantum FNS = FN = AVN = PT correlations define tight Bell inequalities.

Proof. The proof uses Theorem 1 and a nonlocal game with a PT strategy in the (5, 8; 5, 8) Bell scenario. Consider the pentagram in Fig. 2. It has five edges and 10 vertices; four vertices in each edge. In each round of the game G, Alice and Bob are asked to output 1 or −1 to each of the four vertices of one edge (not necessarily the same edge). That is, each party must output four bits. The conditions to win G are the following: (I) The product of the four outputs must be 1, except when the edge is $\{A, B, C, D\}$. In this case, the product must be −1. (II) If the parties are asked different edges, both parties must output the same value for the vertex at the intersection of the edges. (III) If the parties are asked the same edge, Alice's four outputs must be the same as Bob's respective outputs.

The classical winning probability is $\omega_{\rm C}(G) = \frac{23}{25}$. However, the quantum correlation p, produced with two eight-dimensional systems in the state $|\psi\rangle = \frac{1}{2\sqrt{2}} \sum_{i=0}^{7} |ii\rangle$, and measuring, on each eight-dimensional system, $A = X \otimes Z \otimes Z$, $B = Z \otimes X \otimes Z$, $C = Z \otimes Z \otimes X$, $D = X \otimes X \otimes X$, $ab = I \otimes I \otimes Z$, $ac = I \otimes Z \otimes I$, $ad = X \otimes I \otimes I$, $bc = Z \otimes I \otimes I$, $bd = I \otimes X \otimes I$, and $cd = I \otimes I \otimes X$, where X and Z are the corresponding Pauli matrices, and I is the identity [42], gives $\omega^{(p)}(G) = 1$ and thus provides a PT strategy.

However, the Bell inequality defined by the game G is not tight. To show it, let us denote the inputs as follows: $0 : \{A, B, C, D\}$, $1 : \{A, ab, ac, ad\}$, $2 : \{ab, B, bc, bd\}$, $3 :$

$\{ac, bc, C, cd\}$, and $4 : \{ad, bd, cd, D\}$. Then, the Bell inequality associated to G is

$$I_G = p(A = A|0, 1) + p(A = A|1, 0) + \ldots + p(cd = cd|4, 3) + \sum_{x=0}^{4} p(a = b|x = y) \leqslant 23, \tag{7}$$

where, e.g., $p(A = A|0, 1)$ is the probability that Alice's and Bob's output for vertex A are equal when Alice's input is 0 and Bob's input is 1. $p(a = b|x = y)$ is the probability that Alice's and Bob's outputs are equal, one by one, when Alice's and Bob's inputs are the same. Inequality (7) is saturated by 628 local vertices, which span a subspace of dimension 460. However, the dimension of the NS space of the $(5, 8; 5, 8)$ Bell scenario is 1295. Therefore, inequality (7) is not tight. ■

The counterexample used in the proof is not an isolated case. In (see Appendix D within the SM [43]), we present a general method for non-facet-preserving (i.e., non-tight-preserving) lifting Bell inequalities in which it holds the property that, if the original Bell inequality corresponds to a game with a quantum PT strategy, then the lifted Bell inequality still corresponds to a game with a quantum PT strategy.

Where does FNS = FN = AVN = PT occur? Combining Theorem 1 with previous results [44,54,55], we can conclude that QM does not allow for bipartite FNS = FN = AVN = PT with qubits (a qutrit-qutrit is the smallest quantum system needed) [55], or if one of the parties has only two settings [54], or if all measurements have two outcomes [44]. However, QM predicts correlations that are arbitrarily close to FNS = FN = AVN = PT in $(m, 2; m, 2)$ using a qubit-qubit maximally entangled state when m tends to infinity [56]. The question is: When does QM allow for bipartite FNS = FN = AVN = PT with a *finite* number of settings? The simplest example of quantum bipartite FNS = FN = AVN = PT known occurs in $(3, 4; 3, 4)$ [19–21,38–40]. But, is there any simpler example?

Gisin, Méthot, and Scarani [54] made the observation that no result excludes the possibility of quantum PT in the $(3, 3; 3, 2)$ Bell scenario. This led them to raise the question of whether PT can happen in $(3, 3; 3, 2)$. In the light of the equivalences established by Theorem 1, this becomes important in several fields and in relation to several problems. Our third result answers a more general version of this question.

Theorem 2. Quantum mechanics does not allow for FNS = FN = AVN = PT neither in the $(3, 3; 3, 2)$ nor in the $(3, 2; 3, 4)$ Bell scenarios.

Proof. The proof is based on Theorem 1 and in the observation that AVN proofs require correlations whose table of zeros cannot be realized by LHV models. The idea is to identify all tables of zeros that cannot be realized classically unless one of the zeros is removed and then check whether these tables can be realized with a quantum correlation. We will refer to one of such tables as a critical nonlocal table of zeros (CNTZ). For example, a CNTZ in the $(3, 2; 3, 3)$ Bell scenario is the following:

x	$a \backslash b$ (y)	$y=0$, $b=0$	$y=0$, $b=1$	$y=0$, $b=2$	$y=1$, $b=0$	$y=1$, $b=1$	$y=1$, $b=2$	$y=2$, $b=0$	$y=2$, $b=1$	$y=2$, $b=2$
0	0	0	0	_	_	_	_	_	_	_
	1	_	_	_	_	_	0	0	_	_
1	0	_	_	0	_	_	_	_	_	_
	1	_	_	_	_	_	0	0	_	_
2	0	_	_	_	_	_	_	_	0	0
	1	_	_	_	0	0	_	_	_	_

Note that it is impossible to find $f : \{(a|x)\}_{x\in X, a\in A} \cup S_B = \{(b|y)\}_{y\in Y, b\in B} \to \{0, 1\}$ satisfying $\sum_a f(a|x) = 1, \forall x \in X$, and $\sum_b f(b|y) = 1, \forall y \in Y$, without having a pair $\{(a|x), (b|y)\}$ for which $f(a|x) = f(b|y) = 1$ and $p(a, b|x, y) = 0$.

We wrote a Matlab program that produces all CNTZs, modulo relabelings of inputs, outputs, and parties. The version for $(3, 3; 3, 2)$ is in Appendix E within the SM [43]. We run this program on a high-performance computer and obtained 223 nonequivalent CNTZs for the $(3, 3; 3, 2)$ Bell scenario.

To check the quantum realizability of each CNTZ, we used the NPA hierarchy at level 1 (or 2). We found that none of the CNTZs yielded a feasible solution to the corresponding semidefinite programming (SDP) problem in the NPA hierarchy.

Our program for producing all CNTZs relies on the structure of the Bell symmetric group (see Appendix F within the SM [43]). The size of the Bell symmetric group grows rapidly with the addition of more parties, inputs, or outputs. Consequently, the program becomes computationally too demanding for applying it to the $(3, 3; 3, 3)$ and $(3, 4; 3, 3)$ Bell scenarios. Nevertheless, for $(3, 2; 3, 4)$, we still can handle it by using, at a certain step of the program, a sub-group S' of the symmetric group S. This provides a faster convergence and results of a manageable size for the subsequent checking of the quantum realizability. See Appendix E within the SM [43] for details. Again, none of the CNTZs in $(3, 2; 3, 4)$ were found to have a quantum realization. ■

Theorem 2 implies that in $(3, 3; 3, 2)$ and $(3, 2; 3, 4)$ there is a finite gap between the quantum set and the faces of the NS polytope that do not contain local points. This leads to the question of what is the maximum nonlocal content that can be achieved in these scenarios. We have partially answered this question for $(3, 3; 3, 2)$. The local set for the $(3, 3; 3, 2)$ is fully described by a set of 25 classes of tight Bell inequalities [52,57], the facets of the corresponding local polytope. Since we have the half-space representation of the local polytope (i.e., we have the local polytope defined as an intersection of a finite number of half-spaces), we can calculate, for every facet, the corresponding quantum bound (or an upper bound of it) using the Navascués-Pironio-Acín (NPA) hierarchy [58]. First, we have confirmed that for every facet, the quantum bound is strictly smaller than the NS bound. In addition, we have computed the local, quantum, and NS bounds for each facet (see Appendix G within the SM [43]). The maximum nonlocal content allowed by QM in $(3,3;3,2)$ maximally violating a tight Bell inequality is $q_{\rm NL}^{\rm min}=0.598$ (see Appendix G within the SM [43]).

Theorem 2 leads to the question of whether quantum FNS = FN = AVN = PT is possible in $(3,3;3,3)$. Because of the exponential complexity of the polytopes, the tools used in the proof of Theorem 2 are not enough for answering this question. Still, we have computed the NS and quantum bounds for 4801183 classes of local facets in the $(3,3;3,3)$ Bell scenario and found no example of FNS = FN = AVN = PT (see Appendix H within the SM [43]). The lower bound on the number of classes was computed using the tally, i.e., the frequency of distinct coefficients in each inequality, from a total list of 8269146 facets. The quantum bounds were obtained at level $1+AB$ in the NPA hierarchy.

The method used to prove Theorem 2 is a useful tool in itself. The method can be expected to efficiently produce novel quantum FNS = FN = AVN = PT correlations. Why is this important? On the one hand, for the reasons mentioned in (i), (ii), ..., (VI') in the introduction. On the other hand, it is also important for graph theory. The graph of exclusivity [8] of any quantum FNS = FN = AVN = PT correlation has the property that its independence number is strictly smaller than its Lovász and fractional packing numbers, which are equal [59]. Only a few graphs are known with these properties [59]. Our method provides a systematic way to identify new examples. In addition, the method can help to solve another problem [60] that, after the FNS = FN = AVN = PT equivalence, turns out to have fundamental interest in many fields: Is FNS = FN = AVN = PT possible with nonmaximally entangled states?

Conclusions. We have shown that four different resources that are crucial for a wide range of applications and results in quantum information and quantum computing are actually equivalent. The term "resource" is appropriate as FNS = FN = AVN = PT can be quantified (e.g., with respect to the number of local settings that can be removed while preserving FNS = FN = AVN = PT or by the number of local settings required), used, and consumed. The FNS = FN = AVN = PT equivalence provides a unified perspective about problems in several fields and allows us to combine different tools to investigate this extreme form of Bell nonlocality.

We have also shown that not all quantum FNS = FN = AVN = PT correlations define tight Bell inequalities. This solves an open problem and shows that finding all quantum FNS = FN = AVN = PT correlations is not easy even if one has the complete description of the local set. We have also solved another open question and demonstrated that QM does not allow for FNS = FN = AVN = PT in $(3,3;3,2)$ and $(3,2;3,4)$.

In addition, for proving the previous results, we have introduced an efficient method for finding quantum FNS = FN = AVN =PT correlations that is useful in different fields and may help to solve several open problems. Whether FNS = FN = AVN = PT is possible in $(3,3;3,3)$, $(3,3;4,2)$, or $(3,4;3,3)$ remains an open problem.

Acknowledgments. We thank useful conversations with M. Araújo. This work was supported by the EU-funded project FoQaCiA and the MCINN/AEI (Project No. PID2020-113738GB-I00). E.Z.C. acknowledges funding by FCT/MCTES - Fundação para a Ciência e a Tecnologia (Portugal) - through national funds and when applicable cofunding by EU funds under the Project No. UIDB/50008/2020, and FCT through Project No. 2021.03707.CEECIND/CP1653/CT0002. J.R.G.U. acknowledges support from Akademikernas a-kassa (Sweden). L.Y., H.Y.C., and R.R. acknowledge support from the Research Grants Council, University Grants Committee, Hong Kong. Some computations in this work were performed using research computing facilities offered by Information Technology Services, the University of Hong Kong. A.C. acknowledges support from the Wallenberg Center for Quantum Technology (WACQT). Foundations of Quantum Computational Advantage (FoQaCiA) and Ministerio de Ciencia, Innovación y Universidades/Agencia Estatal de Investigación (MCINN/AEI).

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