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Test of the physical significance of Bell non-locality

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Loophole-free violations of Bell inequalities imply that at least one of the assumptions behind local hidden-variable theories must fail. Here, we show that, if only one fails, then it has to fail completely, therefore excluding models that partially constrain freedom of choice or allow for partial retrocausal influences, or allow partial instantaneous actions at a distance. Specifically, we show that (i) any hidden-variable theory with outcome independence (OI) and arbitrary joint relaxation of measurement independence (MI) and parameter independence (PI) can be experimentally excluded in a Bell-like experiment with many settings on high-dimensional entangled states, and (ii) any hidden-variable theory with MI, PI and arbitrary relaxation of OI can be excluded in a Bell-like experiment with many settings on qubit-qubit entangled states.

In the early days of quantum theory, the question of whether there is deeper theory underlying quantum theory was considered "a philosophical question for which physical arguments alone are not decisive"[1]. Bell's theorem[2,3] made it possible to exclude experimentally some of these deeper theories, called hidden-variable (HV) theories[4]. Today, Bell tests[5–9] have convinced us that *some* HV theories cannot explain what we see.

In a Bell test, a source of pairs of particles sends each particle to a different laboratory. In the first laboratory, an observer (Alice) chooses to measure $x \in X$ and obtains $a \in A$. In the second laboratory, a different observer (Bob) chooses to measure $y \in Y$ and obtains $b \in B$. After many repetitions, Alice and Bob compute the joint probability of (a, b) given (x, y), denoted $p(a, b|x, y)$. The set $\{p(a, b|x, y)\}_{x\in X, y\in Y, a\in A, b\in B}$ is called a *correlation* for the *Bell scenario* $(|X|, |A|; |Y|, |B|)$, in which Alice can choose between $|X|$ measurement settings with $|A|$ possible outcomes and Bob between $|Y|$ settings with $|B|$ outcomes.

Bell's theorem asserts that no HV model satisfying some assumptions can reproduce certain quantum correlations. These models are collectively called "local" HV models and are defined as those satisfying the following assumptions[10,11]:

(0) *Hidden variables*. There are hidden variables that associate to each pair of particles a state $\lambda \in \Lambda$ and underlying probability densities $p(a, b|\lambda, x, y)$ and $p(\lambda|x, y)$ so

$$p(a,b|x,y) = \int d\lambda\, p(\lambda|x,y) p(a,b|\lambda,x,y). \tag{1}$$

(1) *Measurement independence* (MI): For every pair of particles, the measurements (x, y) are not correlated with λ. That is, $p(x, y|\lambda) = p(x, y)$, which, through Bayes's theorem, is equivalent to

$$p(\lambda|x,y) = p(\lambda). \tag{2}$$

Therefore, the knowledge of λ gives no information about (x, y), and vice versa.

(2) *Outcome independence* (OI)[11] also referred to as completeness[10,12]: $p(a|\lambda, x, y, b)$ is independent of b, and hence may be written

$$p(a|\lambda,x,y,b) = p(a|\lambda,x,y). \tag{3}$$

Similarly, $p(b|\lambda, x, y, a) = p(b|\lambda, x, y)$.

(3) *Parameter independence* (PI)[11], initially called locality[10]: $p(a|\lambda, x, y)$ is independent of y, and hence may be written as

$$p(a|\lambda,x,y) = p(a|\lambda,x). \tag{4}$$

Similarly, $p(b|\lambda, x, y) = p(b|\lambda, y)$.

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Assumptions (2) and (3) are independent[10] and, together, imply that

$$p(a,b|\lambda,x,y)=p(a|\lambda,x)p(b|\lambda,y), \tag{5}$$

which is Bell's original assumption[2] which is now called local factorizability or local causality.

Assumption (0) is the expression of the belief in a deeper theory underlying quantum theory. MI is motivated by the assumption that each of the observers has freedom of choice[13] or, more generally, by the assumption that which specific measurements are actually performed is not governed by the HVs that govern the particles. OI is based on the assumption that, as it happens in deterministic models [i.e., when $p(a, b|\lambda, x, y) \in \{0, 1\}$], if we would know λ, we would observe that $p(a, b|\lambda, x, y) = p(a|\lambda, x, y)p(b|\lambda, x, y)$[14]. PI is grounded on the assumption that superluminal signalling between one party's choice and the other party's spacelike separated outcome is impossible[10].

Existing experiments are inconclusive about which assumptions fail. As a consequence, possible explanations include HV theories with different degrees of "measurement dependence"[12,14–21] (that may occur due to limitations to freedom of choice[22,23] or to retrocausal influences[24,25]), different amounts of instantaneous "actions at a distance"[26–32], and combinations thereof[33]. At least one of the four assumptions is false. But which one or which ones?[11][pp. 124, 149, 96],[34][p. 66]. The prevalent view is that advancing in the resolution of this problem is not possible "on purely physical grounds but it requires an act of metaphysical judgement"[35]. Here, we challenge this view and present two results. Result 1 shows that there are quantum correlations that cannot be simulated with any HV theory assuming OI but *partial* (as opposed to complete) measurement dependence (MD) or *partial* parameter dependence (PD). Result 2 shows that there are quantum correlations that cannot be simulated with any HV theory assuming MI, PI but *partial* outcome dependence (OD).

In Sec. II A, we introduce the standard ways to quantify MI, PI and OI. Result 1 is presented in Sec.II B, where we also describe an experiment to exclude HV theories with partial MD and PD. Result 2 is presented in Sec. II C, which includes the description of an experiment to exclude HV theories with partial OD. The consequences and applications of the results are discussed in Sec. III.

Results

Relaxing the assumptions

Quantifying measurement dependence. To quantify any lack of MI, and therefore to quantify MD, we have to take into account the distribution of x and y and, therefore, we have to consider the full distribution $p(a, b, x, y)$ rather than only $p(a, b|x, y)$. The full distribution $p(a, b, x, y)$ can be reproduced with an l-measurement dependent (l-MD) HV model[36] if it can be reproduced with an HV model such that, for all $x \in X,\ y \in Y$, and for all λ,

$$p(x,y|\lambda)\ge l\ge 0. \tag{6}$$

If there are only two inputs per party, a value $l = 1/4$ implies that $p(x, y|\lambda)$ must be uniform. Therefore, in this case, $p(a, b, x, y)$ can be reproduced with an HV model with MI in which there is no correlation between the hidden variables of the particles and the measurement settings. However, this is not the case for $0 \le l < 1/4$. We say that $p(a, b, x, y)$ can be reproduced with an HV model with *partial* MD if there is an (l-MD) model for some $l > 0$. We say that $p(a, b, x, y)$ can only be reproduced with an HV model with *complete* MD if $p(a, b, x, y)$ cannot be reproduced with any (l-MD) model for any $l > 0$; the only possible HV models have $p(x, y|\lambda) = 0$ for some pair of settings (x, y) and some λ. If there are only two inputs per party, any $p(a, b, x, y)$ corresponding to any non-signaling correlation can be reproduced with an HV model with complete MD. The complete relaxation of MI using alternative ways of quantifying MD[14,18] matches the above definition of complete MD (see also Supplementary Note1).

Quantifying parameter dependence. A correlation $p(a, b|x, y)$ can be reproduced with an (ε_A, ε_B)-parameter dependent [(ε_A, ε_B)-PD] HV model[14] if it can be reproduced with an HV model such that, for all x, y, y' (y, x, x'), and for all λ,

$$\frac{1}{2}\sum_a |p(a|\lambda,x,y)-p(a|\lambda,x,y')|\le\varepsilon_A\ , \tag{7a}$$

$$\frac{1}{2}\sum_b |p(b|\lambda,x,y)-p(b|\lambda,x',y)|\le\varepsilon_B\ . \tag{7b}$$

Therefore, $p(a, b|x, y)$ can be reproduced with an HV model with PI if, and only if, it can be reproduced with an (ε_A, ε_B)-PD HV model with $\varepsilon_A = \varepsilon_B = 0$. If not, we say that $p(a, b|x, y)$ can be reproduced with an HV model with *partial* PD if it can be reproduced with an (ε_A, ε_B)-PD HV model for some $0 < \varepsilon_A, \varepsilon_B < 1$. Finally, $p(a, b|x, y)$ can only be reproduced with HV models with *complete* PD if $p(a, b|x, y)$ cannot be reproduced with any (ε_A, ε_B)-PD HV model for any ε_A, $\varepsilon_B < 1$. Any non-signaling correlation can be reproduced with HV models with complete PD.

Quantifying outcome dependence. A correlation $p(a, b|x, y)$ can be reproduced with a δ-outcome dependent (δ-OD) HV model[14] if it can be reproduced with an HV model such that, for all x, y, a, a', and for any λ,

$$\frac{1}{2}\sum_b|p(b|\lambda,x,y,a)-p(b|\lambda,x,y,a')|\le\delta. \tag{8}$$

Therefore, $p(a, b|x, y)$ can be reproduced with an HV model with OI if, and only if, it can be reproduced with a δ-OD HV model with $\delta = 0$. We say that $p(a, b|x, y)$ can be reproduced with an HV model with *partial* OD if it can be reproduced with a δ-OD HV model with $0 < \delta < 1$. We say that $p(a, b|x, y)$ can only be reproduced with an HV model with *complete* OD if it cannot be reproduced with any δ-OD HV model for any $\delta < 1$. Any non-signaling correlation can be reproduced with HV models with complete OD.

Result 1: Quantum correlations that cannot be simulated if there is arbitrarily small MI or PI

Consider the bipartite Bell experiment in which Alice and Bob have two measurement options $x, y \in \{0, 1\}$, each of them with 2^N possible results which can be expressed as a string of N bits, $a, b \in \{(0, 0, \ldots, 0), (0, 0, \ldots, 1), \ldots, (1, 1, \ldots, 1)\}$. Suppose that Alice and Bob share the following $2^N \times 2^N$-dimensional entangled state:

$$|\psi\rangle=|\phi\rangle_{A_1,B_1}\otimes\cdots\otimes|\phi\rangle_{A_N,B_N}\ , \tag{9}$$

where

$$|\phi\rangle=a(|01\rangle+|10\rangle)+\sqrt{1-2a^2}|11\rangle, \tag{10}$$

with $a=\frac{\sqrt{5}-1}{2}$, is a two-qubit state with the first qubit in Alice's side and the second qubit in Bob's side. Suppose that Alice's and Bob's measurements are of the form

$$A_{a_1,\ldots,a_N|x}=A_{a_1|x}\otimes\cdots\otimes A_{a_N|x}\ , \tag{11a}$$

$$B_{b_1,\ldots,b_N|y}=B_{b_1|y}\otimes\cdots\otimes B_{b_N|y}\ , \tag{11b}$$

where, here, the tensor product refers to the qubits in each observer's system and the specific form of the factors is given by

$$A_{1|x} = \mathbb{1} - A_{0|x}\ , \tag{12a}$$

$$B_{1|y} = \mathbb{1} - B_{0|y}\ , \tag{12b}$$

where

$$A_{0|0} = B_{0|0} = |0\rangle\langle 0|, \tag{13a}$$

$$A_{0|1} = B_{0|1} = |\varphi\rangle\langle\varphi|, \tag{13b}$$

with $|\varphi\rangle = \frac{1}{\sqrt{1-a^2}}(\sqrt{1-2a^2}|0\rangle - a|1\rangle)$. That is, each of the 2^N-outcome measurements can be seen as N (nonindependent) two-outcome measurements performed simultaneously on a 2^N-dimensional quantum system. These state and measurements produce a correlation with the following properties:

$$p(0,1,a_2,b_2,\ldots,a_N,b_N|0,1) = \ldots = p(a_1,b_1,\ldots,a_{N-1},b_{N-1},0,1|0,1) = 0, \tag{14a}$$

$$p(1,0,a_2,b_2,\ldots,a_N,b_N|1,0) = \ldots = p(a_1,b_1,\ldots,a_{N-1},b_{N-1},1,0|1,0) = 0, \tag{14b}$$

$$p(0,0,a_2,b_2,\ldots,a_N,b_N|1,1) = \ldots = p(a_1,b_1,\ldots,a_{N-1},b_{N-1},0,0|1,1) = 0, \tag{14c}$$

for all $a_1, \ldots, a_N, b_1, \ldots, b_N \in \{0, 1\}$. Eq. (14a) indicates that, if the measurements are $x = 0$ for Alice and $y = 1$ for Bob, then, in the N-bit strings that Alice and Bob obtain as outputs cannot be one position where Alice has 0 and Bob has 1. Similarly, for Eqs. (14b) and (14c). These state and measurements are the ones needed for the parallelised version[37] of the optimal version of the proof of Bell non-locality proposed by Hardy[38].

Let us define

$$p_H^N := \sum_{\substack{a_1,\ldots,a_N,b_1,\ldots,b_N \\ (a_1,b_1)=(0,0)\vee\ldots\vee(a_N,b_N)=(0,0)}} p(a_1,b_1,\ldots,a_N,b_N|0,0), \tag{15}$$

where $\vee$ is the logical OR.

Result 1 can be stated as follows: In any l-MD and $(\varepsilon_A, \varepsilon_B)$-PD HV model satisfying OI and Eqs. (14a), (14b) and (14c), for all $l > 0$ and all N,

$$p_H^N \le \varepsilon_A + \varepsilon_B - \varepsilon_A\varepsilon_B. \tag{16}$$

The proof is in the Supplementary Note 2. Therefore, if $\varepsilon_A < 1$ and $\varepsilon_B < 1$, then $p_H^N < 1$. In contrast, in quantum theory[37], as N tends to infinity,

$$p_H^N \xrightarrow{N\to\infty} 1. \tag{17}$$

Consequently, for any l-MD and $(\varepsilon_A, \varepsilon_B)$-PD HV model with $l > 0$, $\varepsilon_A < 1$, $\varepsilon_B < 1$, satisfying OI, there is N such that quantum theory predicts a value for p_H^N that cannot be simulated.

For example, Table 1 gives the values of $\varepsilon = \varepsilon_A = \varepsilon_B$ that cannot be simulated if nature achieves the quantum value for p_H^N. Notice that the number of excluded HV models grows with N. As N tends to infinity, the only surviving HV models are those with $\varepsilon = 1$.

The correlations defined by Eqs. (10)–(13) are special: they define an extremal non-exposed point of the quantum set of correlations for the two-observer two-setting two-outcome Bell scenario[39]. Any other correlation with this property can also be used in the experiment. The full characterisation of these points is in[40].

Table 1 | Relaxation of PI as a function of the number of parallel copies

N	ε	p_H^N
1	< 0.0461	0.0902
2	< 0.0901	0.1722
3	< 0.1321	0.2469
4	< 0.1722	0.3148
5	< 0.2104	0.3766
6	< 0.2468	0.4328
7	< 0.2816	0.4839
8	< 0.3147	0.5304
9	< 0.3463	0.5727
10	< 0.3765	0.6113

N is the number of parallel copies and 2^N is the number of outputs in the Bell test. $\varepsilon = \varepsilon_A = \varepsilon_B$ quantifies the relaxation of PI and p_H^N is the upper bound in the probability given by Eq. (16) for l-MD and $(\varepsilon, \varepsilon)$-PD HV models satisfying OI. HV models with ε above the threshold indicated in the Table cannot be excluded by the corresponding experiment.

A natural question is what conditions a correlation must satisfy to allow for arbitrarily small MI and PI, and whether there are quantum correlations in Bell scenarios with finite number of inputs that allow for such relaxation. In the Supplementary Note 3, we show a necessary condition - the quantum correlation must necessarily lie on or be arbitrarily close to the nonsignaling boundary. We also illustrate by an explicit example that this condition is not sufficient. We leave as an open question whether there is a finite input-output quantum correlation that proves Result 1.

Experimental test to exclude HV theories with partial MD and PD

So far, we have identified a quantum correlation that cannot be simulated by any l-MD and $(\varepsilon_A, \varepsilon_B)$-PD HV model with $l > 0$, $\varepsilon_A < 1$, $\varepsilon_B < 1$, satisfying OI. This correlation is a point in the set of quantum correlations. The problem is that, due to experimental errors, an actual experiment will fail to exactly produce this point. Here, we reformulate Result 1 in a way that the existence of correlations that cannot be simulated by l-MD and $(\varepsilon_A, \varepsilon_B)$-PD HV models with $l > 0$, $\varepsilon_A < 1$, $\varepsilon_B < 1$, and satisfying OI, can be experimentally tested.

It can be proven (see Supplementary Note 4) that, for any l-MD and $(\varepsilon_A, \varepsilon_B)$-PD HV model with $l > 0$, $\varepsilon_A < 1$, $\varepsilon_B < 1$, satisfying OI, the following Bell-like inequality holds:

$$I_\kappa^N(p) \le \tilde{\varepsilon}_A + \tilde{\varepsilon}_B - \tilde{\varepsilon}_A\tilde{\varepsilon}_B, \tag{18}$$

where

$$\begin{aligned} I_\kappa^N(p_H^N) := &\sum_{\substack{a_1,\ldots,a_N,b_1,\ldots,b_N \\ (a_1,b_1)=(0,0)\vee\ldots\vee(a_N,b_N)=(0,0)}} p((a_1,b_1),\ldots,(a_N,b_N)|0,0) \\ -\kappa &\sum_{\substack{a_1,\ldots,a_N,b_1,\ldots,b_N \\ (a_1,b_1)=(0,1)\vee\ldots\vee(a_N,b_N)=(0,1)}} p((a_1,b_1),\ldots,(a_N,b_N)|0,1) \\ -\kappa &\sum_{\substack{a_1,\ldots,a_N,b_1,\ldots,b_N \\ (a_1,b_1)=(1,0)\vee\ldots\vee(a_N,b_N)=(1,0)}} p((a_1,b_1),\ldots,(a_N,b_N)|1,0) \\ -\kappa &\sum_{\substack{a_1,\ldots,a_N,b_1,\ldots,b_N \\ (a_1,b_1)=(0,0)\vee\ldots\vee(a_N,b_N)=(0,0)}} p((a_1,b_1),\ldots,(a_N,b_N)|1,1), \end{aligned} \tag{19}$$

with

$$\kappa > \frac{N^2}{l(1-\varepsilon)^2}, \quad (20)$$

where $\varepsilon = \max\{\varepsilon_A, \varepsilon_B\}$, and

$$\tilde{\varepsilon}_A = \varepsilon_A + N\sqrt{\frac{2}{l\kappa}}, \quad (21a)$$

$$\tilde{\varepsilon}_B = \varepsilon_B + N\sqrt{\frac{2}{l\kappa}}. \quad (21b)$$

This means that, for any l-MD and (ε_A, ε_B)-PD HV model with $l > 0$, $\varepsilon_A < 1$, $\varepsilon_B < 1$, satisfying OI, for sufficiently large κ, the quantity $I_\kappa^N(p)$ is upper bounded by a value that is always smaller than 1. Furthermore, for fixed N, this bound approaches the bound for (16) when we take large values of κ and is therefore violated by the quantum state and measurements described earlier.

Result 2: Quantum correlations which cannot be simulated if there is arbitrarily small OI

Consider the bipartite Bell experiment in which Alice and Bob have $M + 1$ measurement options $x, y \in \{0, 1, ..., M\}$, each of them with 2 possible results, $a, b \in \{0, 1\}$. Suppose that Alice and Bob share the following two-qubit entangled state:

$$|\phi\rangle = \frac{1}{\sqrt{1+t^2}}(t|00\rangle - |11\rangle), \quad (22)$$

where $t \in [0, 1]$ is the value that maximises

$$\max_{0\le t\le 1} \frac{t^2}{1+t^2}\left(\frac{1-t^{2M}}{1+t^{2M+1}}\right)^2. \quad (23)$$

Alice's and Bob's measurements are of the form $A_{a|x} = |\pi_{a|x}\rangle\langle\pi_{a|x}|$ and $B_{b|y} = |\sigma_{b|y}\rangle\langle\sigma_{b|y}|$, with

$$\begin{aligned} |\pi_{0|x}\rangle &= \cos a_x|0\rangle + \sin a_x|1\rangle, \forall x \in \{0, \dots, M\}, \\ |\pi_{1|x}\rangle &= -\sin a_x|0\rangle + \cos a_x|1\rangle, \forall x \in \{0, \dots, M\}, \end{aligned} \quad (24)$$

and

$$\begin{aligned} |\sigma_{0|y}\rangle &= \cos b_y|0\rangle + \sin b_y|1\rangle, \forall y \in \{0, \dots, M\}, \\ |\sigma_{1|y}\rangle &= -\sin b_y|0\rangle + \cos b_y|1\rangle, \forall y \in \{0, \dots, M\}, \end{aligned} \quad (25)$$

with

$$a_k = b_k = \arctan\left[(-1)^k t^{k+1/2}\right] \forall k \in \{0, \dots, M\}. \quad (26)$$

These state and measurements produce a correlation with the following properties:

$$p(0,0|0,0) = 0, \quad (27a)$$

$$p(0,1|k,k-1) = 0 \quad \forall k \in \{1, \dots, M\}, \quad (27b)$$

$$p(1,0|k-1,k) = 0 \quad \forall k \in \{1, \dots, M\}, \quad (27c)$$

and correspond to the optimal implementation of the "ladder" version of Hardy's proof[21,38,40].

Let us define

$$p_H^M := p(0,0|M,M). \quad (28)$$

Result 2 can be formulated as follows: In any δ-OD HV model that satisfies MI, PI and Eq. (27),

$$p_H^M \le \frac{\delta^q}{2}, \quad (29)$$

where $q = \frac{M+1}{2}$. The proof is provided in the Supplementary Note 5. Therefore, if $\delta < 1$, it follows that $p_H^M < \frac{1}{2}$. In contrast, in quantum theory[21,40], as M approaches infinity,

$$p_H^M \overset{M\to\infty}{\longrightarrow} \frac{1}{2}. \quad (30)$$

Consequently, for any HV model with MI, PI and δ-OD, with $\delta < 1$, there is M such that quantum theory predicts a value for p_H^M that cannot be simulated. In addition, it can be proven (see Supplementary Note 6) that the set of correlations produced by HV with MI, PI and complete OD is the set of nonsignaling correlations.

The above proof is based on the assumption that Eq. (27) hold. In an actual experiment, instead of the zeros Eq. (27), we will obtain small values. Once we have them, we can derive an optimal Bell-like inequality that will allows us to discard any HV model with δ-OD for some $\delta < 1$.

Discussion

The results presented have consequences both for foundations and applications in quantum information processing, communication and computation. For foundations, our results bring us closer to the solution of a problem proposed by Shimony[11][pp. 96, 124, 149][34][p. 66] and which can be formulated as follows: "One of these three premises [MI, PI and OI] must be false and it is important to locate the false one"[11][p. 96]. If we assume that only *one* is false (and that it is the same one for all non-local quantum correlations), then

I. If the assumption that fails is MI, Result 1 shows that, MI has to fail completely because there are quantum correlations that can only be explained with complete MD.
II. If the assumption that fails is PI, Result 1 shows that, PI has to fail completely because the same correlations used in [I] can only be explained with complete PD.
III. If the assumption that fails is OI, Result 2 shows that OI has to fail completely because there are quantum correlations that can only be explained with complete OD.

Each of these solutions to Shimony's problem requires extra causal influences which are not needed if HVs do not exist. These extra causal influences are shown with non-black colours in Fig. 1a–c, respectively. The causal influences if HVs do not exist are shown in Fig. 1d.

More generally, our results allow us to experimentally narrow down the possible explanations of Bell non-locality and the whole quantum theory, since they allow to experimentally excluding large subsets of HV models that are not excluded by previous experiments. Specifically, in principle, any l-MD, (ε_A, ε_B)-PD HV model with $l > 0$, $\varepsilon_A < 1$, $\varepsilon_B < 1$, satisfying OI can be experimentally excluded. Similarly, any δ-OD HV model with $\delta < 1$ and satisfying MI and PI can be, in principle, experimentally excluded. Still, the experiments cannot exclude HV models with complete MD[22] or complete PD[26,32] or complete OD.

Result 1 extends the observation in[36] that there are quantum correlations that cannot be obtained from an l-MD HV model that satisfies OI and PI, for all values of $l > 0$. In[36], the difference between quantum theory and the models with OI, PI and arbitrarily small MI is so small that any relaxation of PI makes the difference to vanish. In this respect, Result 1 makes testable the impossibility of HV models with partial MD and PD.

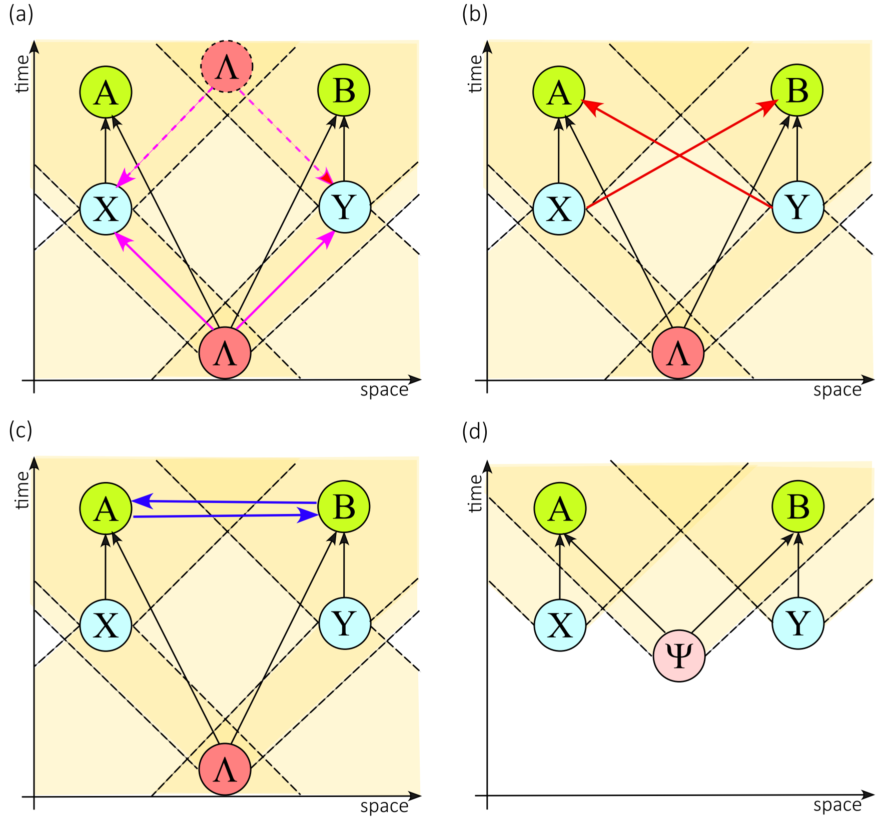

Fig. 1 | Space-time diagrams of the causal influences needed, in every round of the Bell-like test, for each of the possible solutions to Shimony's problem in light of our results. In all diagrams, black arrows represent causal influences common to all possibilities. **a** Complete measurement dependence. It can occur in, essentially, two ways. The first is with complete superdeterminism without retrocausality. In this case, Alice and Bob do not have freedom of choice to choose the measurement settings, X and Y, respectively. Instead, the settings are determined by the distribution of the HVs in the past light cones of X and Y, represented by the lower node Λ. The causal influences between the lower Λ and X and Y are represented by violet continuous arrows. The second way complete MD can occur is with complete retrocausality with freedom of choice. In this case, Alice and Bob have freedom to choose the "nominal" measurement settings X and Y, but the actual measurements are determined by the distribution of the HVs in the future light cones of X and Y, represented by the upper node Λ. The causal influences between the upper Λ and X and Y are represented by violet dashed arrows. **b** Complete parameter dependence. X is decided by Alice and Y is decided by Bob. However, X does not only influence Alice's measurement outcome, represented by A, but also Bob's measurement outcome, represented by B, which is outside the light cone of X. This superluminal influence (or "action at a distance") is represented by a red arrow. Similarly, Y does not only influence B, but also A. **c** Complete outcome dependence. X is decided by Alice and Y is decided by Bob. However, A and B are causally connected despite they are space-like separated. These superluminal influences are represented by blue arrows. **d** No hidden variables. X is decided freely by Alice and Y is decided freely by Bob. A is causally connected only to the quantum state, represented by Ψ, and X. Similarly, B is causally connected only to Ψ and Y.

Result 2 is related to the observation in[31] that there are quantum correlations that cannot be simulated with the assumptions of *causal models* (CM), MI, *causal parameter independence* (CPI) and the complete relaxation of *causal outcome independence* (COI). This observation is not made in the framework of the four assumptions of Bell's theorem (HV, MI, PI, OI) but in the framework of causal models[41]. In general, neither HV and CM, nor PI and CPI, nor OI and COI, are equivalent.

In addition, Results 1 and 2 confirm that there is some interchangeability between MD and (PD+OD)[42,43]. However, our results go beyond that as they show that epsilon of each of MD and (PD+OD) is not enough: complete MD or complete PD or complete OD is needed.

One reason why it is important to exclude HV models with partial (but not complete) MD is that these models have been proposed to explain quantum correlations[12,14–21]. In addition, partial MD or, more precisely partial human's free will, has been proposed in philosophy to resolve the conflict between the concept of an omniscient God and God's commandment not to commit sin[44].

One reason why it is important to exclude HV models with partial (but not complete) actions at a distance is that these models have been proposed to explain quantum correlations[27–32]. A second reason, which also applies to HV models with partial MD, is that excluding larger sets of HV models facilitates the discussion of the remaining models and, in particular, the discussion of the thermodynamics of the HV models[45] that could not be discarded.

Our results are also of practical interest in quantum information processing, quantum communication and quantum computation. In the first place, for a general reason: the results show that quantum correlations do not only offer advantage with respect to *local* correlations, but also with respect to correlations assisted by partial instantaneous actions at a distance or even assuming the existence of partial constraints to freedom of choice or partial retrocausal influences. This can make a big difference in quantum computational advantage. For example, when mapping quantum non-local correlations into the circuit model, the advantage of quantum theory *with respect to local HV theories* translates into a non-oracular quantum advantage[46,47]. Our results show that there is also advantage with respect to non-local correlations with partial MD, PD and OD. This may translate into new forms of quantum computational advantage.

Another reason why our results are of practical interest is device-independent (DI) quantum information processing[48,49]. DI protocols for random number generation[50], quantum key distribution[48], state tomography[51] and self-testing of quantum devices[52] achieve advantage allowing users to monitor the performance of their devices irrespective of noise, imperfections, and lack of knowledge regarding the inner workings - the users simply treat their devices as black boxes with classical inputs and outputs. An obstacle for practical DI protocols is the experimentally challenging requirement of a Bell test with: (I) quantum devices being isolated from each other, (II) with the inputs being chosen with uniform randomness and (III) with the detection loophole[53] closed. Experiments in different platforms[54–56] allow for Bell tests with the detection loophole closed and high DI randomness generation rates. However, in these platforms, the quantum systems are very close, primarily to drive high entanglement generation rates via non-negligible coupling. The problem is that, precisely because the systems are close to one another, they can no longer be regarded as isolated in the sense needed for a Bell test. Sophisticated theoretical techniques have been devised to handle the issues of cross-talk and weak seeds separately. A Bell-like test allowing for arbitrary relaxation of MI and PI provides a simple and elegant solution to the problem of leakage of input information. A Bell-like test allowing for simultaneous relaxation of MI, PI and OI would allow DI randomness generation tolerating weak seeds and cross-talk. We hope that our results will stimulate research in these directions.

Data availability
No data sets were generated or analysed during the current study.

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Author contributions

C.V., R.R., and A.C. contributed equally to this work.

Competing interests

There are no competing interests.

Additional information

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