Pricing European Options with Non-normal Log-returns: a Simulation Study

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Abstract

This study examines the limitations of the Black-Scholes (B-S) model, which assumes normality in log-return distributions, by exploring alternative non-normal distributions that better capture the observed kurtosis and skewness. Through the incorporation of these distributions into a Discrete-Time stochastic process model, the study investigates their impact on option pricing compared to the traditional B-S model. Employing Monte Carlo simulation techniques, call and put option prices are estimated under various log-return distributions adhering to the risk-neutral paradigm, and the simulated prices are compared using relative difference curves. The findings demonstrate that deviations from normality in log-return distributions significantly affect option pricing, emphasizing the need for financial professionals to employ more sophisticated models that accurately represent the statistical profiles of diverse assets.

CCS Concepts

• **Mathematics of computing** → Probability and statistics; Stochastic processes; Markov processes.

Keywords

Option pricing, Black-Scholes model, Monte Carlo Simulation, Stochastic process

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1 Introduction

The identification of the typical log-return distribution of financial assets has important implications for option-pricing models, such as the Black-Scholes (B-S) model. Since its inception in 1973, the B-S model has been the most widely used method in practice. However, its fundamental assumption that the log-returns are normally distributed is considered to be unrealistic.

Empirical studies on financial time series show that the log-return densities of financial assets exhibit fatter tails and are more peaked than those supported by standard Gaussian assumptions, and they may be skewed to the left rather than symmetric. The seminal works by Mandelbrot (1963) revealed that the log-returns of cotton and common stock generally have fatter tails compared with the normal distribution.[1] Liu P and Zheng Y. (2023) observed that the return distribution presents a leptokurtic, fat-tailed, and almost symmetrical shape in China stock market.[2] . Mason and Wilmot (2021) claimed that the presence of fat tails of key energy prices increases the premium associated with delaying investment in new infrastructure.[3] The analysis of the distributional characteristics shows that German real estate returns were not normally distributed and that a logistic distribution would have been a better fit (Lian et al. 2024).[4] Non-normality can lead to biased parameter estimates and unreliable statistical inferences, particularly in the presence of extreme observations. This significantly impacts asset pricing models and volatility forecasting (Templeton and Blank, 2023).[5]

To capture the important kurtosis or skewness characteristics of the actual empirical distribution of asset log returns, it is advantageous to adopt a more nuanced asset pricing model that accommodates distributions beyond the Gaussian framework. Many alternative non-Gaussian distributions have been proposed in the literature. For examples, Markowitz and Usmen (1996) identified the Student-t distribution with about 4.5 degrees of freedom as the best fit to daily log-return data of S&P500.[6] In Hurst and Platen (1997) the Student-t distribution with 3.0~4.5 degrees of freedom was determined as the best fit to daily, regional stock market index returns.[7] The study of Theodossiou and Trigeorgis(2023) demonstrated that the skewed generalized Gaussian distribution (SGGD) provides a good fit to log-return data from representative stock index, exchange rate and metal prices.[8] However, no consensus has been reached about which distribution will generally fit log-returns of a particular class of financial securities in the literature.

This article presents the main types of non-Gaussian distributions proposed in the literature and develops Discrete-Time stochastic process model based on these distributions. Our study endeavors to examine the implications of deviating from standard distributional assumptions on option pricing mechanics, utilizing Monte Carlo method, we estimate the call/put option prices. This process involves simulating sample paths of stock prices when log-returns follow non-normal and normal distributions separately before the

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maturity according to the risk-neutral measure. Then utilizing different strike prices, we derive estimated option prices based on non-normal log-returns and normal log-returns. We further plot the relative difference curves to illustrate how various log-return distributions with different kurtosis and skewness affect option pricing compared to traditional Black-Scholes assumptions. Our study summarizes how the difference of log-returns distributions influences the option prices with different strike prices and provides people a reference when choosing the appropriate distribution to describe the log-returns of a particular financial asset.

This paper is organized as follows. Section 2 presents a spectrum of distributions with different kurtosis and skewness. Section 3 delineates the discrete time stochastic process models with normal and non-normal exponents. This is followed by the relative difference curves of different non-normal distributions. Section 4 provides a synthesis of the study's findings, encapsulating the core insights and implications of the research.

2 Distribution Selection

Voluminous literature shows that the distribution of log returns of many financial assets is skewed, leptokurtic. Many alternative non-Gaussian distributions with different skewness and kurtosis have been proposed to replace the normal distribution, like the Generalized Gaussian Distribution (GGD), the skewed student-t distribution, and the Weibull distribution. These distributions are pictured in Figure1 and 2 to give readers a graphical sense of the effect of kurtosis and skewness on a distribution. In practice, researchers often choose different distributions for different classes of financial assets, and only a distribution that provides good empirical approximation of the stochastic behavior of the desired financial asset can be chosen. In this paper, we choose fat tails (or heavy) tails as synonyms to indicate tails that are fatter (or heavier) than the normal ones, which poses significant implications for both risk assessment and portfolio optimization within financial markets.

We employ symmetric generalized gaussian distribution (also known as the exponential power distribution) to model the stochastic process of those financial assets whose log-return is leptokurtic and symmetric. Figure 1A shows the standardized (zero mean and unit variance) symmetric GGD for various values of shape parameter β ranging between the benchmark cases β =2 (normal) and $\beta = 1$ (Laplace), illustrating the effect of kurtosis (heavy tails and peakedness) alone on the shape of the distribution of asset log-returns. The shape parameter β governs the tails and peakedness of a symmetric GGD. With the decrease of β, the tails become heavier and the peakedness gets more pronounced. In general, the probability density generated by values of $\beta < 2$ (such as $\beta = 1.5$ or 1.2, as depicted in the figure) exhibits a heavier-tailed distribution than the normal distribution, indicating a higher probability of extreme outcomes on both the downside and the upside than that implied by the normal distribution.

For financial assets with log-return that exhibit left skewness and heavy-tails on both ends, we use the skewed Student-t distribution to model the stochastic process. Figure 1B shows the skewed Student-t distribution with different degrees of freedom. As the degrees of freedom increase, the skewed Student's t distribution becomes less leptokurtic, meaning that the probability of extreme outcomes decreases. The heavier tails on both sides may be caused by a higher likelihood of downward and upward jumps in asset prices.

To illustrate the impact of having significant left skewness, we employ the Weibull distribution with specific scale parameter and shape parameter as a replacement for normal distribution, the comparison of these two distributions is shown in Figure 2. The left tail of the PDF is much longer than the right one, and only the left tail is fatter than the normal distribution, while the right one is thinner; in this case, the likelihood of observing extremely negative values is higher than that of extremely positive values having the same absolute size. Rubinstein(1994) found a similar distributional characteristic in the returns of the S&P500 index in the post-October 1987 crash period, and this pattern may be caused by the likelihood of a series of sharp declines due to a bankruptcy or market crashes.[9] Domenico et al. (2023) reveals that the modified Weibull models can explain the occurrence and scaling of large price movements observed in financial markets.[10]

3 Simulation Analysis

3.1 Black-Scholes Model

Let S_t be a sequence of stock prices observed at discrete times t= 0, 1, 2, . . . on a per-day basis, with S_0 being the initial price.

The log-return on day t is calculated as the formula (1):

$$Z_t = \log(\frac{S_t}{S_{t-1}}), t \geq 1 \tag{1}$$

Suppose there are N periods in a year, starting from t=0. Then at the end of the *jth* period (time *j*), The S_t is calculated as the formula (2)

$$S_j = S_0 e^{\sum_{i=1}^{j} Z_i} \tag{2}$$

where $\{Z_i\}_{i=1}^{N}$ are log returns.

In this model, log returns over non-overlapping periods of time are assumed to be *i,i,d.* Especially, assume that $\{Z_i\}_{i=1}^{N} \overset{i.i.d}{\sim} N(\mu_z, \sigma_z^2)$, under certain risk-neutral measure Q.

The variance of annual log returns is σ^2, then the *Var* as calculated by the formula (3), that is:

$$Var_{\mathbb{Q}}\left(\sum_{i=1}^{N} Z_i\right) = \sigma^2 \tag{3}$$

Therefore, variance for one period log return is $\sigma_z^2 = \sigma^2/\text{N}$.

As shown in formula (4), in considering the mean, the annual risk-free interest rate is r, the interest rate per period is r/N (continuous compounding). Since the risk-neutral measure Q is a measure such that the discounted stock price is martingale. Therefore, we need for any j ≥ 0,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[e^{-\frac{r}{N}} S_{j+1} | \mathcal{F}_j\right] = S_j \\ \Leftrightarrow\ & S_j \mathbb{E}_{\mathbb{Q}}\left(e^{Z_{j+1} - \frac{r}{N}} | \mathcal{F}_j\right) = S_j \\ \Leftrightarrow\ & \mathbb{E}_{\mathbb{Q}}\left(e^{Z_{j+1} - \frac{r}{N}}\right) = 1 \end{aligned} \tag{4}$$

where F_j denotes the information set up to period j (formally called $\sigma -$ field).

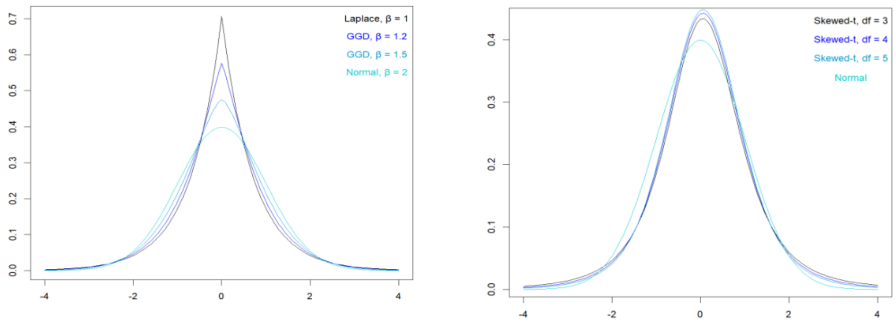

Figure 1. A Figure 1. B

Figure 1: GGD and Skewed-t Distribution compared with Normal Distribution

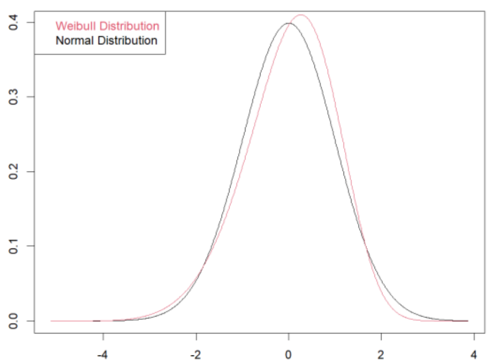

Figure 2: Weibull Distribution Compared with Normal Distribution

Since $Var_{\mathbb{Q}}(Z_t) = \frac{\sigma^2}{N}$, $E_{\mathbb{Q}}(e^{Z_{j+1}-\frac{r}{N}}) = e^{\mathbb{E}_{\mathbb{Q}}(Z_{j+1})+\frac{\sigma^2}{2N}-\frac{r}{N}} = 1$, that brings formula (5):

$$\mathbb{E}_{\mathbb{Q}}(Z_i) = \mu_z = \frac{r}{N} - \frac{\sigma^2}{2N} \tag{5}$$

3.2 Non-normal Stochastic Process Model

Subsequently, the model of the stochastic process for stock prices $\{S_t\}$ with non-normal exponents can be formed. Specifically, assume that under the risk-neutral measure Q, $S_{i+1} = S_i e^{\mu+cX_i}$ in which $\{X_i\} \overset{i.i.d}{\sim} F(x)$ for some non-normal distribution F. The parameters μ and c are selected so that the resulting price has a similar distribution as in the (discrete time) Black-Scholes model. Then, assume N short periods in one year, and it is required that the two models have same volatility, and can be calculated as formula (6)

$$Var\,(cX_i) = \frac{\sigma^2}{N} \Rightarrow c = \frac{\sigma}{\sqrt{N \cdot Var\,(X_i)}} \tag{6}$$

where σ is the annual volatility.

The discounted asset price is a martingale. Thus,

$$\begin{aligned} S_{i-1} &= e^{-\frac{r}{N}} \mathbb{E}_{\mathbb{Q}}\,(S_i|\mathcal{F}_{i-1}) \\ S_{i-1} &= e^{-\frac{r}{N}} \mathbb{E}_{\mathbb{Q}} \left[S_{i-1}e^{\mu+cX_i}|\mathcal{F}_{i-1}\right] \\ \Rightarrow 1 &= e^{\left(\mu-\frac{r}{N}\right)} \mathbb{E}_{\mathbb{Q}} \left(e^{cX_i}\right), \\ 1 &= e^{\left(\mu-\frac{r}{N}\right)} m_X\,(c) \end{aligned} \tag{7}$$

where $m_X(c)$=$E_Q(e^{eX_i})$ is the moment generating function of X_i evaluated at c. This solves for the formula (8).

$$\mu = \frac{r}{N} - \log\left(m_X\,(c)\right) \tag{8}$$

If the explicit forms of $Var(X)$and m_X are unknown or nonexistent, an alternative approach must be adopted to address this issue.

Generate, independently of $\{X_i^{(k)} : 1 \leq i \leq j\}_{k=1}^{B}$, a (large) number of random variates $\{X_i : 1 \leq i \leq M\}$ from F.

Estimate c by formula (9):

$$\hat{c} = \frac{\sigma}{\sqrt{N \cdot S^2\,(X)}} \tag{9}$$

where $S^2(X)$ represent the sample variance of $\{X_i : 1 \leq i \leq M\}$.

Then, $m_X(c)$ is estimated by the *sample mgf* as shown in formula (10).

$$\hat{m}_X\,(c) = \frac{1}{M} \sum_{i=1}^{M} e^{\hat{c}X_i} \tag{10}$$

3.3 Relative difference

Subsequently, consider the difference between the option price in the case using non-normally and the case of normally distributed log-returns. Suppose that the random samples $\{U_i^{(k)}\}_{i=1}^{j}$are generated with a U(0, 1) distribution. We simulate B paths of the stochastic process $\{S_i^{(k)} : 1 \leq i \leq j\}_{k=1}^{B}$, using the non-normal exponent$Y_i^{(k)}$ and normal exponent $Z_i^{(k)}$, respectively. Especially, for the purpose of comparison, for each pair of *(i, k)* $X_i^{(k)}$ and $Z_i^{(k)}$ are generated from the same *U (0,1)* (shown as Figure 3), in this way, we can simulate the relative difference of two approaches under the same randomness.

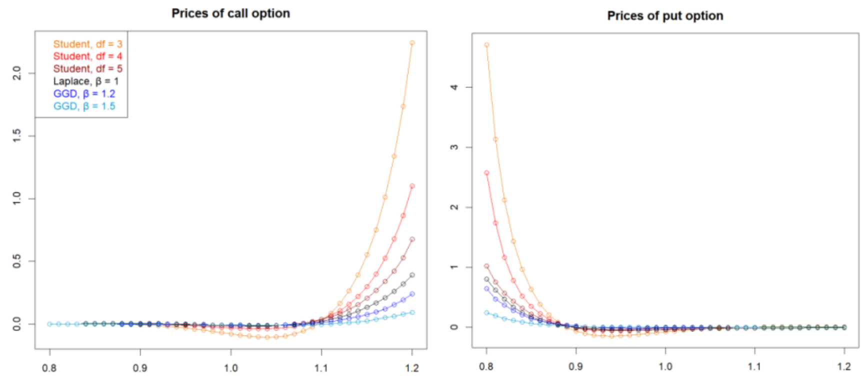

Figure 3: Relative Difference of Student's t Distribution and GGD distribution

Calculate stock prices at maturity j:
$S_j^{(k)}(F) = S_0 e^{\left(\sum_{i=1}^{j} Y_i^{(k)}\right)}$, $S_j^{(k)}(\Phi) = S_0 e^{\left(\sum_{i=1}^{j} Z_i^{(k)}\right)}$ for all k.

Then the simulated option prices can be computed utilizing the following methodology:
$C_0^{(k)}(F) = e^{-\frac{rj}{N}} \left(S_j^{(k)}(F) - K\right)^+$, $C_0^{(k)}(\Phi) = e^{-\frac{rj}{N}} \left(S_j^{(k)}(\Phi) - K\right)^+$ for all k.
$P_0^{(k)}(F) = e^{-\frac{rj}{N}} \left(K - S_j^{(k)}(F)\right)^+$, $P_0^{(k)}(\Phi) = e^{-\frac{rj}{N}} \left(K - S_j^{(k)}(\Phi)\right)^+$ for all k.

Aggregate $\{C_0^{(k)}(F) : 1 \le k \le B\}$, $\{P_0^{(k)}(F) : 1 \le k \le B\}$, $\{C_0^{(k)}(\Phi) : 1 \le k \le B\}$, and$\{P_0^{(k)}(\Phi) : 1 \le k \le B\}$ to obtain the simulated option prices:$\hat{C}_0(F)$, $\hat{C}_0(\Phi)$, $\hat{P}_0(F)$, and $\hat{P}_0(F)$.

Finally, the relative difference is estimated following the computation of the formula (11) and (12):

$$\frac{\hat{C}_0(F) - \hat{C}_0(\Phi)}{\hat{C}_0(\Phi)} \tag{11}$$

$$\frac{\hat{P}_0(F) - \hat{P}_0(\Phi)}{\hat{P}_0(\Phi)} \tag{12}$$

3.4 Analysis of Simulated Relative Difference Results

Assuming an annual framework consisting of N = 252 discrete trading periods, we examine a European option on the stock with a maturity j = 10, a striking price K ranging from \$0.80 to \$1.20, an initial price S_0 = \$1.00, an annual volatility parameter $\sigma = 0.20$,and in a market with a risk-free annual interest rate r = 0.05. We employ the distributions introduced earlier as distributions F and simulate B = 100000 paths of the stochastic process for each distribution. This procedure yields a series of relative difference curves, corresponding to each of the diverse distributions under consideration.

Figure 3 shows the simulated relative difference when asset log returns follow symmetric GGD with different shape parameter β and skewed Student-t distribution with different degrees of freedom. These relative difference curves have the same trend: for either call options with strike prices not higher than S_0 by 10%, or put options with strike prices not lower than S_0 by 10%, the curves are relative flat around 0; for call (put) options, when the strike prices are higher (lower) than S_0 by 10%, the relative difference increases rapidly as the strike prices increase (decrease). That means the Normal SP model performs almost the same as the model with these leptokurtic non-normal exponents, for call (put) options with strike prices not higher (lower) than S_0 by 10%. The normal SP models tends to undervalue the call option whose strike price exceeds S_0 by 10%, and the degree of undervaluation deepens with the rise of the strike price. In the meantime, it underestimates the put option when the strike price is much lower than S_0, and the degree of undervaluation deepens with the drop of the strike price.

The normal distribution underestimates the probability of tail event on both sides compared with GGD and skewed student t distribution. This inclination leads to a simulated stock price distribution which predicts extreme values less frequently than that derive from these leptokurtic non-normal distributions. This difference is further transmitted to option pricing. Consider call options pricing for instance, the main difference between the two models in pricing the call option with strike price K is caused by the distribution of simulated stock prices that are higher than K. Since the non-normal distribution, we choose is similar in shape(bell-shape) to the normal distribution and the parameters μ and c are selected with the purpose of deriving stock price has a similar distribution as in the (discrete time) Black-Scholes model, when K is relatively low, simulated stock prices distribution to the right of K under normal SP model does not exhibit much difference from that under non-normal one. However, as K increases to a certain level (in our study, about 10% higher than the initial price), the fat tail characteristics begins to influence the stock prices distribution to the right of K, that non-normal SP model generate more stocks prices to the right of K and more extremely high stock prices than that of normal SP model, and this effect becomes more significant with the increment of K. This mechanism explains why these relative difference curves increase rapidly at the right end and these non-normal distributions with heavier tails generate curves grow faster.

Figure 4 illustrates the computed relative difference resulting from the assumption that log returns obey a Weibull distribution. For either call options with strike price not higher than the initial price or put options with strike price not lower than the initial price, the curves are relative flat around 0; for call (put) options, when the strike price are higher (lower) than the initial price, the relative

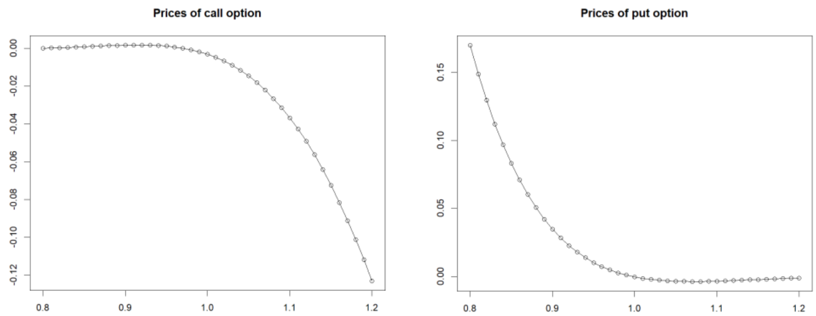

Figure 4: Relative Difference of Weibull Distribution

difference decreases (increases) as the strike price increases (decreases). These patterns suggest that the Normal SP model tend to overprice (underprice) call options with strike price higher (lower) than S_0.

When compared with the Weibull distribution, the normal distribution displays a propensity to underestimate the probability of tail event on the left side and overestimates that on the right side. Consequently, this leads to a simulated stock price distribution which predicts more extremely high values and less extremely low values. In the context of option pricing for a call (put) option, as surpasses S_0, the impact of the difference in the right (left) tail of the two distributions on option pricing becomes more pronounced, highlighting the asymmetric impact of tail events on financial derivatives valuation.

4 Conclusion

The underlying assumption made in most financial models is that log-returns follow a normal distribution. However, this assumption is insufficient when considering two commonly observed characteristics in empirical studies of financial time series for asset log-returns: (1) the presence of leptokurtic distributions, commonly known as 'fat tails', and (2) the potential asymmetry of their distribution, often characterized by negative skewness.

In this paper, we conduct a comprehensive analysis of the impact of kurtosis and skewness on option pricing using Simple Monte Carlo simulation. We specifically focus on three distinct distributions: the Generalized Gaussian Distribution (GGD), the skewed Student-t distribution, and the Weibull distribution. By examining the differences in log-returns distributions, we explore how they influence option pricing for various strike prices. Additionally, we elucidate the biases present in the Black-Scholes model when financial assets' log-returns follow the aforementioned distributions. As a result, our research provides valuable insights and practical recommendations for practitioners seeking to accurately model the log-return behavior of specific financial instruments.

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