

## Comments on "Least Squares Restoration of Multichannel Images"

Michael K. Ng and Wilson C. Kwan

**Abstract**—In this correspondence, we give the correct matrix formulation arising from the constrained optimization of the least squares restoration of multichannel images in the above paper.<sup>1</sup>

**Index Terms**—Image restoration, least squares method.

The degradation of multichannel images can be modeled as solving

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{n} \quad (1)$$

where

- $\mathbf{H}$  degradation matrix;
- $\mathbf{n}$  unknown Gaussian noise or measurement errors;
- $\mathbf{g}$  observed multichannel image;
- $\mathbf{f}$  original multichannel image.

For  $N$  channels of  $M \times M$  pixels each, the observed and original images can be expressed as

$$\mathbf{g} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_N \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_N \end{pmatrix}$$

where each of the  $M^2$ -vectors  $\mathbf{g}_i$  and  $\mathbf{f}_i$  are the observed and the original images in each channel. The multichannel degradation operator  $\mathbf{H}$  is given by

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \cdots & \mathbf{H}_{1N} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \cdots & \mathbf{H}_{2N} \\ \vdots & \cdots & \ddots & \vdots \\ \mathbf{H}_{N1} & \mathbf{H}_{N2} & \cdots & \mathbf{H}_{NN} \end{pmatrix}.$$

Here, the operators  $\mathbf{H}_{ii}$  and  $\mathbf{H}_{ij}$  ( $i \neq j$ ) represent the within-channel and the cross-channel degradation operators, respectively.

As the system (1) is ill-conditioned and generally not positive definite, we solve it by using a constrained minimization technique. The solution  $\hat{\mathbf{f}}$  is determined by minimizing  $\|\mathbf{Q}\mathbf{f}\|_2^2$  subject to  $\|\hat{\mathbf{H}}_i - \mathbf{g}_i\|_2^2 = \|\mathbf{n}_i\|_2^2$  for  $1 \leq i \leq N$ , where

$$\hat{\mathbf{H}}_i = (\mathbf{H}_{i1}, \mathbf{H}_{i2}, \dots, \mathbf{H}_{iN}).$$

Using the method of Lagrange multipliers, the solution is obtained by minimizing

$$\Phi(\mathbf{f}, \lambda) = \sum_{i=1}^N \lambda_i (\|\hat{\mathbf{H}}_i \mathbf{f} - \mathbf{g}_i\|_2^2 - \|\mathbf{n}_i\|_2^2) + \|\mathbf{Q}\mathbf{f}\|_2^2 \quad (2)$$

where  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)^t$  is the Lagrange multiplier vector. The necessary conditions for the solution of (2) are

$$\nabla_f \Phi(\mathbf{f}, \lambda) = \mathbf{0} \quad (3)$$

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The authors are with the Department of Mathematics, The University of Hong Kong, Hong Kong (e-mail: mng@maths.hku.hk).

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<sup>1</sup>N. Galatsanos, A. Katsaggelos, R. Chin, and A. Hillery, "Least squares restoration of multichannel images," *IEEE Trans. Signal Processing*, vol. 39, pp. 2222–2236, Oct. 1991.

$$\nabla_\lambda \Phi(\mathbf{f}, \lambda) = \mathbf{0}. \quad (4)$$

From (3), we have

$$\left[ \sum_{i=1}^N \lambda_i \hat{\mathbf{H}}_i^t \hat{\mathbf{H}}_i + \mathbf{Q}^t \mathbf{Q} \right] \hat{\mathbf{f}} = \sum_{i=1}^N \lambda_i \hat{\mathbf{H}}_i^t \mathbf{g}_i.$$

Since

$$\sum_{i=1}^N \lambda_i \hat{\mathbf{H}}_i^t \hat{\mathbf{H}}_i = \mathbf{H}^t \mathbf{\Lambda} \mathbf{H}$$

where  $\mathbf{\Lambda}$  is defined as

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 \mathbf{I} & 0 & \cdots & 0 \\ 0 & \lambda_2 \mathbf{I} & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \mathbf{I} \end{pmatrix}$$

the matrix system should be

$$[\mathbf{H}^t \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^t \mathbf{Q}] \hat{\mathbf{f}} = \mathbf{\Lambda} \mathbf{H}^t \mathbf{g} \quad (5)$$

instead of

$$[\mathbf{\Lambda} \mathbf{H}^t \mathbf{H} + \mathbf{Q}^t \mathbf{Q}] \hat{\mathbf{f}} = \mathbf{\Lambda} \mathbf{H}^t \mathbf{g} \quad (6)$$

which is stated in (2.14) of the paper. We remark that if there are only the within-channel blur and no cross-channel blur in the degradation, i.e.,  $H_{ij} = 0$ , then both formulae (5) and (6) are identical.

Solving for  $\hat{\mathbf{f}}$  in (5) yields

$$\hat{\mathbf{f}} = [\mathbf{H}^t \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^t \mathbf{Q}]^{-1} \mathbf{\Lambda} \mathbf{H}^t \mathbf{g} \quad (7)$$

which is the solution of (2) with respect to chosen values in  $\lambda$ . We note in the paper that the incorrect solution is given by

$$\hat{\mathbf{f}} = [\mathbf{H}^t \mathbf{H} + \mathbf{\Lambda}^{-1} \mathbf{Q}^t \mathbf{Q}]^{-1} \mathbf{H}^t \mathbf{g}, \quad (8)$$

[see (2.15) of the paper].

As the solution  $\hat{\mathbf{f}}$  must satisfy the following functions:

$$Z_i(\lambda_1, \lambda_2, \dots, \lambda_N) = \|\hat{\mathbf{H}}_i \hat{\mathbf{f}} - \mathbf{g}_i\|_2^2 - \|\mathbf{n}_i\|_2^2, \quad i = 1, 2, \dots, N$$

finding the roots of the functions  $Z_i$  simultaneously yields the desired  $\lambda_i$ . Since the functions  $Z_i$  are nonlinear, Newton's method is therefore used to find these  $\lambda_i$  numerically. Since Newton's method for the estimate of  $\lambda_i$  is derived according to the form of  $\hat{\mathbf{f}}$ , the correct formula of  $\hat{\mathbf{f}}$  is important in the estimation of  $\lambda_i$  and the restoration of multichannel images.

The  $(i, j)$ th element of the Jacobian matrix  $\mathbf{J}$  of this nonlinear system of functions based on the matrix formulation (5) should be given by

$$[\mathbf{J}]_{ij} = \frac{\partial Z_i(\lambda_1, \lambda_2, \dots, \lambda_N)}{\partial \lambda_j} = 2(\hat{\mathbf{H}}_i \hat{\mathbf{f}} - \mathbf{g}_i)^t \hat{\mathbf{H}}_i \frac{\partial \hat{\mathbf{f}}}{\partial \lambda_j}$$

for  $1 \leq i, j \leq N$ . By (7), we obtain

$$\frac{\partial \hat{\mathbf{f}}}{\partial \lambda_j} = [\mathbf{H}^t \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^t \mathbf{Q}]^{-1} \cdot (\mathbf{I}_{jj} - \hat{\mathbf{H}}_j^t \hat{\mathbf{H}}_j [\mathbf{H}^t \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^t \mathbf{Q}]^{-1} \mathbf{\Lambda}) \mathbf{H}^t \mathbf{g}$$

where  $\mathbf{I}_{jj}$  is a zero matrix, except that the  $(j, j)$ th diagonal block is an identity matrix. Hence, we have

$$[\mathbf{J}]_{ij} = 2(\hat{\mathbf{H}}_i \hat{\mathbf{f}} - \mathbf{g}_i)^t \hat{\mathbf{H}}_i [\mathbf{H}^t \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^t \mathbf{Q}]^{-1} \cdot (\mathbf{I}_{jj} - \hat{\mathbf{H}}_j^t \hat{\mathbf{H}}_j [\mathbf{H}^t \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^t \mathbf{Q}]^{-1} \mathbf{\Lambda}) \mathbf{H}^t \mathbf{g}$$

which is the correct formula for the Jacobian matrix.