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Exact dissipative cosmologies with stiff fluid

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Abstract. – The general solution of the gravitational field equations in the flat Friedmann-Robertson-Walker geometry is obtained in the framework of the full Israel-Stewart-Hiscock theory for a bulk viscous stiff cosmological fluid, with bulk viscosity coefficient proportional to the energy density.

Introduction. – Dissipative bulk viscous-type thermodynamical processes are supposed to play a crucial role in the dynamics and evolution of the early Universe. There are many processes capable of producing bulk viscous stresses in the early cosmological fluid-like interaction between matter and radiation, quark and gluon plasma viscosity, different components of dark matter, particle creation, strings and topological defects, etc. [1].

Traditionally, for the description of these phenomena the theories of Eckart [2] and Landau and Lifshitz [3] were used. But Hiscock and Lindblom [4] have shown that the Eckart-type theories suffer from serious drawbacks concerning causality and stability. Regardless of the choice of equation of state, all equilibrium states in these theories are unstable and in addition signals may be propagated through the fluid at velocities exceeding the speed of light. These problems arise due to the first-order nature of the theory, *i.e.* it considers only first-order deviations from the equilibrium. The neglected second-order terms are necessary to prevent non-causal and unstable behavior.

A relativistic consistent second-order theory was found by Israel [5] and developed into what is called transient or extended irreversible thermodynamics [6–8]. In particular, Hiscock and Lindblom [7] have shown that these second-order theories are free of the pathologies of the Eckart-type theories. Therefore, the best currently available theory for analyzing dissipative processes in the Universe is the full Israel-Stewart-Hiscock causal thermodynamics.

Exact general solutions of the field equations for flat homogeneous universes filled with a full causal viscous fluid source for a power law dependence of the bulk viscosity coefficient on the energy density have been obtained recently in [9–11]. The evolution of a homogeneous and isotropic dissipative fluid in the truncated Israel-Stewart theory has been analyzed, by using dynamical systems methods, by Di Prisco, Herrera and Ibanez [12]. They have found

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that almost all solutions inflate, and only a few of them can be considered physical, since they do not satisfy the dominant energy condition.

It is the purpose of this letter to obtain the general solution of the gravitational field equations for the case of a bulk viscous cosmological fluid, with the bulk viscosity coefficient proportional to the energy density, obeying the Zeldovich (stiff) equation of state. In this case the solution can be presented in an exact parametric form. The behavior of the scale factor, deceleration parameter, viscous pressure, viscous pressure-thermodynamic pressure ratio, comoving entropy and Ricci and Kretschmann invariants is also considered.

Geometry, field equations and consequences. – For a Friedmann-Robertson-Walker (FRW) Universe with a line element $\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$ filled with a bulk viscous cosmological fluid the energy-momentum tensor is given by

$$T_i^k = (\rho + p + \Pi)\, u_i u^k - (p + \Pi)\, \delta_i^k, \tag{1}$$

where ρ is the energy density, p the thermodynamic pressure, Π the bulk viscous pressure and u_i the four-velocity satisfying the condition $u_i u^i = 1$. We use units so that $8\pi G = c = 1$.

The gravitational field equations together with the continuity equation $T^k_{i;k} = 0$ imply

$$3H^2 = \rho\,, \qquad 2\dot{H} + 3H^2 = -p - \Pi\,, \qquad \dot{\rho} + 3\,(\rho + p)\, H = -3H\Pi\,, \tag{2}$$

where $H = \dot{a}/a$ is the Hubble parameter. The causal evolution equation for the bulk viscous pressure is given by [13]

$$\tau\dot{\Pi} + \Pi = -3\xi H - \frac{1}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right), \tag{3}$$

where T is the temperature, ξ the bulk viscosity coefficient and τ the relaxation time. Equation (3) arises as the simplest way (linear in Π) to satisfy the H theorem (*i.e.*, for the entropy production to be non-negative, $S^i_{;i} = \Pi^2/\xi T \geq 0$, where $S^i = eN^i - \frac{\tau\Pi^2}{2\xi T}u^i$ is the entropy flow vector, e is the entropy per particle and $N^i = nu^i$ is the particle flow vector) [6, 7].

In order to close the system of equations (2) we have to give the equation of state for p and specify T, ξ and τ. As usual, we assume the following phenomenological laws [13]:

$$p = (\gamma - 1)\,\rho\,, \qquad \xi = \alpha\rho^s, T = \beta\rho^r\,, \qquad \tau = \xi\rho^{-1} = \alpha\rho^{s-1}\,, \tag{4}$$

where $1 \leq \gamma \leq 2$, and $\alpha \geq 0$, $\beta \geq 0$, $r \geq 0$ and $s \geq 0$ are constants. Equations (4) are standard in cosmological models, whereas the equation for τ is a simple procedure to ensure that the speed of viscous pulses does not exceed the speed of light.

The requirement that the entropy is a state function imposes in the present model the constraint $r = (\gamma - 1)/\gamma$ [9] so that $0 \leq r \leq 1/2$ for $1 \leq \gamma \leq 2$.

The growth of the total comoving entropy Σ over a proper time interval (t_0, t) is given by [13]

$$\Sigma(t) - \Sigma(t_0) = -3k_{\mathrm{B}}^{-1}\int_{t_0}^{t}\Pi H a^3 T^{-1}\mathrm{d}t, \tag{5}$$

where k_{B} is the Boltzmann constant.

The Israel-Stewart-Hiscock theory is derived under the assumption that the thermodynamical state of the fluid is close to equilibrium, that is the non-equilibrium bulk viscous pressure should be small when compared to the local equilibrium pressure $|\Pi| \ll p = (\gamma - 1)\,\rho$ [14]. If this condition is violated then one is effectively assuming that the linear theory holds also in the non-linear regime far from equilibrium. For a fluid description of the matter, the condition ought to be satisfied.

To see if a cosmological model inflates or not, it is convenient to introduce the deceleration parameter $q = \frac{\mathrm{d}H^{-1}}{\mathrm{d}t} - 1 = \frac{\rho+3p+3\Pi}{2\rho}$. The positive sign of the deceleration parameter corresponds to standard decelerating models, whereas the negative sign indicates inflation.

With these assumptions the evolution equation for flat homogeneous causal bulk viscous cosmological models is

$$\ddot{H} + 3H\dot{H} + 3^{1-s}\alpha^{-1}H^{2-2s}\dot{H} - (1+r)H^{-1}\dot{H}^2 + \frac{9}{4}(\gamma-2)H^3 + \frac{3^{2-s}}{2\alpha}\gamma H^{4-2s} = 0. \tag{6}$$

By introducing a set of non-dimensional variables h and θ by means of the transformations $H = \alpha_0 h, t = \frac{2}{3\alpha_0}\theta$, with $\alpha_0 = \left(\frac{3^s\alpha}{2}\right)^{\frac{1}{1-2s}}, s \neq \frac{1}{2}$ and using the expression of r as a function of γ, eq. (6) takes the form

$$\frac{\mathrm{d}^2h}{\mathrm{d}\theta^2} + \left[2h + h^{2(1-s)}\right]\frac{\mathrm{d}h}{\mathrm{d}\theta} - (1+r)\,h^{-1}\left(\frac{\mathrm{d}h}{\mathrm{d}\theta}\right)^2 + \frac{2r-1}{1-r}h^3 + \frac{1}{1-r}h^{2(2-s)} = 0. \tag{7}$$

By denoting $n = (1-2s)/(1-r)$ and changing the variables according to $h = y^{1/(1-r)}$, $\eta = \int y^{1/(1-r)}\mathrm{d}\theta$ eq. (7) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}\eta^2} + (2+y^n)\frac{\mathrm{d}y}{\mathrm{d}\eta} + (2r-1)\,y + y^{n+1} = 0. \tag{8}$$

General solution for a stiff cosmological fluid. – We consider the case of a stiff cosmological fluid with equilibrium pressure equal to the energy density by taking the values of the parameters $\gamma = 2$ and $r = 1/2$. By introducing the substitutions $v = 1/u$ and $u = \mathrm{d}y/\mathrm{d}\eta$, eq. (8) can be transformed into a second-type Abel first-order differential equation:

$$\frac{\mathrm{d}v}{\mathrm{d}y} = (2+y^n)\,v^2 + y^{n+1}v^3. \tag{9}$$

A particular solution of eq. (9) for $n = 1$, $s = 1/4$ has been obtained in [15]. We consider now another case of a stiff cosmological fluid with bulk viscosity coefficient ξ linearly proportional to the energy density $\rho, \xi = \alpha\rho$. Consequently, $s = 1$ and $n = -2$ and hence eq. (9) becomes

$$\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{v^3}{y} + \left(2 + \frac{1}{y^2}\right)v^2 = -\frac{v^3}{B(y)} - \left[\frac{\mathrm{d}}{\mathrm{d}y}\frac{A(y)}{B(y)}\right]v^2, \tag{10}$$

where $A(y) = 2y^2 - 1$ and $B(y) = -y$. By introducing a new variable $\sigma = \frac{1}{v} - \frac{A(y)}{B(y)}$, eq. (10) can be written in the general form

$$\frac{\mathrm{d}y}{\mathrm{d}\sigma} = \sigma B(y) + A(y), \tag{11}$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}\sigma} = 2y^2 - \sigma y - 1. \tag{12}$$

Hence we have transformed the initial Abel-type equation into a Riccati differential equation. A particular solution of eq. (12) is given by

$$y = -\frac{\sigma}{1+\sigma^2}, \tag{13}$$

and therefore the general solution of eq. (12) is

$$y(\sigma) = -\sigma\Delta(\sigma) + \frac{\Delta^2(\sigma)\, e^{-\frac{\sigma^2}{2}}}{C - 2\int \Delta^2(\sigma)\, e^{-\frac{\sigma^2}{2}} \mathrm{d}\sigma}, \tag{14}$$

where $\Delta(\sigma) = \left(1+\sigma^2\right)^{-1}$ and C is a constant of integration.

Hence the general solution of the gravitational field equations for a Zeldovich causal bulk viscous fluid-filled flat FRW Universe, with bulk viscosity coefficient proportional to the energy density, can be obtained in the following exact parametric form, with σ taken as parameter:

$$\begin{aligned} t(\sigma) - t_0 &= -\frac{2}{3\alpha_0}\int \frac{\mathrm{d}\sigma}{y(\sigma)}, \qquad H(\sigma) = \alpha_0 y^2(\sigma)\,, \\ a(\sigma) &= a_0 \exp\left[-\frac{2}{3}\int y(\sigma)\,\mathrm{d}\sigma\right], \qquad \rho(\sigma) = 3\alpha_0^2 y^4(\sigma)\,, \qquad (15) \\ q(\sigma) &= 5 - \frac{3}{y^2(\sigma)} - \frac{3\sigma}{y(\sigma)}, \qquad \Pi(\sigma) = 6\alpha_0^2 y^2(\sigma)\left[y^2(\sigma) - \sigma y(\sigma) - 1\right], \\ \left|\frac{\Pi}{p}\right| &= 2\left|1 - \frac{1}{y^2(\sigma)} - \frac{\sigma}{y(\sigma)}\right|, \qquad (16) \\ \Sigma(\sigma) &= \Sigma(\sigma_0) + \frac{4\sqrt{3}\alpha_0 a_0^3}{k_{\mathrm{B}}\beta}\int y(\sigma)\left[y^2(\sigma) - \sigma y(\sigma) - 1\right]\exp\left[-2\int y(\sigma)\,\mathrm{d}\sigma\right]\mathrm{d}\sigma, \qquad (17) \end{aligned}$$

where a_0, t_0 and $\Sigma(\sigma_0)$ are constant of integration and $\alpha_0 = 2/3\alpha$.

The singular or non-singular character of the solution for all times $t \geq 0$ can be checked from the finite (infinite) character of the Ricci invariant $R_{ij}R^{ij}$ and Kretschmann scalar $R_{ijkl}R^{ijkl}$, given by

$$I = R_{ij}R^{ij} = 12\alpha_0^4 y^4\left[\left(3 + 3\sigma y - 5y^2\right)\left(3 + 3\sigma y - 4y^2\right) + y^4\right], \tag{18}$$

$$J = R_{ijkl}R^{ijkl} = 12\alpha_0^4 y^4\left[\left(3 + 3\sigma y - 5y^2\right)^2 + y^4\right]. \tag{19}$$

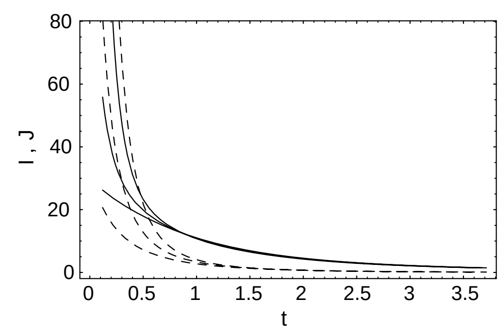

Fig. 1

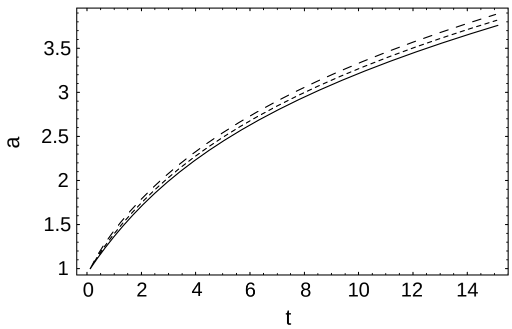

Fig. 2

Fig. 1 – Time evolution of the invariants $I = R_{ij}R^{ij}$ (solid curves) and $J = R_{ijkl}R^{ijkl}$ (dashed curves) for different values of the integration constant C ($\alpha_0 = 3/2$).

Fig. 2 – Time variation of the scale factor a for different values of the integration constant C: $C = -1.2$ (solid curve), $C = -1.1$ (dotted curve) and $C = -1$ (dashed curve) ($\alpha_0 = 3/2$).

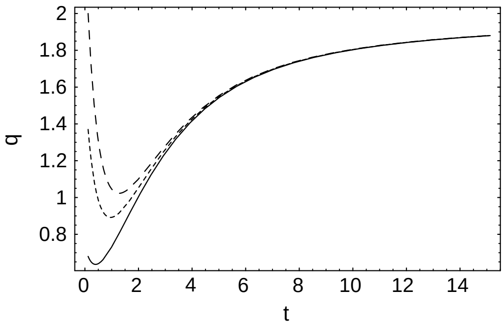

Fig. 3

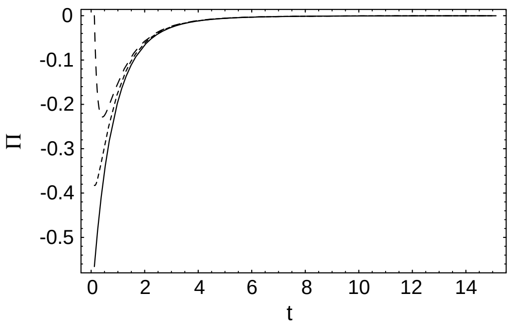

Fig. 4

Fig. 3 – Evolution of the deceleration parameter q for different values of the integration constant C: $C = -1.2$ (solid curve), $C = -1.1$ (dotted curve) and $C = -1$ (dashed curve) ($\alpha_0 = 3/2$).

Fig. 4 – Time behavior of the bulk viscous pressure Π for different values of the integration constant C: $C = -1.2$ (solid curve), $C = -1.1$ (dotted curve) and $C = -1$ (dashed curve) ($\alpha_0 = 3/2$).

Discussions and final remarks. – The evolution of the causal bulk viscous Zeldovich fluid filled flat Universe starts generally its evolution from a singular state, as can be seen from the singular behavior of the invariants I and J, presented, for different values of the integration constant C, in fig. 1. Generally, the dynamics of the Universe depends on the numerical values of C.

The behavior of the scale factor is presented in fig. 2, for some specific values of the integration constant. The evolution is expansionary for all times. At the initial moment the scale factor is zero, while the energy density tends to infinity.

The dynamics of the deceleration parameter, shown in fig. 3, indicates, for the chosen range of the integration constant, a non-inflationary behavior for all times, with $q > 0, \forall t \geq 0$.

The bulk viscous pressure Π is negative during the cosmological evolution, $\Pi < 0, \forall t \geq 0$,

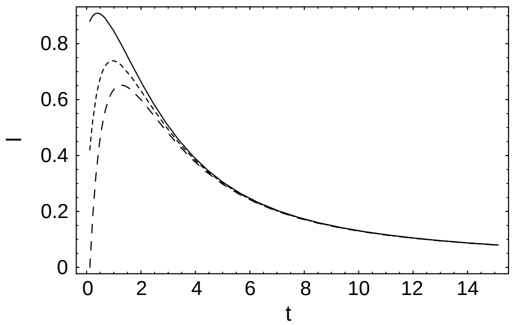

Fig. 5

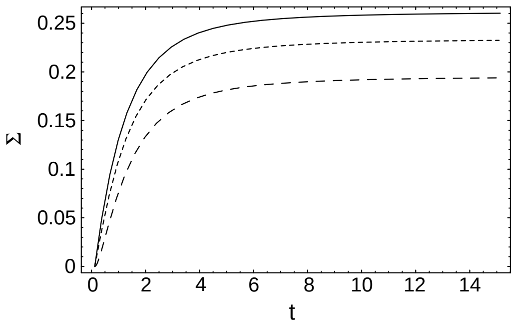

Fig. 6

Fig. 5 – Variation with time of the ratio $l = \frac{\Pi}{p}$ for different values of the integration constant C: $C = -1.2$ (solid curve), $C = -1.1$ (dotted curve) and $C = -1$ (dashed curve) ($\alpha_0 = 3/2$).

Fig. 6 – Dynamics of the comoving entropy Σ for different values of the integration constant C: $C = -1.2$ (solid curve), $C = -1.1$ (dotted curve) and $C = -1$ (dashed curve). We have used the normalizations $\alpha_0 = 3/2$ and $\frac{4\sqrt{3}\alpha_0 a_0^3}{k_{\mathrm{B}}\beta} = 1$.

as expected from a thermodynamic point of view. In the large-time limit, as can be seen from fig. 4, the viscous pressure tends to zero. In the same limit the bulk viscosity coefficient also becomes negligibly small.

The ratio l of the bulk and thermodynamic pressures, $l = |\Pi/p|$, is presented in fig. 5. For all times the condition $l < 1$ holds and therefore the model is thermodynamically consistent.

During the cosmological evolution a large amount of comoving entropy is produced, with the entropy Σ increasing in time and tending in the large-time limit to a constant value presented in fig. 6.

Due to the specific equation of state satisfied by the bulk viscosity coefficient, the relaxation time τ is a constant in the present model.

Most of the known exact solutions of the gravitational field equations with a causal bulk viscous cosmological fluid do not satisfy the required conditions of the thermodynamic consistency, leading to an inflationary behavior and violating the condition of smallness of bulk viscous pressure. The solution represented by eqs. (15)-(17) can consistently describe the early dynamics of a superdense post-inflationary era in the evolution of the matter-dominated Universe, when, as expected, the bulk viscous dissipative effects play an important role.

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