Density matrix negativity for two oscillators in an Agarwal bath

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(Received 19 December 2001; published 17 May 2002)

A system of two harmonic oscillators is placed in an Agarwal bath. The resulting quantum master equations are studied with the help of quantum characteristic functions. The density matrix positivity is investigated in view of the recent interest in searching for a sound quantum dissipation theory. An analytical criterion is derived for density matrix negativity for two uncoupled oscillators. It is found that, for an initial two-oscillator squeezed state with a real squeezing parameter s, density matrix negativity occurs for two uncoupled oscillators at temperatures lower than $\hbar\omega/(k_B \ln\coth|s|)$ with ω the oscillator frequency and k_B the Boltzmann factor. As a by-product an analytical expression is also obtained for determining the quantum separability of two uncoupled oscillators. The effects of interoscillator coupling on density matrix negativity are discussed.

DOI: 10.1103/PhysRevE.65.056120 PACS number(s): 05.30.−d, 03.67.Lx, 02.50.Ga

I. INTRODUCTION

In this paper we extend a previous discussion [1] on one primary oscillator in a dissipative bath to a system of two coupled primary oscillators. The aim is to study the density matrix positivity of the two-oscillator system under the influence of an Agarwal bath.

Three important attributes are desired for any bath theories of quantum dissipation, namely, translational symmetry, approach to eventual thermal equilibrium, and complete positivity for the reduced density matrix of the system. The bath model first proposed by Agarwal [2] (the Agarwal bath) guarantees translational invariance and thermal equilibration. For many initial conditions and moderate-to-high temperatures density matrix positivity is also supplied by the Agarwal bath. For these reasons the Agarwal bath model has seen a wide range of applications in physics and chemistry. However, the Agarwal bath model is not totally free of density matrix pathology, which usually occurs at low temperatures. For the case of one primary oscillator, density matrix negativity was found for a subset of initial conditions at low bath temperatures [3]. Quite recently, a pair of harmonic oscillators has emerged as an important paradigm for infinite dimensional systems, also known as systems of continuous variables, in quantum information and quantum communication theories. To our knowledge, systems of two harmonic oscillators dissipated by an Agarwal bath have only recently been investigated by us [4] in the context of quantum separability. One of the purposes here is to show the occurrence of density matrix negativity for a system of two harmonic oscillators in an Agarwal bath, and subsequently to determine the physical regimes within which density matrix positivity, translational symmetry, and the approach to thermal equilibrium coexist. Due to the wide range of applicability of the Agarwal bath in various branches of physics and chemistry, it is highly desirable to develop analytical criteria for determining density matrix negativity for both one-oscillator and two-oscillator systems. These aims have been fully achieved in this paper for initial one-mode and two-mode squeezed states.

The paper is organized as follows. In Sec. II we introduce quantum master equations for two oscillators in an Agarwal bath. In Sec. III the problem of the density matrix negativity of a one-oscillator system is reviewed, and an analytical criterion for density matrix negativity is derived. In Sec. IV the dissipative dynamics of two coupled oscillators in an Agarwal bath is examined, and an analytical expression is obtained for determining the density matrix negativity of two uncoupled oscillators initially in a highly entangled two-mode squeezed state. A discussion is presented in Sec. V with analytical results concerning quantum separability of bipartite continuous-variable systems.

II. MODEL

We start with a model Hamiltonian describing only one primary oscillator of frequency ω_0 and mass m coupled to a bath of secondary oscillators of frequency ω_k and mass m_k $(k\neq 0)$

$$\hat{H}_A=\hbar\omega_0 a^\dagger a+\sum_k \hbar\omega_k b_k^\dagger b_k+q\sum_k g_k q_k^b, \tag{2.1}$$

where q^s and q_k^b are the coordinate observables for the system and the bath oscillators, respectively, which are related to the corresponding boson operators by

$$q^s=\left(\frac{\hbar}{2m\omega_0}\right)^{1/2}(a^\dagger+a),$$

$$q_k^b=\left(\frac{\hbar}{2m_k\omega_k}\right)^{1/2}(b_k^\dagger+b_k), \tag{2.2}$$

and g_k are the coupling coefficients. The bath oscillators can be various phonon modes in a solid, or modes of vacuum radiation fields into which an excited atom decays via spontaneous emission. Adopting the rotating-wave approximation (RWA) widely used in fields such as quantum optics, the model Hamiltonian reduces to

$$\hat{H}_{\mathrm{RWA}}=\hbar\omega_0 a^\dagger a+\hbar\sum_k \omega_k b_k^\dagger b_k+\sum_k g_k(b_k^\dagger a+b_k a^\dagger). \tag{2.3}$$

We note that the rotating-wave approximation neglects the rapidly oscillating terms of Eq. (2.1). For simplicity we shall set $\hbar=1$ in the rest of the paper.

Agarwal has obtained a Schrödinger-representation master equation for the reduced density operator ρ in the limit of an infinite number of bath oscillators $[\Sigma_k \to \int d\omega_k f(\omega_k)]$ [2]

$$\frac{\partial\rho}{\partial t}+i\omega[a^\dagger,a,\rho]=\mathcal{L}_A(a,a^\dagger)\rho \equiv -\gamma\overline{n}[a+a^\dagger,[a+a^\dagger,\rho]] -\gamma(a[a+a^\dagger,\rho]-[a+a^\dagger,\rho]a^\dagger-2\rho), \tag{2.4}$$

where $\gamma=\pi f(\omega_0)|g_c(\omega_0)|^2$ is the damping constant, $f(\omega)$ is the density of bath oscillators, $g_c(\omega)$ is the continuum form of g_k, $\overline{n}=(e^{\omega/k_BT}-1)^{-1}$, ω is the renormalized frequency of ω_0 [5],

$$\omega=\omega_0+\mathcal{P}\int_0^\infty d\omega\,\frac{f(\omega)|g_c(\omega)|^2}{\omega-\omega_0}, \tag{2.5}$$

and $\mathcal{P}$ stands for the Cauchy principal part. Approximations assumed in deriving Eq. (2.4) include the Born approximation, which treats the bath effects in the lowest order, and the short memory hypothesis for the bath.

It is straightforward to generalize the one-oscillator Agarwal master equation (2.4) to a system of two oscillators. If there are two primary oscillators coupled to each other in the system with a Hamiltonian

$$\hat{H}_{\rm two}=\omega(a_1^\dagger a_1+a_2^\dagger a_2)+J(a_1^\dagger a_2+a_2^\dagger a_1), \tag{2.6}$$

where J is the coupling strength, and each oscillator is dissipated by the bath modes in the form of Eq. (2.4), the master equation for the reduced density matrix for the two oscillators can be written as

$$\frac{\partial\rho}{\partial t}+i[\hat{H}_{\rm two},\rho]=\sum_{i=1,2}\mathcal{L}_A(a_i,a_i^\dagger)\rho. \tag{2.7}$$

Following our previous treatment [1] we adopt a method of solution which utilizes the quantum characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)$ [6],

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)=\mathrm{Tr}(\rho e^{\lambda_1 a_1^\dagger}e^{-\lambda_1^* a_1}e^{\lambda_2 a_2^\dagger}e^{-\lambda_2^* a_2}), \tag{2.8}$$

where the trace is taken over the system of two oscillators (cf. Appendix A). Instead of dealing directly with the density matrix, we derive an equation of motion for the characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)$ which is then solved by the method of characteristics.

III. ONE PRIMARY OSCILLATOR

In this section our purpose is twofold. First, we briefly illustrate our method of solution for the simpler problem of one oscillator in an Agarwal bath [cf. Eq. (2.4)]. Second, we reconsider density matrix negativity for the one-oscillator case, thereby establishing a basis for later comparisons. An equation for the one-oscillator characteristic function $\chi^1(\lambda,\lambda^*)$ follows from the quantum master equation (2.4):

$$\frac{\partial\chi^1}{\partial t}+[-i\omega\lambda+\gamma(\lambda+\lambda^*)]\frac{\partial\chi^1}{\partial\lambda} +[i\omega\lambda^*+\gamma(\lambda+\lambda^*)]\frac{\partial\chi^1}{\partial\lambda^*} =-\gamma\overline{n}(\lambda+\lambda^*)^2\chi^1. \tag{3.1}$$

Equation (3.1) is solved by the method of characteristics [7]. Assuming the characteristic function

$$\chi^1(\lambda,\lambda^*)=\exp\left[\sum_{mn}C_{mn}(t)\lambda^m(-\lambda^*)^n\right], \tag{3.2}$$

where $C_{mn}(t)$ are the coefficients to be determined, one arrives at the set of differential equations for C_{mn}

$$\dot{C}_{10}=(i\omega-\gamma)C_{10}+\gamma C_{01}, \tag{3.3}$$

$$\dot{C}_{01}=(-i\omega-\gamma)C_{01}+\gamma C_{10}, \tag{3.4}$$

$$\dot{C}_{20}=2(i\omega-\gamma)C_{20}-\gamma(\overline{n}-C_{11}), \tag{3.5}$$

$$\dot{C}_{11}=2\gamma(\overline{n}-C_{11})+2\gamma(C_{02}+C_{20}), \tag{3.6}$$

$$\dot{C}_{02}=-2(i\omega+\gamma)C_{02}-\gamma(\overline{n}-C_{11}). \tag{3.7}$$

Analytical solutions to the above equations for all initial conditions can be found in Ref. [1] where illustrations of second order moments as functions of time are also given for an initial coherent state and an initial squeezed state. For the case of a squeezed-state start, i.e., the wave function at $t=0$ reads

$$|\psi(0)\rangle=|\xi\rangle=S(\xi)|0\rangle, \tag{3.8}$$

where $\xi=re^{i\theta}$, and

$$S(\xi)=\exp\left(\frac{1}{2}\xi^* a^2-\frac{1}{2}\xi a^{\dagger 2}\right), \tag{3.9}$$

the second order coefficients at $t=0$ are (cf. Appendix C)

$$C_{11}(t=0)=\sinh^2 r, \tag{3.10}$$

$$C_{20}(t=0)=-\frac{1}{4}e^{-i\theta}\sinh 2r, \tag{3.11}$$

$$C_{02}(t=0)=-\frac{1}{4}e^{i\theta}\sinh 2r. \tag{3.12}$$

To probe density matrix negativity, Talkner proposed an operator b [3]

$$b=e^r q+ie^{-r}p \tag{3.13}$$

where

$$p=a+a^{\dagger},\quad iq=a-a^{\dagger}. \tag{3.14}$$

One can show that

$$b|\xi\rangle=0. \tag{3.15}$$

It follows that at $t=0$

$$\mathrm{Tr}(b^{\dagger}b\rho)=\mathrm{Tr}(b^{\dagger}b|\xi\rangle\langle\xi|)=0. \tag{3.16}$$

For a small $\Delta t>0$,

$$\begin{aligned}\mathrm{Tr}(b^{\dagger}b\rho)&=\mathrm{Tr}(b^{\dagger}b\Delta\rho)=\Delta t\,\mathrm{Tr}(b^{\dagger}b\mathcal{L}_A|\xi\rangle\langle\xi|)\\&=\Delta t\,\gamma[e^{-2r}(2\bar{n}+1)-1].\end{aligned} \tag{3.17}$$

This implies that when

$$e^{2r}>2\bar{n}+1, \tag{3.18}$$

$\mathrm{Tr}(b^{\dagger}b\rho)$ turns negative at small t, indicating density matrix negativity. However, when this happens, the variance product $\Delta p\Delta q$ does not necessarily drop below the uncertainty limit. The requirement of the uncertainty principle is much weaker than that of density matrix positivity. This can be understood as follows. The uncertainty principle for the pair of operators p and q has the form

$$\Delta p\Delta q\geqslant 1, \tag{3.19}$$

while the physical density matrix requires

$$\langle p^2\rangle\langle q^2\rangle\geqslant|\langle pq\rangle|^2. \tag{3.20}$$

Equation (3.20) is a necessary and sufficient condition for the density matrix positivity of Gaussian states as shown by Talkner [3]. For an initial squeezed state

$$\langle p\rangle=0,\quad \langle q\rangle=0, \tag{3.21}$$

the positivity requirement is then

$$\Delta p\Delta q\geqslant|\langle pq\rangle|=\sqrt{1+|\langle a^2\rangle-\langle a^{\dagger 2}\rangle|^2}. \tag{3.22}$$

Therefore the positivity requirement is stronger than the uncertainty lower bound (3.19). In terms of the second order moments the positivity requirement can be written as

$$(2C_{11}+1)^2-4(C_{02}+C_{20})^2\geqslant 4|C_{02}-C_{20}|^2+1. \tag{3.23}$$

Introducing an auxiliary function $\zeta(t)$

$$\zeta(t)\equiv C_{11}^2+C_{11}-(C_{02}+C_{20})^2-|C_{02}-C_{20}|^2, \tag{3.24}$$

the positivity requirement is equivalent to

$$\zeta(t)\geqslant 0. \tag{3.25}$$

At $t=0$, for an initial squeezed state with a real squeezing parameter $\xi=r$, the auxiliary function $\zeta(t=0)$ equals zero. It follows that positivity of the density matrix ρ depends on the sign of the time derivative of $\zeta(t)$ at $t=0$. A derivation of $\zeta(t)$ and its time derivative is given in Appendix B from the solutions to Eqs. (3.5)–(3.7) [1]. Here we list an analytical expression for $\zeta(t)$:

$$\begin{aligned}\zeta(t)=&\,\bar{n}(\bar{n}+1)+e^{-4\gamma t}[\bar{n}^2-(2\bar{n}+1)\sinh^2 r]+e^{-2\gamma t}(\omega^2-\gamma^2)^{-1}(2\bar{n}+1)(\sinh^2 r-\bar{n})[\omega^2-\gamma^2\cos(2t\sqrt{\omega^2-\gamma^2})]\\&-\frac{1}{2}\gamma e^{-2\gamma t}(\omega^2-\gamma^2)^{-1/2}(2\bar{n}+1)\sinh 2r\,\sin(2t\sqrt{\omega^2-\gamma^2}).\end{aligned} \tag{3.26}$$

It is easily verified that $\zeta(0)=0$. The asymptotic value of $\zeta(t)$ at long times is apparently $\bar{n}(\bar{n}+1)$, which implies that for finite temperatures at some point in time the density matrix will regain positivity regardless of its anomalies at earlier times. The time derivative of $\zeta(t)$ at $t=0$ follows from Eq. (3.26):

$$\dot{\zeta}(0)=\gamma[(2\bar{n}+1)e^{-2r}-1]. \tag{3.27}$$

We hence recover the negativity criterion (3.18), first discussed by Talkner [3], from the set of exact solutions [1] of second order moments governed by Eqs. (3.5)–(3.7). However, far more information is revealed in the analytical expression (3.26) obtained for $\zeta(t)$, which allows negativity determination for all t. In addition, we prove in Appendix B that the density matrix is positive for all t if the temperature is high such that $2\bar{n}+1>e^{2r}$.

To gain a better understanding of the density matrix evolution in the whole time domain, we display in Fig. 1 the auxiliary function $\zeta(t)$ for $\gamma/\omega=0.1$, and three temperatures, namely, zero temperature $\bar{n}=0$ (top panel), $\bar{n}=(e-1)^{-1}$ (or $T=\omega/k_B$, middle panel), and the critical temperature $\bar{n}=(e^2-1)/2$ (bottom panel). The initial squeezed state has a real squeezing parameter $\xi=r=1$. The zero temperature case exhibits the worst density matrix negativity with $\zeta(t)$ remaining negative until approximately $\omega t=0.630\,43$. For $T=\omega/k_B$, $\zeta(t)$ becomes positive at approximately $\omega t=0.3745$. The critical temperature allows no density matrix negativity, and neither do any higher temperatures $\bar{n}>(e^2-1)/2$ as shown in Appendix B. We note that the duration of negativity has a slight dependence on γ. For example, for $\gamma/\omega=0.2$ and $T=\omega/k_B$ $(T=0)$, $\zeta(t)$ takes $\omega t=0.3763$ $(\omega t=0.640\,65)$ to recover from negativity. In general, for low temperatures such that $2\bar{n}+1<e^{2r}$, density matrix

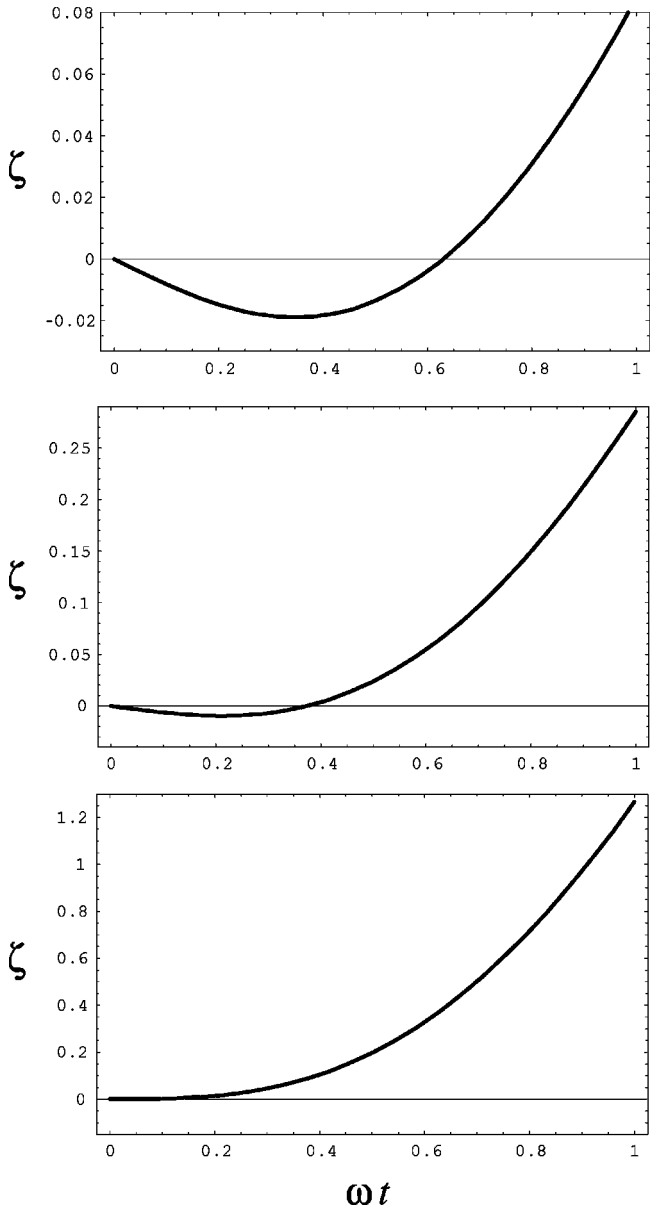

FIG. 1. Negativity of the one-oscillator density matrix is illustrated by the auxiliary function $\zeta(t)$ plotted as a function of ωt for $r=1$, $\gamma/\omega=0.1$, and three temperatures: $\bar{n}=0$ (top panel), $\bar{n}=1/(e-1)$ (middle panel), and $\bar{n}=(e^2-1)/1$ (bottom panel).

negativity may recur for a period of time. This will be discussed in detail in the next section.

IV. TWO PRIMARY OSCILLATORS

One of the most frequently mentioned examples in quantum optics and quantum information theories is the two-oscillator squeezed state, which was also utilized in a recent experimental realization of continuous-variable teleportation [8]. In this section we shall investigate the time evolution of the density matrix of a system of two coupled harmonic oscillators in an Agarwal bath initially in a two-oscillator squeezed state. Our focus is to solve a set of general equations for the time evolution of the Gaussian wave packets with arbitrary initial means and variances, and to investigate density matrix anomalies for the particular time evolution starting from an initial two-mode squeezed state. Results will be compared with the squeezed-state start of the one-oscillator case.

From the master equation for the reduced density matrix for the two oscillators one can derive an equation of motion for the characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)$:

$$\frac{\partial\chi}{\partial t}+\sum_{i=1,2}\left[\left(-i\omega\lambda_i\frac{\partial\chi}{\partial\lambda_i}+i\omega\lambda_i^*\frac{\partial\chi}{\partial\lambda_i^*}\right)+\gamma(\lambda_i+\lambda_i^*)\left(\frac{\partial\chi}{\partial\lambda_i}+\frac{\partial\chi}{\partial\lambda_i^*}\right)+\gamma\bar{n}(\lambda_i+\lambda_i^*)^2\chi\right]-iJ\left(\lambda_2\frac{\partial}{\partial\lambda_1}+\lambda_1\frac{\partial}{\partial\lambda_2}\right)\chi+iJ\left(\lambda_2^*\frac{\partial}{\partial\lambda_1^*}+\lambda_1^*\frac{\partial}{\partial\lambda_2^*}\right)\chi=0. \tag{4.1}$$

We assume that the characteristic function has the form

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,t)=\exp\left[\sum_{mnkl}C_{mn,kl}\lambda_1^m(-\lambda_1^*)^n\lambda_2^k(-\lambda_2^*)^l\right]. \tag{4.2}$$

The Gaussian wave packets are obtained by restricting $m+n+k+l\leqslant 2$ in the summation over m,n,k,l. Aside from the on-site first and second order moments introduced for each individual oscillator, four more cross moments, namely, $C_{10,01}$, $C_{01,10}$, $C_{10,10}$, and $C_{01,01}$, are added to account for cross correlations of the two oscillators. In the absence of a direct coupling between the two oscillators, i.e., $J=0$, a case which is of high relevance to the usual setups of quantum teleportation, on-site second order moments of each oscillator evolve according to Eqs. (3.5)–(3.7). However, quantum correlations between the two oscillators from an initial highly entangled state (such as a two-mode squeezed state) will persist for a significant period of time with cross moments playing an important role in determining the properties of the two-oscillator system density matrix. Various operator averages can be calculated from the first and second moments; for example, for the first oscillator, one has

$$\langle a_1\rangle=C_{01,00}, \tag{4.3}$$

$$\langle a_1^\dagger\rangle=C_{10,00}, \tag{4.4}$$

$$\langle a_1^2\rangle=2C_{02,00}+(C_{01,00})^2, \tag{4.5}$$

$$\langle a_1^{\dagger 2}\rangle=2C_{20,00}+(C_{10,00})^2, \tag{4.6}$$

$$\langle a_1^\dagger a_1\rangle=C_{11,00}+C_{10,00}C_{01,00} \tag{4.7}$$

(similar relations exist for the second oscillator); and for cross correlations

$$\langle a_1^\dagger a_2\rangle=C_{10,00}C_{00,01}+C_{10,01}, \tag{4.8}$$

$$\langle a_2^\dagger a_1\rangle = C_{01,00}C_{00,10} + C_{01,10}, \tag{4.9}$$

$$\langle a_1^\dagger a_2^\dagger\rangle = C_{10,00}C_{00,10} + C_{10,10}, \tag{4.10}$$

$$\langle a_1 a_2\rangle = C_{01,00}C_{00,01} + C_{01,01}. \tag{4.11}$$

Equations of motion for the moments can be derived from Eq. (4.1). Although for our purpose in this paper only second order moments are important to the density matrix positivity, we nonetheless discuss in passing first order moments, which are governed by four coupled differential equations:

$$\dot{C}_{10,00} = (i\omega - \gamma)C_{10,00} + \gamma C_{01,00} + iJC_{00,10}, \tag{4.12}$$

$$\dot{C}_{01,00} = (-i\omega - \gamma)C_{01,00} + \gamma C_{10,00} - iJC_{00,01}, \tag{4.13}$$

$$\dot{C}_{00,10} = (i\omega - \gamma)C_{00,10} + \gamma C_{00,01} + iJC_{10,00}, \tag{4.14}$$

$$\dot{C}_{00,01} = (-i\omega - \gamma)C_{00,01} + \gamma C_{00,10} - iJC_{01,00}. \tag{4.15}$$

Equations (4.12)–(4.15) obey the Ehrenfest theorem, which connects the time dependence of expectation values of canonically conjugate variables with the Hamiltonian equations of classical mechanics:

$$\frac{d}{dt}(C_{10,00} + C_{01,00} + C_{00,10} + C_{00,01}) = i(\omega + J)(C_{10,00} - C_{01,00} + C_{00,10} - C_{00,01}). \tag{4.16}$$

The equations that govern the time evolution of second order moments were first derived in Ref. [4]:

$$\dot{C}_{20,00} = 2(i\omega - \gamma)C_{20,00} - \gamma(\bar{n} - C_{11,00}) + iJC_{10,10}, \tag{4.17}$$

$$\dot{C}_{11,00} = 2\gamma(\bar{n} - C_{11,00}) + 2\gamma(C_{02,00} + C_{20,00}) + iJ(C_{01,10} - C_{10,01}), \tag{4.18}$$

$$\dot{C}_{02,00} = -2(i\omega + \gamma)C_{02,00} - \gamma(\bar{n} - C_{11,00}) - iJC_{01,01}, \tag{4.19}$$

$$\dot{C}_{10,10} = 2i\omega C_{10,10} + 2iJ(C_{20,00} + C_{00,20}) + \gamma(C_{10,01} + C_{01,10} - 2C_{10,10}), \tag{4.20}$$

$$\dot{C}_{10,01} = iJ(C_{00,11} - C_{11,00}) + \gamma(C_{01,01} + C_{10,10} - 2C_{10,01}), \tag{4.21}$$

$$\dot{C}_{01,10} = -iJ(C_{00,11} - C_{11,00}) + \gamma(C_{01,01} + C_{10,10} - 2C_{01,10}), \tag{4.22}$$

$$\dot{C}_{01,01} = -2i\omega C_{01,01} - 2iJ(C_{02,00} + C_{00,02}) + \gamma(C_{10,01} + C_{01,10} - 2C_{01,01}), \tag{4.23}$$

$$\dot{C}_{00,20} = 2(i\omega - \gamma)C_{00,20} - \gamma(\bar{n} - C_{00,11}) + iJC_{10,10}, \tag{4.24}$$

$$\dot{C}_{00,11} = 2\gamma(\bar{n} - C_{00,11}) + 2\gamma(C_{00,02} + C_{00,20}) - iJ(C_{01,10} - C_{10,01}), \tag{4.25}$$

$$\dot{C}_{00,02} = -2(i\omega + \gamma)C_{00,02} - \gamma(\bar{n} - C_{00,11}) - iJC_{01,01}. \tag{4.26}$$

For a state with initial zero first order moments, the dissipative dynamics and the properties of the Gaussian density matrix as a function of time are completely determined by the above ten coupled equations. Solutions for on-site moments were already listed in Ref. [1] for all initial Gaussian wave packets including that of the two-mode squeezed state. For zero direct coupling $J=0$, analytical solutions for the cross moments are given in Appendix D.

To further understand the physical significance of the second order moments, let us define S as the sum of four out of the ten second order moments:

$$S \equiv C_{10,10} + C_{01,01} + C_{11,00} + C_{00,11}, \tag{4.27}$$

and A as the sum of the remaining six second order moments:

$$A \equiv C_{20,00} + C_{02,00} + C_{00,02} + C_{00,20} + C_{10,01} + C_{01,10}. \tag{4.28}$$

Consider two Einstein-Podolsky-Rosen type operators $\hat{u}$ and $\hat{v}$:

$$\sqrt{2}\hat{u} = a_1 + a_1^\dagger + a_2 + a_2^\dagger, \tag{4.29}$$

$$i\sqrt{2}\hat{v} = a_1 - a_1^\dagger - a_2 + a_2^\dagger. \tag{4.30}$$

The variances of the operators $\hat{u}$ and $\hat{v}$ can be conveniently expressed in terms of S and A:

$$\langle(\Delta\hat{u})^2\rangle = 1 + S + A, \tag{4.31}$$

$$\langle(\Delta\hat{v})^2\rangle = 1 + S - A. \tag{4.32}$$

From the set of coupled equations for second moments one can derive equations of motion for S in Eq. (4.27) and A in Eq. (4.28):

$$\dot{S} = -2\gamma(S - 2\bar{n}) + 2\gamma A + 2iJ(C_{20,00} - C_{02,00} + C_{00,20} - C_{00,02}) + 2i\omega(C_{10,10} - C_{01,01}), \tag{4.33}$$

$$\dot{A} = -2\gamma A + 2\gamma(S - 2\bar{n}) + 2iJ(C_{10,10} - C_{01,01}) + 2i\omega(C_{20,00} - C_{02,00} + C_{00,20} - C_{00,02}). \quad (4.34)$$

It follows that S asymptotically goes to $2\bar{n}$ at long times, and A to zero. Adding the above two equations,

$$\frac{d}{dt}(S+A) = 2i(J+\omega)(C_{20,00} - C_{02,00} + C_{00,20} - C_{00,02} + C_{10,10} - C_{01,01}). \quad (4.35)$$

Our initial state here is the highly entangled two-oscillator squeezed state

$$e^{-s(a_1^\dagger a_2^\dagger - a_1 a_2)}|0\rangle \quad (4.36)$$

with s a real number and $|0\rangle$ the vacuum state for both oscillators. The characteristic function at time $t=0$ has the form

$$\chi(\lambda_1, \lambda_1^*, \lambda_2, \lambda_2^*, 0) = \exp[-\tfrac{1}{2}\sinh 2s(\lambda_1\lambda_2 + \lambda_1^*\lambda_2^*) - \sinh^2 s(|\lambda_1|^2 + |\lambda_2|^2)]. \quad (4.37)$$

Details of the derivation of Eq. (4.37) can be conveniently found in Appendix C. At $t=0$ there are only four nonzero second moments, namely,

$$C_{11,00}(0) = C_{00,11}(0) = \sinh^2 s, \quad (4.38)$$

$$C_{10,10}(0) = C_{01,01}(0) = -\frac{1}{2}\sinh 2s. \quad (4.39)$$

For zero direct coupling $J=0$, analytical expressions of all nonzero second order moments can be obtained for an initial two-mode squeezed state (cf. Appendix E). Those expressions will be used to derive an analytical criterion for density matrix negativity for two uncoupled oscillators. Analytical expressions of S and A for an initial two-squeezed state and zero coupling $J=0$ are also listed in Appendix E.

The criterion for density matrix positivity is best illustrated in the matrix forms of the second order moments. The second order terms in the exponent of Eq. (4.2) can be written in a matrix form:

$$\frac{1}{2}(\lambda_1^* \ \lambda_1 \ \lambda_2^* \ \lambda_2) \times \begin{pmatrix} -C_{11,00} & 2C_{02,00} & -C_{10,01} & C_{10,10} \\ 2C_{20,00} & -C_{11,00} & C_{10,10} & -C_{10,01} \\ -C_{10,01} & C_{01,01} & -C_{00,11} & 2C_{00,02} \\ C_{10,10} & -C_{01,10} & 2C_{00,20} & -C_{00,11} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_1^* \\ \lambda_2 \\ \lambda_2^* \end{pmatrix}. \quad (4.40)$$

One can also write the second order exponent in the form

$$\frac{1}{2}(\lambda_1^R \ \lambda_1^I \ \lambda_2^R \ \lambda_2^I) M (\lambda_1^R \ \lambda_1^I \ \lambda_2^R \ \lambda_2^I)^T, \quad (4.41)$$

where the new basis is related to $(\lambda_1^* \ \lambda_1 \ \lambda_2^* \ \lambda_2)^T$ by

$$\begin{pmatrix} \lambda_1^R \\ \lambda_1^I \\ \lambda_2^R \\ \lambda_2^I \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{i}{2} & \frac{i}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{i}{2} & \frac{i}{2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_1^* \\ \lambda_2 \\ \lambda_2^* \end{pmatrix} \quad (4.42)$$

and M is a real symmetric matrix

$$M = \begin{pmatrix} V_1 & V_{12} \\ V_{12}^T & V_2 \end{pmatrix} \quad (4.43)$$

with

$$V_1 = 2\begin{pmatrix} C_{20,00} + C_{02,00} - C_{11,00} & i(C_{20,00} - C_{02,00}) \\ i(C_{20,00} - C_{02,00}) & -C_{02,00} - C_{20,00} - C_{11,00} \end{pmatrix}, \quad (4.44)$$

$$V_2 = 2\begin{pmatrix} C_{00,20} + C_{00,02} - C_{00,11} & i(C_{00,20} - C_{00,02}) \\ i(C_{00,20} - C_{00,02}) & -C_{00,02} - C_{00,20} - C_{00,11} \end{pmatrix}, \quad (4.45)$$

and

$$V_{12} = \begin{pmatrix} C_{10,10} + C_{01,01} - C_{10,01} - C_{01,10} & i(C_{10,10} - C_{01,01} + C_{10,01} - C_{01,10}) \\ i(C_{10,10} - C_{01,01} - C_{10,01} + C_{01,10}) & -C_{10,10} - C_{01,01} - C_{10,01} - C_{01,10} \end{pmatrix}. \quad (4.46)$$

The corresponding Wigner characteristic function $\chi^W(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)$ can also be written in the matrix form

$$\chi^W(\lambda_1^R,\lambda_1^I,\lambda_2^R,\lambda_2^I)=\exp\left[-\frac{1}{2}(\lambda_1^R\ \lambda_1^I\ \lambda_2^R\ \lambda_2^I)\times(I-M)(\lambda_1^R\ \lambda_1^I\ \lambda_2^R\ \lambda_2^I)^T\right] \tag{4.47}$$

with I the identity matrix.

The criterion for density matrix positivity can be written in the form [9]

$$Z(t)>0 \tag{4.48}$$

where the auxiliary function $Z(t)$ is defined as

$$Z(t)\equiv \mathrm{Det}[I-V_1]\mathrm{Det}[I-V_2]+(1-\mathrm{Det}[V_{12}])^2 - \mathrm{Tr}[(I-V_1)KV_{12}K(I-V_2)V_{12}^TK]-\mathrm{Det}[I-V_1] - \mathrm{Det}[I-V_2] \tag{4.49}$$

with the matrix K

$$K=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.50}$$

For Gaussian states, which are the focus of the discussion here, Eq. (4.48) is a necessary and sufficient condition for density matrix positivity [9]. Since the determinant of $I-M$ can be written as

$$\mathrm{Det}[I-M]=\mathrm{Det}[I-V_1]\mathrm{Det}[I-V_2]+\mathrm{Det}^2[V_{12}] - \mathrm{Tr}[(I-V_1)KV_{12}K(I-V_2)V_{12}^TK], \tag{4.51}$$

the positivity criterion is equivalent to

$$\mathrm{Det}[M-I]+2\,\mathrm{Det}[V_{12}]-\mathrm{Det}[V_1-I]-\mathrm{Det}[V_2-I]<1. \tag{4.52}$$

For an initial two-mode squeezed state the auxiliary function $Z(0)$ equals zero. Unlike the case of one primary oscillator, it turns out that $\dot{Z}(0)$ vanishes as well (cf. Appendix F). Therefore, for the density matrix to exhibit negativity at small t, one needs to show that

$$\ddot{Z}(0)<0. \tag{4.53}$$

Fortunately, for the set of initial conditions with four nonzero second order moments (4.38) and (4.39), and zero direct coupling $J=0$, an analytical expression for $Z(t)$ can be obtained from the ten second order moments after some effort:

$$Z(t)=16\{\bar{n}(\bar{n}+1)+e^{-4\gamma t}[\bar{n}^2-(2\bar{n}+1)\sinh^2 s] + e^{-2\gamma t}(2\bar{n}+1)(\sinh^2 s-\bar{n})(\omega^2-\gamma^2)^{-1}[\omega^2-\gamma^2\cos(2t\sqrt{\omega^2-\gamma^2})]\}^2 - 4\gamma^2e^{-4\gamma t}(\omega^2-\gamma^2)^{-1}(2\bar{n}+1)^2\sinh^2 2s\sin^2(2t\sqrt{\omega^2-\gamma^2}). \tag{4.54}$$

This analytical expression for $Z(t)$ is one of our main results, and allows positivity determination of the two-oscillator density matrix for all t. Expanded up to third order in t around $t=0$, the auxiliary function $Z(t)$ has a surprisingly simple form:

$$Z(t\approx 0)=16\gamma^2t^2(1-4\gamma t)(2\bar{n}+1-e^{-2s})\times(2\bar{n}+1-e^{2s}). \tag{4.55}$$

The criterion (4.53) based on the second derivative $\ddot{Z}(t)$ at $t=0$ reads

$$(2\bar{n}+1-e^{-2s})(2\bar{n}+1-e^{2s})>0. \tag{4.56}$$

For $s>0$ ($s<0$), this implies that $2\bar{n}+1>e^{2s}$ ($2\bar{n}+1>e^{-2s}$) in order to maintain density matrix positivity at the beginning of the time evolution. For any lower temperatures, density matrix negativity will be encountered at small t. In the remainder of the section, without loss of generality we shall assume $s>0$.

The next question is how long density matrix negativity lasts and if it recurs for low temperatures. To obtain information on the duration and recurrence of negativity, we denote the roots of $Z(t)=0$ by t_0:

$$(\sinh^2 s-\bar{n})(\omega^2-\gamma^2)^{-1}[\omega^2-\gamma^2\cos(2t_0\sqrt{\omega^2-\gamma^2})] + \bar{n}(2\bar{n}+1)^{-1}[(\bar{n}+1)e^{2\gamma t_0}+\bar{n}e^{-2\gamma t_0}] = e^{-2\gamma t_0}\sinh^2 s\pm\frac{1}{2}\gamma\sinh 2s(\omega^2-\gamma^2)^{-1/2} \times\sin(2t_0\sqrt{\omega^2-t^2}). \tag{4.57}$$

Note the second term on the left hand side of Eq. (4.57) which contains $e^{2\gamma t_0}$ and are hence unbounded. For finite temperatures, $\bar{n}\neq 0$, this term makes sure that Eq. (4.57) cannot be satisfied for large t_0. Therefore, for finite temperatures, the density matrix retains positivity (with no crossovers to negativity) at long times since $Z(\infty)=16\bar{n}^2(\bar{n}+1)^2>0$. At zero temperature, the equation for t_0 can be simplified as

$$\frac{\omega^2-\gamma^2\cos(2t_0\sqrt{\omega^2-\gamma^2})}{\omega^2-\gamma^2}$$

$$=e^{-2\gamma t_0}\pm\frac{\gamma\coth s}{\sqrt{\omega^2-\gamma^2}}\sin(2t_0\sqrt{\omega^2-\gamma^2}). \quad (4.58)$$

Depending on the size of s relative to energies ω and γ, the equation admits one to an infinite number of solutions. This becomes apparent if Eq. (4.58) is rewritten as

$$\frac{\omega^2}{\omega^2-\gamma^2}-e^{-2\gamma t_0}=\frac{\gamma^2}{\omega^2-\gamma^2}\sqrt{1+\frac{\omega^2-\gamma^2}{\gamma^2\tanh^2 s}}\times\sin(\alpha\pm 2t_0\sqrt{\omega^2-\gamma^2}), \quad (4.59)$$

where

$$\alpha=\arctan\frac{\gamma\tanh s}{\sqrt{\omega^2-\gamma^2}}. \quad (4.60)$$

For a sufficiently small squeezing parameter s, such that

$$\tanh s\leqslant\frac{\gamma}{\sqrt{\omega^2+\gamma^2}}, \quad (4.61)$$

there exist an infinity number of t_0 that satisfy Eq. (4.59). It follows that at zero temperature density matrix negativity recurs to $t=\infty$ if Eq. (4.61) is satisfied. For a large squeezing parameter s such that $\tanh s\sim 1$, there is at least one approximate nonzero solution

$$t_0\approx\sqrt{\frac{1.5(\coth s-1)}{(\omega^2-\gamma^2)\coth s+\gamma^2}}. \quad (4.62)$$

The reader is referred to Appendix F for further discussions on t_0 and $Z(t)$.

Since for an initial two-mode squeezed state the critical temperature above which no density matrix negativity occurs is given by $2\bar{n}+1=e^{2|s|}$, a condition closely resembling $2\bar{n}+1=e^{2r}$ for the one-oscillator case with an initial one-mode squeezed state, we are tempted to compare as well the two auxiliary functions, namely, $\zeta(t)$ for one oscillator and $Z(t)$ for two oscillators. As it turns out, the equation $\zeta(t_0)=0$ for the one-oscillator case will have the same form as Eq. (4.57) if the positive sign is taken for the second term on the right hand side of Eq. (4.57), and the squeezing parameter s is simply replaced by r:

$$(\sinh^2 r-\bar{n})(\omega^2-\gamma^2)^{-1}[\omega^2-\gamma^2\cos(2t_0\sqrt{\omega^2-\gamma^2})]+\bar{n}(2\bar{n}+1)^{-1}[(\bar{n}+1)e^{2\gamma t_0}+\bar{n}e^{-2\gamma t_0}]$$

$$=e^{-2\gamma t_0}\sinh^2 r+\frac{1}{2}\gamma\sinh 2r(\omega^2-\gamma^2)^{-1/2}\times\sin(2t_0\sqrt{\omega^2-\gamma^2}). \quad (4.63)$$

Therefore, the results we have obtained so far for t_0 in this section can be applied to the one-oscillator case provided that s is substituted by r, and only the positive sign is taken for the second term on the right hand side of Eq. (4.57). The number of t_0 that satisfy Eq. (4.57), however, is about twice the number of t_0 that satisfy Eq. (4.63) for the same bath temperature and the same values for s and r.

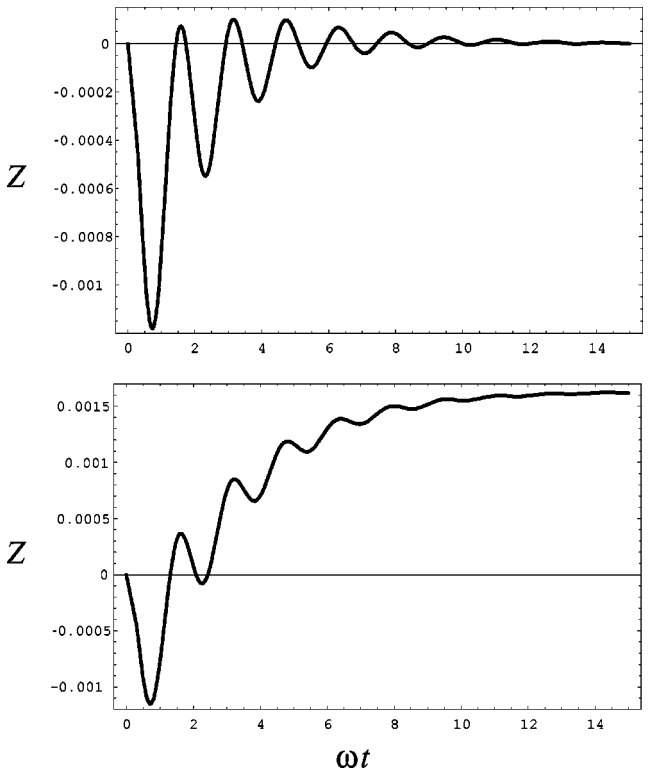

FIG. 2. Negativity of the two-oscillator density matrix is illustrated by the auxiliary function $Z(t)$ plotted as a function of ωt for $s=0.1$, $\gamma/\omega=0.1$, and two temperatures: $\bar{n}=0$ (upper panel) and $\bar{n}=0.01$ (lower panel).

The similarity between $\zeta(t)$ for one oscillator and $Z(t)$ for two oscillators can also be exploited to obtain a proof of the $Z(t)$ non-negativity for two uncoupled oscillators and $2\bar{n}+1\geqslant e^{2|s|}$ which can be based entirely on the proof of the $\zeta(t)$ non-negativity in Appendix B. It is then concluded that for $2\bar{n}+1\geqslant e^{2|s|}$ and $J=0$, the two-oscillator density matrix from an initial two-mode squeezed state does not encounter negativity for all t.

In Fig. 2 the auxiliary function $Z(t)$ of two uncoupled oscillators is plotted for $s=0.1$, $\gamma/\omega=0.1$, and two temperatures $\bar{n}=0$ (upper panel) and $\bar{n}=0.01$ (lower panel). For zero temperature $\bar{n}=0$, density matrix negativity recurs for the entire time interval displayed. Raising the temperature to $\bar{n}=0.01$ drastically reduces the negativity recurrence so that density matrix negativity reappears briefly only once (near $\omega t=2.3$) as shown in the lower panel.

The case of nonzero interoscillator coupling has to be investigated numerically at this stage as analytical solutions to the ten coupled equations remain elusive. In Fig. 3 we display t_0 as a function of the interoscillator coupling J for $s=1/2$, $\gamma/\omega=0.1$, and $\bar{n}=1/2$. For this set of s and $\bar{n}$, only one nonzero t_0 exists. It is found that positive J's reduce the amount of time t_0 the density matrix spends in negativity

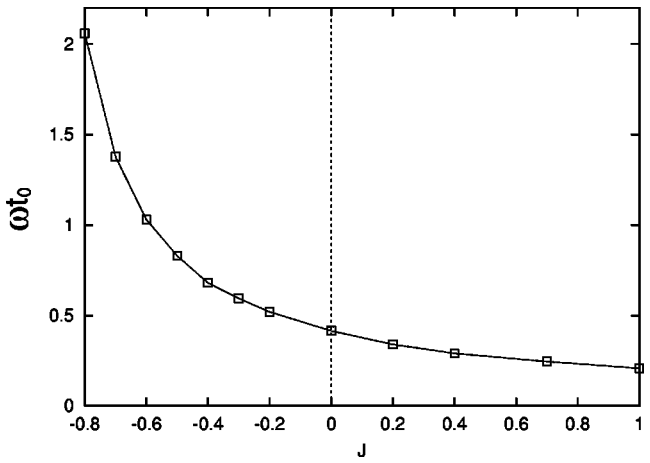

FIG. 3. The duration of negativity t_0 as a function of the interoscillator coupling strength J for $s=1/2$, $\gamma/\omega=0.1$, and $\bar{n}=1/2$.

while negative J's prolong t_0. The coupling J, however, does not change the critical temperature below which density matrix negativity commences.

V. DISCUSSION

As an extension of our previous work [1] we have examined the detailed behavior of density matrices for two harmonic oscillators evolving under the influence of a form of dissipative bath first proposed by Agarwal [2]. The Agarwal bath has the appearance of the perfect bath from the physical perspective: it preserves the translational invariance of the system, and delivers the eventual thermal equilibrium to the dissipative process. For one primary oscillator the Agarwal bath model has been applied to a variety of fields despite the well-known fact that it violates the positivity requirement of the reduced density matrix for certain initial conditions [3,10–12]. Recently, in the emerging literature of quantum information and communication [8,9,13–17] an enormous amount of interest has arisen in bipartite canonical systems of continuous variables, which are often synonymous with a pair of harmonic oscillators. The dynamics of a coupled pair of harmonic oscillators in a dissipative bath have also been an important problem in physics and chemistry. In this paper we have identified density matrix anomalies for two oscillators in an Agarwal bath initially in a two-mode squeezed state. Analytical criteria have also been developed for determining density matrix negativity.

Among our findings an analytical form of $Z(t)$ is obtained for zero direct coupling $J=0$, which can be used to probe the density matrix negativity of two-oscillator systems. For an initial two-oscillator squeezed state with a real squeezing parameter s, density matrix negativity occurs for two uncoupled oscillators at temperatures lower than $\hbar\omega/(k_B \ln\coth|s|)$. An analytical expression was also found for $\zeta(t)$ for a one-oscillator system initially in a one-mode squeezed state with a real squeezing parameter r. Proofs of sustained density matrix positivity have been given for bath temperatures higher than $\hbar\omega/(k_B \ln\coth r)$ for the one-oscillator system, and $\hbar\omega/(k_B \ln\coth|s|)$ for the two-oscillator system. The effects of the direct interoscillator coupling J on density matrix negativity were studied numerically.

The origin of the density matrix negativity documented in this paper lies in the Markovian approximation in the Agarwal master equation [2]. Suárez, Silbey, and Oppenheim have studied the validity of the Markovian approximation in the context of relaxation theory of two-level systems [18]. It was shown that anomalies in the Markovian evolution in the weak coupling limit can be remedied by supplementing a slippage in the initial conditions. It is concluded that nonlocal memory effects need to be taken into account at the beginning of the evolution, and only after this transient time can the reduced dynamics be properly described by Markovian evolution with the density matrix free of negativity [19]. The merits (and shortcomings) of the Agarwal bath have been recently compared in detail [20] with those of other various relaxation approaches.

Entanglement (or inseparability) highlights quantum nonlocality, and is of great importance to quantum information theory. A separable density matrix can be represented by a mixture of direct-product states. Measurements made on states that are separable exhibit only classical correlations. A bipartite density matrix is separable if it can be written in the form

$$\rho=\sum_i p_i \rho_A^i \otimes \rho_B^i, \tag{5.1}$$

where $p_i>0$, and $\Sigma_i p_i=1$. An inseparability criterion concerning positivity of the partial transpose (PPT) of the density matrix was proposed by Peres [13], and later shown by Horodecki *et al.* to be a necessary and sufficient condition for inseparability of 2×2 or 2×3 systems [14]. When it is applied to Gaussian states in a two-oscillator system, the density matrix is inseparable if and only if [9]

$$\Xi(t)<0, \tag{5.2}$$

where the auxiliary function $\Xi(t)$ is defined as

$$\begin{aligned}\Xi(t)\equiv&\,\mathrm{Det}[I-V_1]\mathrm{Det}[I-V_2]+(1-|\mathrm{Det}[V_{12}]|)^2\\&-\mathrm{Tr}[(I-V_1)KV_{12}K(I-V_2)V_{12}^TK]\\&-\mathrm{Det}[I-V_1]-\mathrm{Det}[I-V_2].\end{aligned} \tag{5.3}$$

For an initial two-mode squeezed state with a real squeezing parameter s and zero direct coupling $J=0$,

$$\mathrm{Det}[V_{12}]=-e^{-4\gamma t}\sinh^2 2s\leqslant 0. \tag{5.4}$$

It follows that in order for the two-oscillator system to be separable, it is required that

$$0 \leqslant \Xi(t) = Z(t) - 4e^{-4\gamma t}\sinh^2 2s = 16\{\bar{n}(\bar{n}+1) + e^{-4\gamma t}[\bar{n}^2 - (2\bar{n}+1)\sinh^2 s] + e^{-2\gamma t}(2\bar{n}+1)(\sinh^2 s - \bar{n})(\omega^2-\gamma^2)^{-1}[\omega^2 - \gamma^2\cos(2t\sqrt{\omega^2-\gamma^2})]\}^2 - 4e^{-4\gamma t}\sinh^2 2s[1 + (2\bar{n}+1)^2\gamma^2(\omega^2-\gamma^2)^{-1}\sin^2(2t\sqrt{\omega^2-\gamma^2})]. \quad (5.5)$$

Therefore, as a by-product, our examination of the density matrix negativity here has resulted in an analytical expression for the determination of quantum separability of the bipartite system.

Gaussian states evolving under the Agarwal bath are adequately described by normal-ordered characteristic functions. Our approach can be readily extended to tripartite systems of harmonic oscillators. Trimode entanglement has attracted much attention lately with the advent of experiments of continuous-variable teleportation [14]. Separability properties of multiparty systems are intrinsically more complex and intriguing. Work along this direction is currently underway.

ACKNOWLEDGMENT

Support from the Hong Kong Research Grant Council (RGC) is gratefully acknowledged.

APPENDIX A: THE CHARACTERISTIC FUNCTIONS

The two-oscillator characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)$ [Eq. (2.8)] is a direct generalization of the characteristic function for one oscillator $\chi^1(\lambda,\lambda^*)$ [21]:

$$\chi^1(\lambda,\lambda^*) = \mathrm{Tr}(\rho e^{\lambda a^\dagger} e^{-\lambda^* a}). \quad (A1)$$

For one oscillator, the quantum characteristic function $\chi^1(\lambda,\lambda^*)$ is the Fourier transform of the phase space distribution function $P_1(z,z^*)$ of the density matrix ρ,

$$\chi^1(\lambda,\lambda^*) = \int d^2z \exp(\lambda z^* - \lambda^* z) P_1(z,z^*). \quad (A2)$$

The phase space distribution function $P_1(z,z^*)$ is also called the Glauber-Sudarshan P representation of the density matrix ρ, which plays the role of the quasiprobability:

$$\rho = \int d^2z\, P_1(z,z^*)|z\rangle\langle z|. \quad (A3)$$

For the case of two oscillators similar relations exist between the characteristic function $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)$ and the phase space distribution function $P(z_1,z_1^*,z_2,z_2^*)$ of the density matrix ρ:

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*) = \int d^2z_1\, d^2z_2 \exp(\lambda_1 z_1^* - \lambda_1^* z_1 + \lambda_2 z_2^* - \lambda_2^* z_2) \times P(z_1,z_1^*,z_2,z_2^*). \quad (A4)$$

The often-used Wigner distribution function $W(z_1,z_1^*,z_2,z_2^*)$ is the Fourier transform of the Wigner characteristic function $\chi^{\mathrm{W}}(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)$ which is related to $\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*)$ by a simple factor [22]:

$$\chi^{\mathrm{W}}(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*) = \int d^2z_1\, d^2z_2 \exp(\lambda_1 z_1^* - \lambda_1^* z_1 + \lambda_2 z_2^* - \lambda_2^* z_2) \times W(z_1,z_1^*,z_2,z_2^*) \quad (A5)$$

$$= e^{-|\lambda_1|^2/2 - |\lambda_2|^2/2}\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*). \quad (A6)$$

The Wigner distribution function $W(z_1,z_1^*,z_2,z_2^*)$ gives a description of quantum states in close resemblance to the phase space classical descriptions, which is related to the Weyl classical-quantum correspondence [23].

APPENDIX B: DENSITY MATRIX NEGATIVITY FOR ONE PRIMARY OSCILLATOR

In this appendix we describe derivations of the auxiliary function $\zeta(t)$ and its time derivative from the analytical solutions to Eqs. (3.5)–(3.7) [1]. A proof that $\zeta(t) \geqslant 0$ for $2\bar{n} + 1 \geqslant e^{2r}$ is also given here.

The auxiliary function $\zeta(t)$ can be written as

$$\zeta(t) = C_{11}(C_{11}+1) - 4|C_{02}|^2. \quad (B1)$$

Making use of the expressions for C_{11} and C_{02} from Ref. [1] with the initial conditions $C_{11}(0) = \sinh^2 r$ and $C_{02}(0) = -\sinh 2r/4$, one quickly arrives at Eq. (3.26) after some straightforward simplifications. The time derivative of $\zeta(t)$ which follows from Eq. (3.26) takes the form

$$\frac{d\zeta}{dt} = -2\gamma e^{-4\gamma t}[1+2\bar{n}(\bar{n}+1)] + 2\gamma e^{-4\gamma t}(2\bar{n}+1)\cosh 2r - 2\gamma e^{-2\gamma t}(2\bar{n}+1)(\sinh^2 r - \bar{n})(\omega^2-\gamma^2)^{-1}$$
$$\times[\omega^2 - \gamma^2\cos(2t\sqrt{\omega^2-\gamma^2})] - \gamma^2 e^{-2\gamma t}(2\bar{n}+1)(2\bar{n}+1-e^{2r})(\omega^2-\gamma^2)^{-1/2}\sin(2t\sqrt{\omega^2-\gamma^2})$$
$$-\gamma e^{-2\gamma t}(2\bar{n}+1)\sinh 2r\cos(2t\sqrt{\omega^2-\gamma^2}). \tag{B2}$$

At $t=0$, only $C_{11}(C_{11}+1)$ and $\mathrm{Re}^2(C_{02})$ contribute to $d\zeta(t)/dt$ as the time derivative of $\mathrm{Im}^2(C_{02})$ vanishes at $t=0$. One obtains a simple expression [Eq. (3.27)] for $d\zeta/dt$ at $t=0$.

At the critical temperature $2\bar{n}+1=e^{2r}$, $d\zeta/dt$ adopts the simple form

$$\frac{d\zeta}{dt} = \gamma\omega^2 e^{-2\gamma t}e^{2r}\sinh 2r(\omega^2-\gamma^2)^{-1}[1-\cos(2t\sqrt{\omega^2-\gamma^2})]. \tag{B3}$$

It is apparent that, for $2\bar{n}+1=e^{2r}$, $d\zeta/dt\geqslant 0$. It then follows from $\zeta(0)=0$ that at the critical temperature ($2\bar{n}+1=e^{2r}$) density matrix positivity is guaranteed [or, equivalently, $\zeta(t)\geqslant 0$] for all t.

For temperatures higher than the critical temperature ($2\bar{n}+1>e^{2r}$), we can write $d\zeta/dt$ in the form

$$\frac{d\zeta}{dt} = -\gamma e^{-4\gamma t}(2\bar{n}+1-e^{2r})(2\bar{n}+1-e^{-2r}) + \gamma e^{-2\gamma t}(2\bar{n}+1)(2\bar{n}+1-e^{2r}+\sinh 2r)(\omega^2-\gamma^2)^{-1}[\omega^2-\gamma^2\cos(2t\sqrt{\omega^2-\gamma^2})]$$
$$-\gamma^2 e^{-2\gamma t}(2\bar{n}+1)(2\bar{n}+1-e^{2r})(\omega^2-\gamma^2)^{-1/2}\sin(2t\sqrt{\omega^2-\gamma^2}) - \gamma e^{-2\gamma t}(2\bar{n}+1)\sinh 2r\cos(2t\sqrt{\omega^2-\gamma^2})$$
$$= \gamma\omega^2 e^{-2\gamma t}\sinh 2r(2\bar{n}+1)(\omega^2-\gamma^2)^{-1}[1-\cos(2t\sqrt{\omega^2-\gamma^2})] + \gamma e^{-4\gamma t}e^{-2r}(2\bar{n}+1-e^{2r}) + \gamma e^{-2\gamma t}(2\bar{n}+1)$$
$$\times(2\bar{n}+1-e^{2r})\xi(t), \tag{B4}$$

where the auxiliary function $\xi(t)$ is defined as

$$\xi(t) = \frac{\omega^2-\gamma^2\cos(2t\sqrt{\omega^2-\gamma^2})}{\omega^2-\gamma^2} - \frac{\gamma\sin(2t\sqrt{\omega^2-\gamma^2})}{\sqrt{\omega^2-\gamma^2}} - e^{-2\gamma t}$$
$$= \frac{\gamma\omega}{\omega^2-\gamma^2}[1-\sin(\beta+2t\sqrt{\omega^2-\gamma^2})] + \frac{\omega}{\omega+\gamma} - e^{-2\gamma t} \tag{B5}$$

with

$$\beta = \arctan\frac{\gamma}{\sqrt{\omega^2-\gamma^2}}. \tag{B6}$$

For $2\bar{n}+1>e^{2r}$, in order to prove that $d\zeta/dt$ is non-negative for all t, it is sufficient to show that $\xi(t)$ is non-negative for all t. This is obviously true for $t\geqslant t_c=(2\gamma)^{-1}\ln[(\omega+\gamma)/\omega]$. For small t, it is sufficient to look at a Taylor expansion of $\xi(t)$:

$$\xi(t) = \frac{4}{3}\gamma\omega^2 t^3 - \frac{2}{3}\gamma^2\omega^2 t^4 + O(t)^5. \tag{B7}$$

In order for the leading term, which is positive for $t>0$, to be much greater than the absolute value of the second term, it is required that

$$\gamma t \ll 2. \tag{B8}$$

Obviously, the value of t_c fulfills this requirement:

$$\gamma t_c = \frac{1}{2}\ln\left(1+\frac{\gamma}{\omega}\right) \ll 2 \tag{B9}$$

in the physical regime of weak coupling $\gamma<\omega$. The auxiliary function $\xi(t)$ also has this peculiar property of having positive first and second time derivatives for small t. For example,

$$\dot{\xi}(t) = 2\gamma e^{-2\gamma t} - \frac{2\gamma\omega}{\sqrt{\omega^2-\gamma^2}}\cos(\beta+2t\sqrt{\omega^2-\gamma^2})$$
$$= 4\gamma\omega^2 t^2 - \frac{8}{3}\gamma^2\omega^2 t^3 + O(t)^4. \tag{B10}$$

Since $\xi(0)=0$, the fact that $\dot{\xi}(t)>0$ for $0<t<t_c$ also shows that $\xi(t)\geqslant 0$.

APPENDIX C: CHARACTERISTIC FUNCTIONS FOR SQUEEZED STATES

In this appendix we derive expressions for the characteristic functions for both one-mode and two-mode squeezed states. For the one-oscillator case of a squeezed-state start, the characteristic function at $t=0$ can be calculated from

$$\chi(\lambda,\lambda^*,t=0)=\mathrm{Tr}(|\xi\rangle\langle\xi|e^{\lambda a^\dagger}e^{-\lambda^* a})$$
$$=\langle 0|S^\dagger(\xi)e^{\lambda a^\dagger}e^{-\lambda^* a}S(\xi)|0\rangle. \quad \text{(C1)}$$

Taking advantage of the fact that $S(\xi)S^\dagger(\xi)=1$, a pair of $S(\xi)$ and $S^\dagger(\xi)$ can be inserted between $e^{\lambda a^\dagger}$ and $e^{-\lambda^* a}$:

$$\chi(\lambda,\lambda^*,t=0)=\langle 0|S^\dagger(\xi)e^{\lambda a^\dagger}S(\xi)S^\dagger(\xi)e^{-\lambda^* a}S(\xi)|0\rangle. \quad \text{(C2)}$$

The right hand side of Eq. (C2) can be evaluated from

$$S^\dagger(\xi)e^{\lambda a^\dagger}S(\xi)=\exp(\lambda a^\dagger\cosh r-\lambda a e^{-i\theta}\sinh r), \quad \text{(C3)}$$

$$S^\dagger(\xi)e^{-\lambda^* a}S(\xi)=\exp(-\lambda^* a\cosh r+\lambda^* a^\dagger e^{i\theta}\sinh r), \quad \text{(C4)}$$

which follow from

$$S^\dagger(\xi)aS(\xi)=a\cosh r-a^\dagger e^{i\theta}\sinh r, \quad \text{(C5)}$$

$$S^\dagger(\xi)a^\dagger S(\xi)=a^\dagger\cosh r-a e^{-i\theta}\sinh r. \quad \text{(C6)}$$

Therefore one obtains

$$\chi(\lambda,\lambda^*,t=0)=\exp\left(-|\lambda|^2\sinh^2 r-\frac{1}{4}\lambda^2 e^{-i\theta}\sinh 2r\right.$$
$$\left.-\frac{1}{4}\lambda^{*2}e^{i\theta}\sinh 2r\right). \quad \text{(C7)}$$

For the two-oscillator squeezed state the normal-ordered characteristic function reads by definition

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,0)$$
$$=\mathrm{Tr}[e^{-s(a_1^\dagger a_2^\dagger-a_1a_2)}|0\rangle\langle 0|$$
$$\times e^{s(a_1^\dagger a_2^\dagger-a_1a_2)}e^{\lambda_1 a_1^\dagger}e^{-\lambda_1^* a_1}e^{\lambda_2 a_2^\dagger}e^{-\lambda_2^* a_2}]. \quad \text{(C8)}$$

We first make use of the identity [24]

$$e^{s(a_1^\dagger a_2^\dagger-a_1a_2)}=e^{\Gamma a_1^\dagger a_2^\dagger}e^{-\ln(\cosh s)(a_1^\dagger a_1+a_2^\dagger a_2+1)}e^{-\Gamma a_1a_2}, \quad \text{(C9)}$$

where $\Gamma=\tanh s$. Since $|0\rangle$ is the vacuum state for both oscillators, i.e.,

$$e^{-\ln(\cosh s)(a_1^\dagger a_1+a_2^\dagger a_2+1)}e^{-\Gamma a_1a_2}|0\rangle=e^{-\ln(\cosh s)}|0\rangle, \quad \text{(C10)}$$

one arrives at

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,0)=e^{-2\ln(\cosh s)}\langle 0|e^{-\Gamma a_1a_2}e^{\lambda_1 a_1^\dagger}$$
$$\times e^{-\lambda_1^* a_1}e^{\lambda_2 a_2^\dagger}e^{-\lambda_2^* a_2}e^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle. \quad \text{(C11)}$$

We would like to move the two exponential operators $e^{\lambda_1 a_1^\dagger}$ and $e^{\lambda_2 a_2^\dagger}$ to the left of the double annihilation operator $e^{-\Gamma a_1a_2}$. We need to apply to Eq. (C11) three times the identity

$$e^Ae^B=e^{[A,B]}e^Be^A, \quad \text{(C12)}$$

provided that

$$[[A,B],A]=[[A,B],B]=0. \quad \text{(C13)}$$

It then follows from Eq. (C11) that

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,0)=e^{-2\ln(\cosh s)}\langle 0|e^{-\Gamma\lambda_1 a_2}e^{-\Gamma a_1a_2}e^{\lambda_2 a_2^\dagger}e^{-\lambda_1^* a_1}e^{-\lambda_2^* a_2}e^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle \quad \text{(C14)}$$
$$=e^{-2\ln(\cosh s)}\langle 0|e^{-\Gamma\lambda_1 a_2}e^{\lambda_2 a_2^\dagger}e^{-\Gamma\lambda_2 a_1}e^{-\Gamma a_1a_2}e^{-\lambda_1^* a_1}e^{-\lambda_2^* a_2}e^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle \quad \text{(C15)}$$
$$=e^{-2\ln(\cosh s)}\langle 0|e^{-\Gamma\lambda_1\lambda_2}e^{-\Gamma\lambda_1 a_2}e^{-\Gamma\lambda_2 a_1}e^{-\Gamma a_1a_2}e^{-\lambda_1^* a_1}e^{-\lambda_2^* a_2}e^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle. \quad \text{(C16)}$$

We are now left to evaluate

$$\langle 0|e^{\alpha^* a_1}e^{\beta^* a_2}e^{-\Gamma a_1a_2}e^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle \quad \text{(C17)}$$

with

$$\alpha^*=-\Gamma\lambda_2-\lambda_1^*, \quad \text{(C18)}$$

$$\beta^*=-\Gamma\lambda_1-\lambda_2^*. \quad \text{(C19)}$$

Expanding all four exponential operators, one has

$$\langle 0|e^{\alpha^* a_1+\beta^* a_2}e^{-\Gamma a_1 a_2}e^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle$$

$$=\langle 0|e^{\alpha^* a_1}e^{\beta^* a_2}e^{-\Gamma a_1 a_2}\sum_{n=0}^{\infty}(-\Gamma)^n|n,n\rangle \tag{C20}$$

$$=\langle 0|e^{\alpha^* a_1}e^{\beta^* a_2}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-\Gamma)^{l+n}n!}{l!(n-l)!}|n-l,n-l\rangle \tag{C21}$$

$$=\langle m,m'|\sum_{m=0}^{\infty}\sum_{m^*=0}^{\infty}\frac{(\alpha^*)^m(\beta^*)^{m'}}{\sqrt{m!m'!}}\times\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-\Gamma)^{l+n}n!}{l!(n-l)!}|n-l,n-l\rangle \tag{C22}$$

$$=\sum_{m=0}^{\infty}\sum_{m^*=0}^{\infty}\frac{(\alpha^*)^m(\beta^*)^{m'}}{\sqrt{m!m'!}}\times\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-\Gamma)^{l+n}n!}{l!(n-l)!}\delta_{m,n-l}\delta_{m',n-l} \tag{C23}$$

$$=\sum_{n=0}^{\infty}(-\Gamma)^{2n}\sum_{l=0}^{n}(-1)^{n-l}\left(\frac{\alpha^*\beta^*}{\Gamma}\right)^{n-l}\times\frac{n!}{l!(n-l)!(n-l)!} \tag{C24}$$

$$=\sum_{n=0}^{\infty}L_n\left(\frac{\alpha^*\beta^*}{\Gamma}\right)(\Gamma^2)^n, \tag{C25}$$

where $L_n(x)$ is the Laguerre polynomial of order n [25,26]. The series in Eq. (C25) is just the generating function of Laguerre polynomials:

$$g(x,z)=(1-z)^{-1}e^{-xz/(1-z)}=\sum_{n=0}^{\infty}L_n(x)z^n,\qquad |z|<1. \tag{C26}$$

Therefore

$$\langle 0|e^{\alpha^* a_1+\beta^* a_2}e^{-\Gamma a_1 a_2}e^{-\Gamma a_1^\dagger a_2^\dagger}|0\rangle=\frac{1}{1-\Gamma^2}\exp\left(-\frac{\alpha^*\beta^*\Gamma}{1-\Gamma^2}\right), \tag{C27}$$

where the prefactor $(1-\Gamma^2)^{-1}$ exactly cancels out the prefactor $e^{-2\ln(\cosh s)}$ in Eq. (C14). Substituting α^* and β^*, one finally obtains

$$\chi(\lambda_1,\lambda_1^*,\lambda_2,\lambda_2^*,0)=\exp\left(-\Gamma\lambda_1\lambda_2-\frac{\alpha^*\beta^*\Gamma}{1-\Gamma^2}\right) \tag{C28}$$

$$=\exp\left[-\frac{\Gamma}{1-\Gamma^2}(\lambda_1\lambda_2+\lambda_1^*\lambda_2^*)-\frac{\Gamma^2}{1-\Gamma^2}(|\lambda_1|^2+|\lambda_2|^2)\right], \tag{C29}$$

where the coefficients can be simplified as

$$\frac{\Gamma}{1-\Gamma^2}=\frac{1}{2}\sinh 2s, \tag{C30}$$

$$\frac{\Gamma^2}{1-\Gamma^2}=\sinh^2 s. \tag{C31}$$

APPENDIX D: CROSS MOMENTS FOR TWO PRIMARY OSCILLATORS

From the four coupled equations for the cross moments which are decoupled from the rest of the ten second order moments ($J=0$), one obtains solutions for $C_{10,10}$, $C_{01,01}$, $C_{10,01}+C_{01,10}$, and $C_{10,01}-C_{01,10}$ applicable to all initial Gaussian wave packets:

$$C_{10,10}=\frac{e^{-2t(\gamma+\sqrt{\gamma^2-\omega^2})}}{4(\gamma^2-\omega^2)}\{\gamma^2[B_3(e^{2t\sqrt{\gamma^2-\omega^2}}-1)^2+B_1(e^{2t\sqrt{\gamma^2-\omega^2}}+1)^2]+\gamma B_2(e^{2t\sqrt{\gamma^2-\omega^2}}-1)[\sqrt{\gamma^2-\omega^2}(e^{2t\sqrt{\gamma^2-\omega^2}}+1)+i\omega(e^{2t\sqrt{\gamma^2-\omega^2}}-1)]-2\omega B_1[\omega(e^{4t\sqrt{\gamma^2-\omega^2}}+1)-i\sqrt{\gamma^2-\omega^2}(e^{4t\sqrt{\gamma^2-\omega^2}}-1)]\}, \tag{D1}$$

$$C_{01,01}=\frac{e^{-2t(\gamma+\sqrt{\gamma^2-\omega^2})}}{4(\gamma^2-\omega^2)}\{\gamma^2[B_1(e^{2t\sqrt{\gamma^2-\omega^2}}-1)^2+B_3(e^{2t\sqrt{\gamma^2-\omega^2}}+1)^2]+\gamma B_2(e^{2t\sqrt{\gamma^2-\omega^2}}-1)[\sqrt{\gamma^2-\omega^2}(e^{2t\sqrt{\gamma^2-\omega^2}}+1)-i\omega(e^{2t\sqrt{\gamma^2-\omega^2}}-1)]-2\omega B_3[\omega(e^{4t\sqrt{\gamma^2-\omega^2}}+1)+i\sqrt{\gamma^2-\omega^2}(e^{4t\sqrt{\gamma^2-\omega^2}}-1)]\}, \tag{D2}$$

$$C_{10,01}+C_{01,10}=\frac{e^{-2t(\gamma+\sqrt{\gamma^2-\omega^2})}}{2(\gamma^2-\omega^2)}\{\gamma^2B_2(e^{4t\sqrt{\gamma^2-\omega^2}}+1)-2\omega^2B_2e^{2t\sqrt{\gamma^2-\omega^2}}(e^{2t\sqrt{\gamma^2-\omega^2}}-1)[i\omega(B_1-B_3)(e^{2t\sqrt{\gamma^2-\omega^2}}+1)\times(e^{2t\sqrt{\gamma^2-\omega^2}}-1)+(B_1+B_3)\sqrt{\gamma^2-\omega^2}(e^{2t\sqrt{\gamma^2-\omega^2}}+1)]\}, \tag{D3}$$

$$C_{10,01}-C_{01,10}=B_4e^{-2\gamma t}. \tag{D4}$$

Here the B's are to be determined by the initial conditions.

APPENDIX E: ANALYTICAL EXPRESSIONS FOR THE $J=0$ CASE

For zero direct coupling $J=0$, analytical expressions of nonzero second order moments are easily obtained for an initial two-mode squeezed state:

$$C_{20,00}=C_{00,20}=C^*_{02,00}=C^*_{00,02}$$
$$=\frac{1}{2}\gamma e^{-2\gamma t}(\omega^2-\gamma^2)^{-1}(2\bar{n}+1-\cosh 2s)$$
$$\times\sinh(t\sqrt{\gamma^2-\omega^2})[\sqrt{\gamma^2-\omega^2}\cosh(t\sqrt{\gamma^2-\omega^2})$$
$$+i\omega\sinh(t\sqrt{\gamma^2-\omega^2})], \tag{E1}$$

$$C_{11,00}=C_{00,11}=\bar{n}-\frac{1}{2}e^{-2\gamma t}(2\bar{n}+1-\cosh 2s)$$
$$\times(\omega^2-\gamma^2)^{-1}[\omega^2-\gamma^2\cosh(2t\sqrt{\gamma^2-\omega^2})], \tag{E2}$$

$$C_{10,10}=C^*_{01,01}=-\frac{1}{2}e^{-2\gamma t}\sinh 2s[\cosh(2t\sqrt{\gamma^2-\omega^2})$$
$$+i\omega(\gamma^2-\omega^2)^{-1/2}\sinh(2t\sqrt{\gamma^2-\omega^2})], \tag{E3}$$

$$C_{10,01}=C_{01,10}=-\frac{1}{2}\gamma e^{-2\gamma t}(\gamma^2-\omega^2)^{-1}\sinh 2s$$
$$\times\sinh(2t\sqrt{\gamma^2-\omega^2}). \tag{E4}$$

The above expressions are used to derive an analytical criterion for density matrix negativity for two uncoupled oscillators.

It follows that analytical expressions of A and S can also be obtained for an initial two-squeezed state and zero coupling $J=0$:

$$A=\gamma e^{-2\gamma t}(\omega^2-\gamma^2)^{-1/2}(2\bar{n}+1-e^{-2s})$$
$$\times\sin(2t\sqrt{\omega^2-\gamma^2}), \tag{E5}$$

$$S=2\bar{n}-e^{-2\gamma t}\sinh 2s\cos(2t\sqrt{\omega^2-\gamma^2})$$
$$-e^{-2\gamma t}(2\bar{n}+1-\cosh 2s)(\omega^2-\gamma^2)^{-1}$$
$$\times[\omega^2-\gamma^2\cos(2t\sqrt{\omega^2-\gamma^2})]. \tag{E6}$$

APPENDIX F: NEGATIVITY CRITERION FOR TWO UNCOUPLED OSCILLATORS

For an initial two-mode squeezed state with a squeezing parameter s, there exists an analytical expression for $Z(t)$ for $J=0$. A series expansion of $Z(t)$ for $J=0$ around $t=0$ up to fifth order in t reveals that

$$Z(t\approx 0)=16\left[(\gamma t)^2-4(\gamma t)^3+\frac{28}{3}(\gamma t)^4-16(\gamma t)^5\right]$$
$$\times(2\bar{n}+1-e^{-2s})(2\bar{n}+1-e^{2s})$$
$$+\frac{32}{3}\omega^2\gamma^2t^4(2\bar{n}+1)(\cosh 4s-1)$$
$$+\frac{64}{3}\omega^2\gamma^3t^5(2\bar{n}+1)[2\bar{n}^2\cosh 2s$$
$$-(2\bar{n}+1)\sinh^2 s(7+9\cosh 2s)]+O(t)^6. \tag{F1}$$

Since the power series of $Z(t)$ starts from second order in t, $Z(0)=\dot{Z}(0)=0$. If $2\bar{n}+1<e^{2|s|}$, $\ddot{Z}(0)<0$, and density matrix negativity occurs at small t. As t increases, the negativity problem will be alleviated because the third time derivative d^3Z/d^3t at $t=0$ is positive, and the fourth time derivative d^4Z/d^4t at $t=0$ can also be positive for weak coupling $\omega\gg\gamma$.

Taking $Z(t)$ to zero puts a constraint on t_0. From Eq. (4.57) the solutions for t_0 often come in pairs. Two solutions with a pair labeled by t_0^a and t_0^b are related to one another by

$$t_0^a(s)=t_0^b(-s). \tag{F2}$$

This means that one needs only to solve the problem for positive s if $J=0$. However, the symmetry no longer stands when a finite J is introduced.

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