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A concise algorithm for computing the factor of safety using the Morgenstern–Price method

D.Y. Zhu, C.F. Lee, Q.H. Qian, and G.R. Chen

Abstract: A concise algorithm is proposed in this paper for the calculation of the factor of safety of a slope using the Morgenstern–Price method. Based on force and moment equilibrium considerations, two expressions are derived for the factor of safety F_s and the scaling factor λ, respectively, both in relatively simple forms. With this algorithm and assumed initial values of F_s and λ, the solutions for F_s and λ are found to converge within a few iterations. Compared to other procedures, the present algorithm possesses the advantages of simplicity and high efficiency in application. It is rather straightforward to implement this algorithm into a computer program.

Key words: slope, stability, factor of safety, limit equilibrium method.

Résumé : Dans cet article, on propose un algorithme concis pour le calcul de coefficient de sécurité de talus avec la méthode Morgenstern–Price. En se basant sur des considérations de force et d'équilibre limite, on dérive deux expressions pour le coefficient de sécurité F_s et le coefficient d'échelle λ respectivement, toutes deux dans des formes relativement simples. Avec cet algorithme et des valeurs initiales supposées de F_s et de λ, on trouve que les solutions pour F_s et λ convergent après quelques itérations. En comparaison d'autres procédures, le présent algorithme possède l'avantage d'être efficace et simple d'application. C'est plutôt facile d'entrer cet algorithme dans un ordinateur.

Mots clés : talus, stabilité, coefficient de stabilité, méthode d'équilibre limite.

[Traduit par la Rédaction]

Introduction

Slope-stability problems are usually analyzed using limit equilibrium methods of slices. When evaluating the stability conditions of soil slopes of simple configuration, circular potential slip surfaces are usually assumed, wherein the ordinary method (Fellenius 1936) or the simplified Bishop method (Bishop 1955) can be used, the latter being preferred because it is generally considered to be more vigorous. In many situations, however, the actual surfaces of rupture are found to deviate significantly from the circular shape, or the potential slip surfaces are predefined by planes of weakness in rock slopes. In such cases, a number of methods of slices can be used to accommodate the noncircular shape of slip surfaces (Janbu 1954; Lowe and Karafiath 1960; Morgenstern and Price 1965; Spencer 1967; US Army Corps of Engineers 1967). Of these, the Morgenstern–Price method (Morgenstern and Price 1965) is commonly used because it completely satisfies the equilibrium conditions and involves the least numerical difficulty. The basic assumption underlying the Morgenstern–Price method is that the ratio of normal to shear interslice forces across the sliding mass is represented by an interslice force function that is the product of a specified function $f(x)$ and an unknown scaling factor λ. Based on the vertical force equilibrium conditions for individual slices and the moment equilibrium condition for the whole sliding mass, two equilibrium equations are derived involving the two unknowns, namely, the factor of safety F_s and the scaling factor λ, thereby rendering the problem determinate. Unfortunately, solving for F_s and λ is often complex, since the equilibrium equations are highly nonlinear and in complicated form. Some sophisticated iterative procedures (Morgenstern and Price 1967; Fredlund and Krahn 1977; Zhu et al. 2001) have been developed for solution purposes. Although these procedures can give converged solutions to F_s and λ, they are not easily accessible to most geotechnical designers who have to rely on commercial

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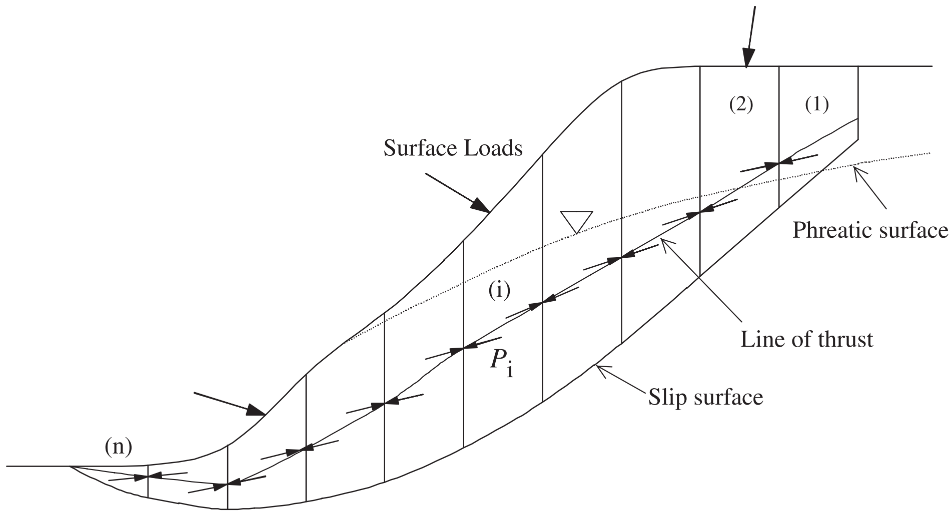
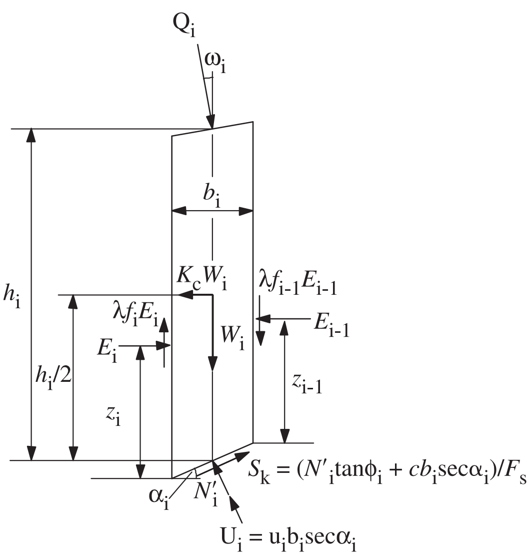
Fig. 1. (*a*) Sliding body. (*b*) Typical slice. P_i, interslice force.



(a)



(b)

packages of the "black box" type. Moreover, when performing analyses of slope stability with established procedures for the Morgenstern–Price method, tedious computations are often involved, resulting in unduly long computation times. This situation necessitates the improvement of the algorithm for computing F_s and λ associated with the Morgenstern–Price method.

In this paper, the two equilibrium equations used in the Morgenstern–Price method are re-derived to obtain two expressions for the factor of safety F_s and the scaling factor λ. This makes the algorithm for calculating the factor of safety more concise and more easily implemented into a computer program.

Equilibrium equations

Consider a sliding mass bounded by the ground surface and a noncircular slip surface, which is subject to self-weight, seismic forces, water pressures, and surface loads, as shown in Fig. 1. As with the other methods of slices, the sliding mass is divided into a number of vertical slices. A typical slice is shown in Fig. 1*b*, with height h_i, width b_i, and base inclination α_i. The *i*th slice is subject to 10 sets of forces as follows: (*i*) self-weight W_i; (*ii*) seismic force $K_c W_i$, where K_c is the horizontal seismic coefficient; (*iii*) external force Q_i, making angle ω_i with the vertical (positive as indicated in Fig. 1); (*iv*) resultant water force $U_i = u_i b_i \sec\alpha_i$, where u_i is the average water pressure; (*v*) effective normal force on the base N'_i; (*vi*) mobilized shear resistance $S_i = (N'_i \tan\phi'_i + c'_i b_i \sec\alpha_i)/F_s$, where ϕ'_i is the effective friction angle, c'_i is the cohesion along the base, and F_s is the factor of safety, which is assumed to be constant along the whole sliding surface; in the case of total stress analysis, the strength parameters are in terms of total stress with zero water pressure; (*vii*) normal interslice forces E_i and E_{i-1} acting on the left and right boundaries of the slice at vertical distances z_i and z_{i-1} from the bottom, respectively; and (*viii*) shear interslice forces $\lambda f_i E_i$ and $\lambda f_{i-1} E_{i-1}$; commensurate with the Morgenstern–Price method, the ratio between the normal and shear interslice forces is assumed to be described by a function $\lambda f(x)$.

Considering the force equilibrium of the *i*th slice, and resolving perpendicular to the slip surface,

$$[1a] \quad N_i' = (W_i + \lambda f_{i-1} E_{i-1} - \lambda f_i E_i + Q_i \cos\omega_i)\cos\alpha_i + (-K_c W_i + E_i - E_{i-1} + Q_i \sin\omega_i)\sin\alpha_i - U_i$$

and resolving parallel to the slip surface,

$$[1b] \quad (N_i' \tan\phi_i' + c_i' b_i \sec\alpha_i)/F_s = (W_i + \lambda f_{i-1} E_{i-1} - \lambda f_i E_i + Q_i \cos\omega_i)\sin\alpha_i - (-K_c W_i + E_i - E_{i-1} + Q_i \sin\omega_i)\cos\alpha_i$$

Substituting eq. [1*a*] into eq. [1*b*] yields

$$[2] \quad E_i[(\sin\alpha_i - \lambda f_i \cos\alpha_i)\tan\phi_i' + (\cos\alpha_i + \lambda f_i \sin\alpha_i)F_s] = E_{i-1}[(\sin\alpha_i - \lambda f_{i-1}\cos\alpha_i)\tan\phi_i' + (\cos\alpha_i + \lambda f_{i-1}\sin\alpha_i)F_s] + F_s T_i - R_i$$

in which

$$[3a] \quad R_i = [W_i \cos\alpha_i - K_c W_i \sin\alpha_i + Q_i \cos(\omega_i - \alpha_i) - U_i] \times \tan\phi_i' + c_i' b_i \sec\alpha_i$$

$$[3b] \quad T_i = W_i \sin\alpha_i + K_c W_i \cos\alpha_i - Q_i \sin(\omega_i - \alpha_i)$$

Actually, R_i is the sum of the shear resistances contributed by all the forces acting on the slices except the normal shear interslice forces, and T_i is the sum of the components of these forces tending to cause instability.

Equation [2] is rearranged in the form

$$[4] \quad E_i \Phi_i = \psi_{i-1} E_{i-1} \Phi_{i-1} + F_s T_i - R_i$$

in which

$$[5a] \quad \Phi_i = (\sin\alpha_i - \lambda f_i \cos\alpha_i)\tan\phi_i' + (\cos\alpha_i + \lambda f_i \sin\alpha_i)F_s$$

$$[5b] \quad \Phi_{i-1} = (\sin\alpha_{i-1} - \lambda f_{i-1}\cos\alpha_{i-1})\tan\phi_{i-1}' + (\cos\alpha_{i-1} + \lambda f_{i-1}\sin\alpha_{i-1})F_s$$

$$[5c] \quad \psi_{i-1} = [(\sin\alpha_i - \lambda f_{i-1}\cos\alpha_i)\tan\phi_i' + (\cos\alpha_i + \lambda f_{i-1}\sin\alpha_i)F_s]/\Phi_{i-1}$$

With the condition $E_0 = 0$ and $E_n = 0$ (where E_0 and E_n are the interslice forces at the upper and lower ends, respectively) from eq. [4], the force equilibrium equation is derived in the form of an expression for the factor of safety F_s:

$$[6] \quad F_s = \frac{\sum_{i=1}^{n-1}\left(R_i \prod_{j=i}^{n-1} \psi_j\right) + R_n}{\sum_{i=1}^{n-1}\left(T_i \prod_{j=i}^{n-1} \psi_j\right) + T_n}$$

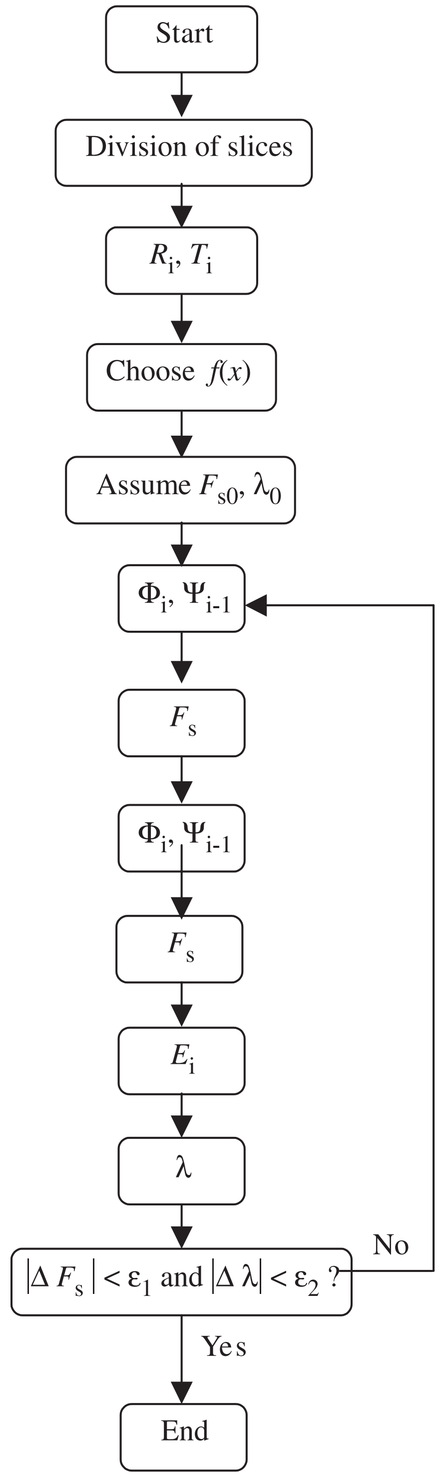


Fig. 2. Flow chart of the computation algorithm. ε_1, ε_2; limits of tolerance for F_s and λ, respectively.

Equation [6] is of an implicit nature because the variable F_s appears on both sides, and thus iteration is required for solving this equation.

Now consider the moment equilibrium of the *i*th slice. Taking moments of all the forces acting on the slice about the centre of the base,

Fig. 3. Slope profile for example 1. Conversions as follows: 1 ft = 0.3048 m, 1 lb/ft^3 = 0.1571 kN/m^3, 1 lb/ft^2 = 0.0479 kPa. *r*, radius of the slip circle.

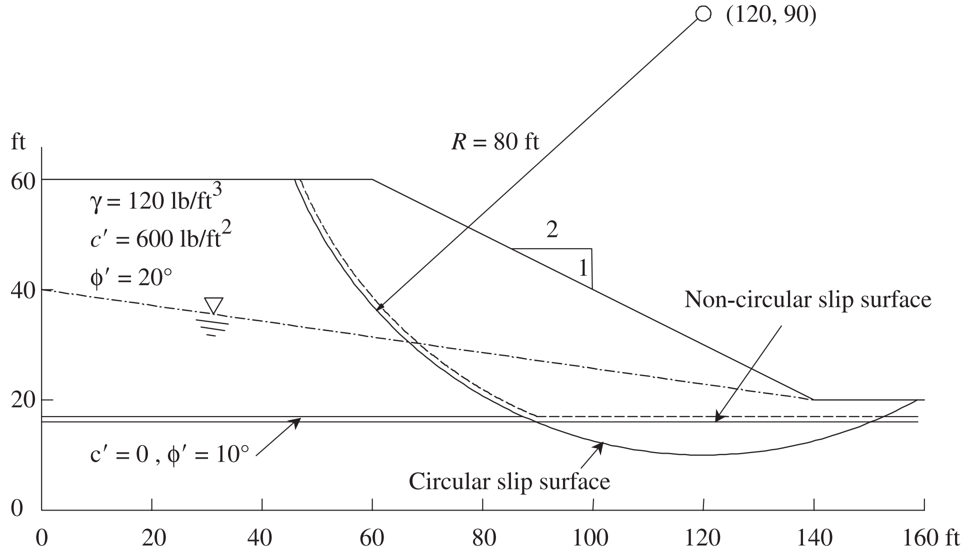


Table 1. Comparison of F_s and λ values computed using the present method with those reported by Fredlund and Krahn (1977) (example 1).

		Constant interslice function				Half-sine interslice function			
		F_s		λ		F_s		λ	
Case	Description	Fredlund and Krahn	Present method	Fredlund and Krahn	Present method	Fredlund and Krahn	Present method	Fredlund and Krahn	Present method
1	Circular slip surface; dry slope	2.073	2.075	0.237	0.258	2.076	2.074	0.318	0.324
2	Noncircular slip surface; dry slope	1.373	1.381	0.185	0.188	1.370	1.371	0.187	0.228
3	Circular slip surface; r_u = 0.25	1.761	1.760	0.255	0.250	1.764	1.760	0.304	0.314
4	Noncircular slip surface; r_u = 0.25	1.118	1.119	0.139	0.163	1.118	1.109	0.130	0.195
5	Circular slip surface; piezometric line	1.830	1.831	0.247	0.240	1.832	1.831	0.290	0.299
6	Noncircular slip surface; piezometric line	1.245	1.261	0.121	0.144	1.245	1.254	0.101	0.165

Note: Two hundred slices were used in the computation. r_u, ratio of pore-water pressure.

Table 2. Values of F_s and λ at each step (half-sine interslice force function is used) (example 1).

	Case 1		Case 2	
Step	F_s	λ	F_s	λ
1	1.0000	0	1.0000	0
2	2.0037		1.3560	
3	1.8701	0.3231	1.3304	0.2161
4	2.0918		1.3707	
5	2.0724	0.3240	1.3686	0.2277
6	2.0746		1.3709	
7	2.0745	0.3240	1.3708	0.2283
8	2.0744		1.3709	
9	2.0744	0.3240	1.3709	0.2283

[7] $$E_i\left(z_i - \frac{b_i}{2}\tan\alpha_i\right) = E_{i-1}\left(z_{i-1} + \frac{b_i}{2}\tan\alpha_i\right) - \lambda\frac{b_i}{2}(f_iE_i + f_{i-1}E_{i-1}) + K_cW_i\frac{h_i}{2} - Q_i\sin\omega_i h_i$$

Table 3. Soil properties used in example 2.

Layer	γ (kN/m^3)	c' (kPa)	ϕ′ (°)
1	19.0	0.0	26.0
2	18.8	21.5	20.0
3	18.0	15.5	26.0
4	18.5	28.0	22.0
5	19.0	50.0	10.0

Assume that

[8] $$M_i = E_iz_i \qquad M_{i-1} = E_{i-1}z_{i-1}$$

in which M_i and M_{i-1} are termed interslice moments (Zhu et al. 2001). Substitution of eq. [8] into eq. [7] gives

[9] $$M_i = M_{i-1} - \lambda\frac{b_i}{2}(f_iE_i + f_{i-1}E_{i-1}) + \frac{b_i}{2}(E_i + E_{i-1})\tan\alpha_i + K_cW_i\frac{h_i}{2} - Q_i\sin\alpha_i h_i$$

Table 4. Comparison of computed values of F_s and λ (example 2).

Water pressure	Earthquake effect	$f(x') = 1$: F_s	$f(x') = 1$: λ	$f(x') = \sin(x')$: F_s	$f(x') = \sin(x')$: λ	Procedure
Yes	Yes	0.827	0.3273	0.791	0.4472	Zhu et al. 2001
		0.829	0.3296	0.793	0.4500	Present method
		0.834	0.3278	0.798	0.4479	Geo-Slope International Ltd. 1998
Yes	No	1.023	0.2512	1.000	0.3307	Zhu et al. 2001
		1.028	0.2530	1.004	0.3332	Present method
		1.032	0.2518	1.008	0.3315	Geo-Slope International Ltd. 1998
No	Yes	1.081	0.3526	1.045	0.4800	Zhu et al. 2001
		1.081	0.3551	1.046	0.4836	Present method
		1.084	0.3522	1.048	0.4797	Geo-Slope International Ltd. 1998
No	No	1.341	0.2707	1.316	0.3591	Zhu et al. 2001
		1.342	0.2725	1.317	0.3618	Present method
		1.345	0.2703	1.320	0.3587	Geo-Slope International Ltd. 1998

Table 5. Effect of the choice of interslice function on the computed factor of safety (example 2).

μ	ν	λ	F_s
0	—	0.2725	1.342
	0.5	0.3315	1.351
0.5	1.0	0.3188	1.329
	2.0	0.3713	1.315
	0.5	0.3856	1.359
1.0	1.0	0.3618	1.317
	2.0	0.4471	1.311
	0.5	0.4769	1.374
2.0	1.0	0.4390	1.300
	2.0	0.5684	1.320
	0.5	0.5504	1.387
3.0	1.0	0.5058	1.290
	2.0	0.6709	1.329
	0.5	0.6123	1.397
4.0	1.0	0.5644	1.283
	2.0	0.7631	1.336
	0.5	0.6665	1.405
5.0	1.0	0.6165	1.279
	2.0	0.8483	1.342

As $M_0 = E_0 z_0 = 0$ and $M_n = E_n z_n = 0$, the moment equilibrium equation is derived in the form of an explicit expression for the scaling factor λ:

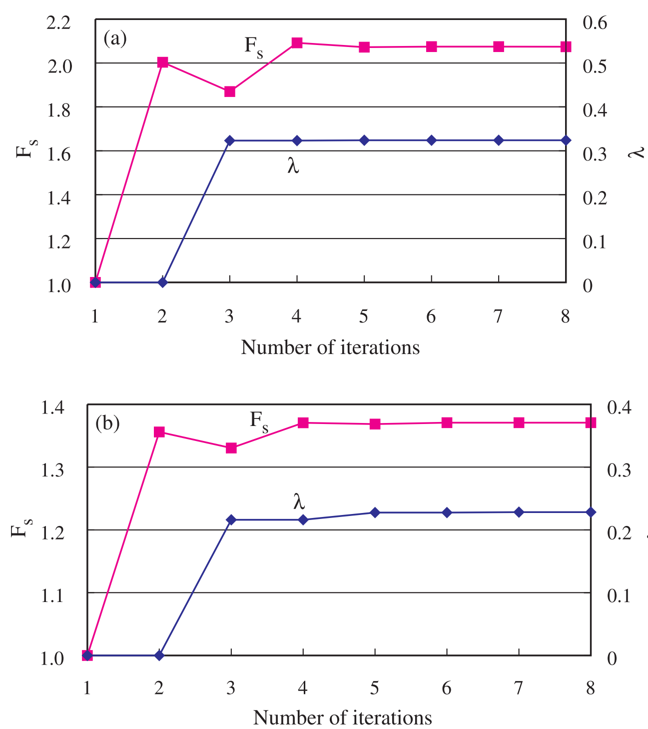
$$[10] \quad \lambda = \frac{\sum_{i=1}^{n}[b_i(E_i + E_{i-1})\tan\alpha_i + K_c W_i h_i + 2Q_i \sin\omega_i h_i]}{\sum_{i=1}^{n}[b_i(f_i E_i + f_{i-1}E_{i-1})]}$$

Iterative algorithm for calculating F_s and λ

The expressions for the factor of safety F_s and the scaling factor λ (eqs. [6] and [10], respectively) were derived in the foregoing section, with the former in an implicit form and the latter in an explicit form. To calculate F_s and λ, an iterative algorithm should be employed in the following steps:

(1) Divide the sliding body into a number of slices.

Fig. 4. Diagram showing the process of iteration for example 1 (half-sine interslice force function is used): (*a*) case 1; (*b*) case 2.



(2) Calculate R_i and T_i using eqs. [3*a*] and [3*b*], respectively, for all the slices.

(3) Specify the form of the interslice function $f(x)$. An extension of the half-sine function is suggested herein for examining the sensitivity of the choice of $f(x)$ on the calculated factor of safety:

$$[11] \quad f(x) = \sin^{\mu}\left[\pi\left(\frac{x-a}{b-a}\right)^{\nu}\right]$$

in which a and b are abscissa of the left and right ends of the failure surface, respectively; and μ and ν are spec-

Fig. 5. Slope profile for example 2, showing soil layers 1–5.

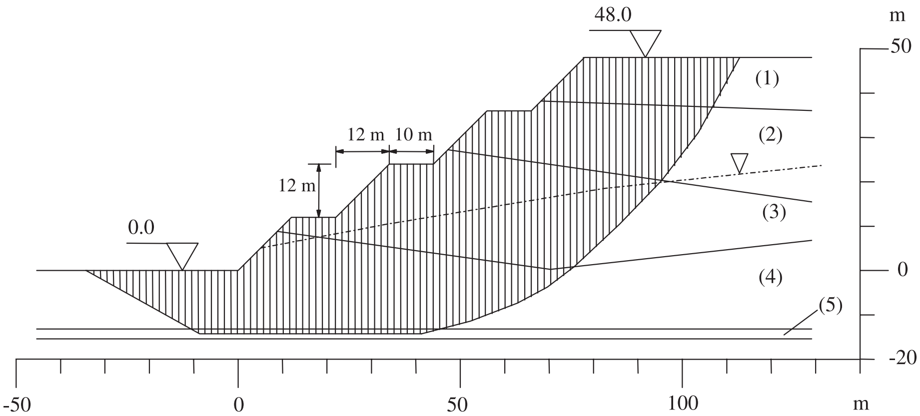
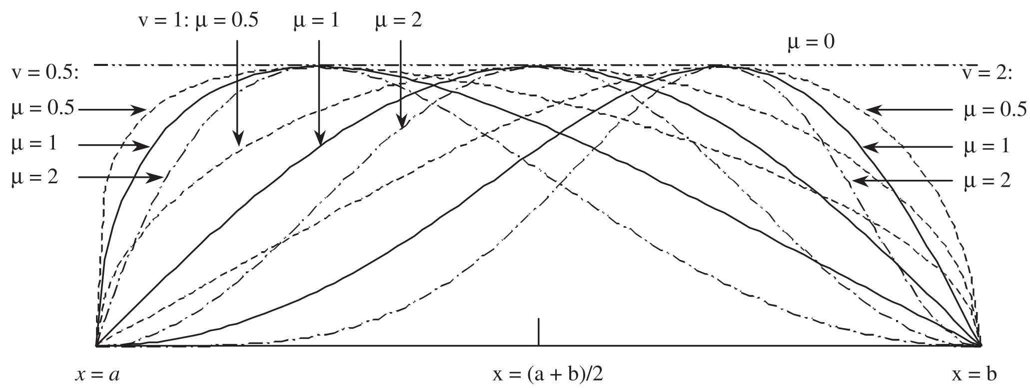


Fig. 6. Curves of interslice force function, $f(x) = \sin^{\mu}\{\pi[(x - a)/(b - a)]^{\nu}\}$, with various values of the exponents.



ified non-negative values, with $\mu = 0$–5.0 and $\nu = 0.5$–2.0, in general.

(4) Assume initial values of F_s and λ. The initial choice of F_s and λ might have an influence on the number of iterations required for convergence but would have no effect on the final values of F_s and λ. Equation [4] shows that the effective transfer of the thrust force from one slice to another requires that

$$\Phi_i = (\sin\alpha_i - \lambda f_i \cos\alpha_i)\tan\phi_i' + (\cos\alpha_i + \lambda f_i \sin\alpha_i)F_s > 0$$

Thus, for any specified value of λ, the choice of the initial F_s should at least satisfy the criterion as follows:

$$[12] \qquad F_s > -\frac{\sin\alpha_i - \lambda f_i \cos\alpha_i}{\cos\alpha_i + \lambda f_i \sin\alpha_i}\tan\phi'$$

In general, it can be assumed that $F_s = 1$ and $\lambda = 0$ for the first iteration.

(5) Calculate Φ_i and ψ_{i-1} using eqs. [5*a*]–[5*c*] for all slices.
(6) Calculate F_s using eq. [6].
(7) With the calculated value of F_s and the prescribed value of λ, repeat steps 5 and 6 once more for improved values of Φ_i, ψ_{i-1}, and F_s.
(8) Calculate E_i using eq. [4] for all slices.
(9) Calculate λ using eq. [10].
(10) With the updated values of F_s and λ, return to step 3 and proceed to step 8 until the differences in values of F_s and λ between two consecutive iterations are within specified limits of tolerance, ε_1 and ε_2.

The flow chart illustrating this algorithm is presented in Fig. 2.

Example 1

A benchmark example previously studied by Fredlund and Krahn (1977) is reexamined herein. The slope profile and soil parameters are presented in Fig. 3. Six computation cases are considered and represent six combinations of slip surfaces (two in total) and water-pressure conditions (three in total). A comparison of the computed factors of safety for this problem is shown in Table 1. It can be seen from Table 1 that for the circular slip surface, the present method gives values of the factor of safety identical to those reported by Fredlund and Krahn. For the noncircular slip surface, there are minor, but practically negligible, differences, possibly as a result of the imperfect reproduction of the slope profile. It is clear from Table 2 and Fig. 4 that the values of F_s and λ have converged within fewer than 10 iterations with a tolerance as small as 0.0001.

Example 2

This example has been previously analysed using a Newton–Raphson procedure proposed by the authors (Zhu et al. 2001), and a comparison was made with the Slope/W software (Geo-Slope International Ltd. 1998). The slope profile used is shown in Fig. 5 and the soil properties are presented in Table 3. A horizontal seismic coefficient of 0.1 is adopted. It is reanalysed here using the present algorithm, resulting in nearly identical results, as shown in Table 4. Six to eight iterations are required by the present algorithm with a tolerance of 0.0001. It is evident that the present algorithm is simpler and easier to use in practice.

To further examine the sensitivity of the choice of interslice function to the computed factor of safety, the general form of interslice force function (eq. [11]) is used in an analysis of this example without water pressure and seismic force, and with 10 different combinations of the exponents μ and ν, which cover a wide range of possible distributions of the ratio of shear to normal interslice forces across the sliding mass. The curves of the 10 interslice force functions are plotted in Fig. 6, and the results are presented in Table 5, showing that the maximum difference is within 10% in values of factors of safety associated with these different interslice functions. Thus, for this particular example, the choice of interslice function has a relatively minor influence on the calculated factor of safety. This is probably due to the vigorous nature of the Morgenstern–Price method, which satisfies all the equilibrium conditions.

Conclusion

The Morgenstern–Price method is reformulated in this paper, resulting in two simple expressions for the factor of safety, F_s, and scaling factor, λ. Solving these two equations with the proposed algorithm requires fewer than 10 iterations, even if the tolerance on the factor of safety is as small as 0.0001. The initial values of F_s and λ can be assumed to be unity and zero, respectively. It has also been demonstrated that the choice of the interslice function has only a minor effect on the calculated factor of safety. The proposed algorithm thus renders the Morgenstern–Price method no more complex than the simplified Bishop method, and its implementation into a computer program is rather straightforward.

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