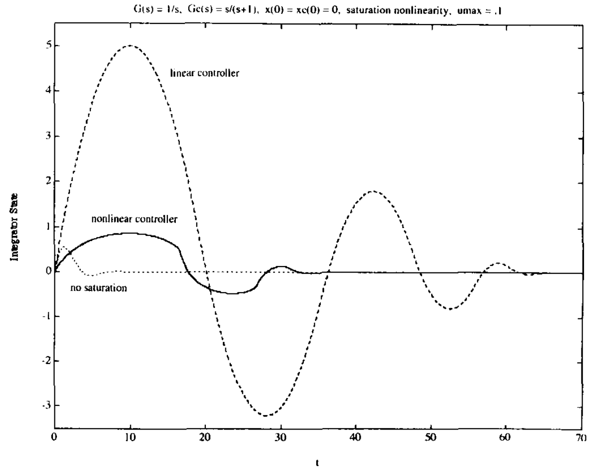

Fig. 4.

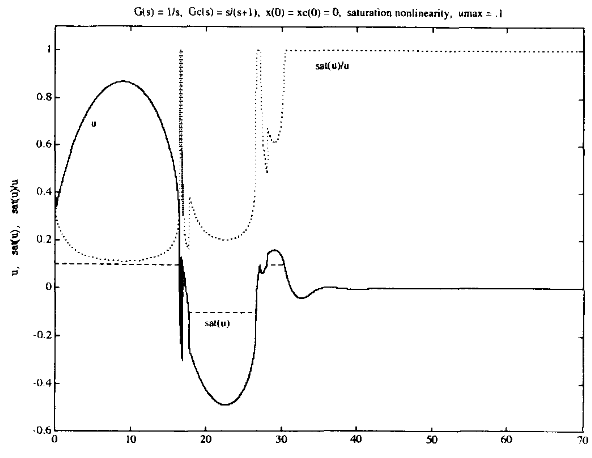

Fig. 5.

approach assumes that the linear plant and compensator are positive real, while the class of input nonlinearities that can be addressed is quite general. To guarantee global asymptotic stability, the linear compensator is modified to form a nonlinear compensator that counteracts the effects of the input nonlinearity by recovering the passivity of the plant. We demonstrated special cases of this result by simulating control systems having quadratic and saturation nonlinearities. Future extensions will focus on extending the result to larger classes of linear plants and compensators.

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Analysis on the Laguerre Formula for Approximating Delay Systems

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***Abstract*—This note provides a detailed analysis on the commonly employed Laguerre formula for approximating delay systems. Convergence proofs are provided, and error bounds are constructed with respect to the L_∞ and L_2 norms.**

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I. Introduction

Recently there have been many research works on the approximations of delay systems of the form

$$G(s) = e^{-sT} g(s) \tag{1}$$

where $g(s)$ is a rational transfer function [1]–[4]. In particular, L_∞ and L_2 convergence are guaranteed where $G(s)$ is approximated by $R_{mn}(sT)g(s)$ with $R_{mn}(sT)$ as the $[m/n]$ Padé approximant of e^{-sT}. The use of Padé approximation for e^{-sT} has been well known, but the so called *Laguerre formula* given by

$$L_n(sT) = \left(\frac{1-\frac{sT}{2n}}{1+\frac{sT}{2n}}\right)^n \tag{2}$$

is also sometimes employed [5], [6]. It is referred to as the Laguerre formula since it has only one pole, of multiplicity n, which resembles to the Laplace transform of Laguerre functions [6]. Due to the simplicity of (2), it can be realized by analog elements a lot easier than using Padé approximants. This is especially true when the order of the Padé approximation becomes large. The present note provides a detailed analysis on the approximation of $G(s)$ by

$$L_n(sT)g(s). \tag{3}$$

It will be shown that the approximation scheme converges in the frequency domain under the L_∞ and L_2 norms and corresponding error bounds will be given. Scalar $G(s)$ will be considered here although the multivariable case can be extended naturally as in [2], [3], and [4].

II. Main Results

First, we give a characterization of the frequency response error for

$$E_n(sT) := e^{-sT} - L_n(sT) \tag{4}$$

with $n \in \mathbb{N} = \{1, 2, \cdots\}$, the set of natural numbers.

Proposition 1: For $\omega \geq 0$, $n \in \mathbb{N}$

$$|E_n(j\omega T)| \leq \min\left(2, \frac{n\phi_n^3(\omega T)}{6\sqrt{4+\phi_n^2(\omega T)}}\right) \leq \min\left(2, \frac{n}{12}\phi_n^3(\omega T)\right) \tag{5}$$

where $\phi_n(\omega T) := (\omega T/n)$.

Proof: We have

$$\begin{aligned}
|E_n(j\omega T)| &= \left| e^{-j\omega T} - \left(\frac{1-\frac{\omega T}{2n}j}{1+\frac{\omega T}{2n}j}\right)^n \right| \\
&= \left| (e^{-j\omega T/n})^n - \left(\frac{1-\frac{\omega T}{2n}j}{1+\frac{\omega T}{2n}j}\right)^n \right| \\
&= \left| e^{-j\omega T/n} - \frac{1-\frac{\omega T}{2n}j}{1+\frac{\omega T}{2n}j} \right| \cdot \left| \sum_{r=1}^{n} (e^{-j\omega T/n})^{n-r} \left(\frac{1-\frac{\omega T}{2n}j}{1+\frac{\omega T}{2n}j}\right)^{r-1} \right| \\
&\leq \left| e^{-j\omega T/n} - \frac{1-\frac{\omega T}{2n}j}{1+\frac{\omega T}{2n}j} \right| \cdot \sum_{r=1}^{n} \left| (e^{-j\omega T/n})^{n-r} \left(\frac{1-\frac{\omega T}{2n}j}{1+\frac{\omega T}{2n}j}\right)^{r-1} \right| \\
&= \left| e^{-j\omega T/n} - \frac{1-\frac{\omega T}{2n}j}{1+\frac{\omega T}{2n}j} \right| n.
\end{aligned} \tag{6}$$

With $\phi_n = (\omega T/n)$, we have

$$\begin{aligned}
&\left| e^{-j\phi_n} - \frac{1-\frac{\phi_n}{2}j}{1+\frac{\phi_n}{2}j} \right| \\
&= \left| \frac{\left(\cos\phi_n + \frac{\phi_n}{2}\sin\phi_n - 1\right) + j\left(\frac{\phi_n}{2} - \sin\phi_n + \frac{\phi_n}{2}\cos\phi_n\right)}{1+j\frac{\phi_n}{2}} \right| \\
&= \sqrt{\frac{8 + 2\phi_n^2 + (2\phi_n^2 - 8)\cos\phi_n - 8\phi_n\sin\phi_n}{4+\phi_n^2}} \\
&\leq \frac{\phi_n^2}{6\sqrt{4+\phi_n^3}} \\
&\leq \frac{\phi_n^3}{12}
\end{aligned}$$

where the details of the last two inequalities can be found in Appendix A. Therefore

$$|E_n(j\omega T)| \leq \frac{n\phi_n^3}{6\sqrt{4+\phi_n^2}} \leq \frac{n\phi_n^3}{12} \tag{7}$$

but we also have $|L_n(j\omega T)| = 1$, $|e^{-j\omega T}| = 1$ and hence $|E_n(j\omega T)| \leq 2$. Thus the result follows. □

Remark 1: The upper bound for $|E_n(j\omega T)|$ given by

$$\frac{(\omega T)^3}{12n^2}$$

corresponds to the first nonzero term of the Maclaurin series expansion of $|E_n(j\omega T)|$. Equalities in (7) hold when and only when $\omega = 0$ which implies $\phi_n = 0$. In fact, it is possible to calculate exactly the value of ω such that the monotonic increasing function

$$\frac{n\phi_n^3}{6\sqrt{4+\phi_n^2}}$$

reaches a value of two. That is, if $\omega = \alpha$ is such that

$$\frac{n\phi_n^3(\alpha T)}{6\sqrt{4+\phi_n^2(\alpha T)}} = 2$$

if and only if

$$n^2\left(\frac{\alpha T}{n}\right)^6 = 144\left(4 + \left(\frac{\alpha T}{n}\right)^2\right)$$

then it can be shown that (see the equation at the bottom of the page).

Although the bound

$$\frac{n\phi_n^3}{6\sqrt{4+\phi_n^2}}$$

in (7) is more accurate than that given by $(n/12)\phi_n^3$, the latter is easier for the development of error bounds. The following corollary of Proposition 1 provides a form of the upper bound of $|E_n(j\omega T)|$ which is more useful in the present work.

Corollary 1: For $\omega \geq 0$, $n \in \mathbb{N}$,

$$|E_n(j\omega T)| \leq \begin{cases} 2\left(\frac{\omega T}{\beta n^{2/3}}\right)^3 & \text{for } \omega T \leq \beta n^{2/3} \\ 2 & \text{for } \omega T \geq \beta n^{2/3} \end{cases} \tag{8}$$

where $\beta := 2(3^{1/3})$.

$$\alpha = \begin{cases} \frac{1}{T}\sqrt{8\sqrt{3}\cos\frac{\pi}{18}} & n = 1 \\ \frac{576^{1/6}}{T} n^{2/3}\left[1 + \left(\frac{9}{2n^2}\right)^{1/3}\left\{\left(1+\sqrt{1-\frac{4}{3n^2}}\right)^{1/3} + \left(1-\sqrt{1-\frac{4}{3n^2}}\right)^{1/3}\right\}\right]^{1/6} & n \geq 2 \end{cases}$$

□

Proof: Observe that $((\omega T)^3/12n^2) = 2$ when $\omega T = \beta n^{(2/3)}$ and the result follows immediately from Proposition 1. □

Suppose the rational part, $g(s)$, of $G(s)$ has relative degree k and has a continuous upper bound along the imaginary axis given by

$$|g(j\omega)| \leq \begin{cases} \frac{M_l}{\omega^m} & \text{for } \omega \leq \omega_c \\ \frac{M_h}{\omega^k} & \text{for } \omega \geq \omega_c \end{cases} \tag{9}$$

where $M_l,\ M_h \geq 0,\ \omega_c > 0$ such that

$$M_l \omega_c^k = M_h \omega_c^m \tag{10}$$

and $k \in \mathbb{N}, m \in \mathbb{Z}$ and $m \leq 3$ where $\mathbb{Z}$ is the set of integers. The constraint on m is due to the fact that $|E_n(j\omega T)|$ increases with a rate of $\mathcal{O}(\omega^3)$ around $\omega = 0$, thus in order that $|E_n(j\omega T)g(j\omega)|$ is to be well defined at $\omega = 0, m$ cannot be greater than three. The condition on $g(s)$ can be satisfied by most strictly proper systems, namely, systems with no poles on the imaginary axis except a possibility of no more than three poles at the origin.

For a given transfer function $F(s)$, we define

$$\|F(s)\|_\infty := \sup_{0 \leq \omega < \infty} |F(j\omega)|$$

and

$$\|F(s)\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega}$$

as the L_∞ and L_2 norms of $F(s)$ if they exist. In this case, we say that $F(s) \in L_\infty$ (respectively, $F(s) \in L_2$) if $\|F(s)\|_\infty < \infty$ (respectively $\|F(s)\|_2 < \infty$). Theorem 1 below gives L_∞ and L_2 error bounds for $E_n(sT)g(s)$.

Theorem 1: Let $G(s) = e^{-sT}g(s)$ with $g(s)$ satisfies (9), (10) for some constants $M_l \geq 0$, $M_h \geq 0$, $k \in \mathbb{N}$, $\omega_c > 0$, and $m \in \mathbb{Z}$ ($m \leq 3$). If $G(s)$ is approximated by $L_n(sT)g(s)$, and

a) if $\omega_c T \geq \beta n^{2/3}$, then

$$\|G - L_n g\|_\infty \leq \begin{cases} 2M_l\left(\frac{T}{\beta n^{2/3}}\right)^m & \text{for } m \geq 0 \\ 2\frac{M_l}{\omega_c^m} = 2\frac{M_h}{\omega_c^k} & \text{for } m \leq 0 \end{cases} \tag{11}$$

if $\omega_c T \leq \beta n^{2/3}$, then

$$\|G - L_n g\|_\infty \leq \begin{cases} 2M_l\omega_c^{3-m}\left(\frac{T}{\beta n^{2/3}}\right)^3 = 2M_h\omega_c^{3-k}\left(\frac{T}{\beta n^{2/3}}\right)^3 \\ \quad \text{for } k \geq 3 \\ 2M_h\left(\frac{T}{\beta n^{2/3}}\right)^k \\ \quad \text{for } k \leq 3 \end{cases} \tag{12}$$

b) if $\omega_c T \geq \beta n^{2/3}$, then

$$\|G - L_n g\|_2^2 \leq \frac{8M_h^2}{\pi\omega_c^{2k-1}(2m-1)} \cdot \left[\left(\frac{\omega_c T}{\beta n^{2/3}}\right)^{2m-1}\frac{3}{7-2m} + \frac{m-k}{2k-1}\right] \tag{13}$$

if $\omega_c T \leq \beta n^{2/3}$, then

$$\|G - L_n g\|_2^2 \leq \frac{8M_h^2}{\pi\omega_c^{2k-1}(7-2k)}\left[\left(\frac{\omega_c T}{\beta n^{2/3}}\right)^{2k-1}\frac{3}{2k-1} + \left(\frac{\omega_c T}{\beta n^{2/3}}\right)^6\frac{m-k}{7-2m}\right] \tag{14}$$

where $\beta = 2(3)^{1/3}$.

Proof: First notice that $|E_n(j\omega T)||g(j\omega)|$ is guaranteed to be well defined at $\omega = 0$ and nondecreasing for $0 < \omega T < \min(\beta n^{2/3},\ \omega_c T)$.

For the L_∞ case, when $\omega_c T \geq \beta n^{2/3}$ the product of the upper bounds of $|E_n(j\omega T)|$ (Corollary 1) and $|g(j\omega)|$ in $[\beta n^{2/3}/T,\ \omega_c]$ is nonincreasing if $m \geq 0$ and nondecreasing if $m \leq 0$. In the former case, the maximum value of the product of the upper bounds is achieved at $\omega = \beta n^{2/3}/T$, but for $m \leq 0$ this occurs at $\omega = \omega_c$. The error bounds in (11) are then obtained by evaluating the product of the upper bounds of $|E_n(j\omega T)|$ and $|g(j\omega)|$ at appropriate ω. When $\omega_c T \leq \beta n^{2/3}$, the corresponding product in $[\omega_c,\ \beta n^{2/3}/T]$ is nonincreasing (respectively, nondecreasing) if $k \geq 3$ (respectively, $k \leq 3$), and this gives a maximum value at $\omega = \omega_c$ (respectively, $\omega = \beta n^{2/3}/T$). Similarly, the upper bounds in (12) can be calculated.

For the L_2 case, when $\omega_c T \geq \beta n^{2/3}$ with $\omega_0 := \beta n^{2/3}/T$, the upper bound in (13) can be calculated by the definition of L_2 error

$$\begin{aligned}
\|G - L_n g\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |E_n(j\omega T)|^2|g(j\omega)|^2\, d\omega \\
&\leq \frac{1}{\pi}\int_0^{\omega_0}\left[2\left(\frac{\omega T}{\beta n^{2/3}}\right)^3\frac{M_l}{\omega^m}\right]^2 d\omega \\
&\quad + \frac{1}{\pi}\int_{\omega_0}^{\omega_c}\left[\frac{2M_l}{\omega^m}\right]^2 d\omega + \frac{1}{\pi}\int_{\omega_c}^{\infty}\left[\frac{2M_h}{\omega^k}\right]^2 d\omega \\
&= \frac{4M_l^2}{\pi\omega_0^{2m-1}(7-2m)} \\
&\quad + \frac{4M_l^2}{\pi}\left[\frac{1}{\omega_c^{2m-1}(1-2m)} - \frac{1}{\omega_0^{2m-1}(1-2m)}\right] \\
&\quad + \frac{4M_h^2}{\pi\omega_c^{2k-1}(2k-1)} \\
&= \frac{4M_l^2}{\pi}\left(\frac{T}{\beta n^{2/3}}\right)^{2m-1}\left(\frac{1}{7-2m} - \frac{1}{1-2m}\right) \\
&\quad + \frac{4}{\pi}\left[\frac{M_l^2}{(1-2m)\omega_c^{2m-1}} + \frac{M_h^2}{(2k-1)\omega_c^{2k-1}}\right] \\
&= \frac{24M_l^2}{\pi(7-2m)(2m-1)}\left(\frac{T}{\beta n^{2/3}}\right)^{2m-1} \\
&\quad + \frac{4}{\pi}\left[\frac{M_l^2}{\omega_c^{2m-1}(1-2m)} + \frac{M_h^2}{\omega_c^{2k-1}(2k-1)}\right] \\
&= \frac{8M_h^2}{\pi\omega_c^{2k-1}(2m-1)} \\
&\quad \cdot \left[\left(\frac{\omega_c T}{\beta n^{2/3}}\right)^{2m-1}\frac{3}{7-2m} + \frac{m-k}{2k-1}\right]
\end{aligned}$$

using (10), which is the required result.

Similarly, when $\omega_c T \leq \beta n^{2/3}$, the upper bound is obtained by calculating

$$\begin{aligned}
\|G - L_n g\|_2^2 &\leq \frac{1}{\pi}\int_0^{\omega_c}\left[2\left(\frac{\omega T}{\beta n^{2/3}}\right)^3\frac{M_l}{\omega^m}\right]^2 d\omega \\
&\quad + \frac{1}{\pi}\int_{\omega_c}^{\omega_0}\left[2\left(\frac{\omega T}{\beta n^{2/3}}\right)^3\frac{M_h}{\omega^k}\right]^2 d\omega \\
&\quad + \frac{1}{\pi}\int_{\omega_0}^{\infty}\left[\frac{2M_h}{\omega^k}\right]^2 d\omega \\
&= \frac{4M_l^2}{\pi\omega_c^{2m-1}(7-2m)}\left(\frac{\omega_c T}{\beta n^{2/3}}\right)^6 + \frac{4M_h^2}{\pi}\left(\frac{T}{\beta n^{2/3}}\right)^6 \\
&\quad \cdot \left[\frac{\omega_0^{7-2k}}{7-2k} - \frac{\omega_c^{7-2k}}{7-2k}\right] + \frac{4M_h^2}{\pi\omega_0^{2k-1}(2k-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4M_h^2}{\pi}\left(\frac{T}{\beta n^{2/3}}\right)^{2k-1}\left[\frac{1}{2k-1}+\frac{1}{7-2k}\right] \\
&\quad + \frac{4}{\pi}\left(\frac{\omega_c T}{\beta n^{2/3}}\right)^6\left[\frac{M_l^2}{\omega_c^{2m-1}(7-2m)}\right. \\
&\quad \left. - \frac{M_h^2}{\omega_c^{2k-1}(7-2k)}\right] \\
&= \frac{24M_h^2}{\pi(7-2k)(2k-1)}\left(\frac{T}{\beta n^{2/3}}\right)^{2k-1} \\
&\quad + \frac{4}{\pi}\left(\frac{\omega_c T}{\beta n^{2/3}}\right)^6\left[\frac{M_l^2}{\omega_c^{2m-1}(7-2m)}\right. \\
&\quad \left. - \frac{M_h^2}{\omega_c^{2k-1}(7-2k)}\right] \\
&= \frac{8M_h^2}{\pi\omega_c^{2k-1}(7-2k)}\left[\left(\frac{\omega_c T}{\beta n^{2/3}}\right)^{2k-1}\frac{3}{2k-1}\right. \\
&\quad \left. + \left(\frac{\omega_c T}{\beta n^{2/3}}\right)^6\frac{m-k}{7-2m}\right]
\end{aligned}
$$

using (10). Hence the result. □

Remark 2: From (12), we can see that

$$\|G - L_n g\|_\infty = \begin{cases}\mathcal{O}(n^{-(2k/3)}) & \text{for } k \leq 3\\ \mathcal{O}(n^{-2}) & \text{for } k \geq 3\end{cases}$$

In other words, the fastest convergence rate achievable is $\mathcal{O}(n^{-2})$ at most for an arbitrarily large roll-off rate of $g(s)$. The larger the value of k implies the smoother the impulse response of $g(s)$ at $t = 0$. The optimal rate of convergence of an N^{th} order rational approximants is $\mathcal{O}(N^{-k})$ (see [2]), thus the Laguerre approximation formula is far away from achieving the optimal rate of convergence especially when k is large. The same observation also occurred in the L_2 case where from (14) we have

$$\|G - L_n g\|_2 = \begin{cases}\mathcal{O}(n^{-(2k-1)/3}) & \text{for } k \leq 3\\ \mathcal{O}(n^{-2}) & \text{for } k > 3\end{cases}.$$

The corresponding optimal rate of convergence with an N^{th} order rational approximation is $\mathcal{O}(N^{-k+1/2})$ which again depends on the smoothness of the impulse response of $g(s)$ at $t = 0$. It can be seen that in both L_∞ and L_2 norms considered, there is a maximum achievable convergence rate $\mathcal{O}(n^{-2})$ which is rather slow. As a result, a high order n for $L_n(sT)$ is required for accurate approximations. Also, the upper bound of $|E_n(j\omega T)|$ in (8) used in the derivation is tight since a) it matches exactly the error rate at $\omega = 0$ as explained in Remark 1, b) the upper bound value of 2 is always achievable. Therefore, as long as the upper bound of $|g(j\omega)|$ in (9) truly reflects the low- and high-frequency characteristics of $|g(j\omega)|$, the convergence rates provided above are the best that can be obtained.

The Laguerre formula has a nice property that $L_n(sT)g(s)$ matches the Maclaurin series of $e^{-sT}g(s)$ up to and including the term s^2. This gives zero steady-state errors between $e^{-sT}g(s)$ and $L_n(sT)g(s)$ for polynomial time-inputs of the form $c_0 + c_1 t + c_2 t^2$ where $c_i, i = 1, 2, 3$ are some real constants (see [7]). □

We shall now illustrate the error bound formulas by means of an example.

Example: Consider

$$G(s) = \frac{e^{-s}}{(s+1)^2} \equiv e^{-sT}g(s)$$

and hence $g(s) = 1/(s+1)^2, T = 1$ with

$$|g(j\omega)| \leq \begin{cases}1 & \text{for } \omega \leq 1\\ \frac{1}{\omega^2} & \text{for } \omega \geq 1.\end{cases}$$

TABLE I
L_∞ AND L_2 ERRORS AND BOUNDS

n	$\|E_n(s)g(s)\|_\infty$	$\|E_n(s)g(s)\|_2$
1	9.891×10^{-2}	1.087×10^{-1}
	(2.404×10^{-1})	(1.869×10^{-1})
2	5.023×10^{-2}	6.473×10^{-2}
	(9.539×10^{-2})	(9.389×10^{-2})
3	3.246×10^{-2}	4.619×10^{-2}
	(5.556×10^{-2})	(6.265×10^{-2})
4	2.349×10^{-2}	3.594×10^{-2}
	(3.786×10^{-2})	(4.700×10^{-2})

Notice that in this case we have $M_l = M_h = \omega_c = T = 1, k = 2$ and $m = 0$ in earlier notation. Since $1 = \omega_c < \beta n^{2/3}$ for $n \in \mathbb{N}$, the L_∞ and L_2 bounds for the approximation error transfer function $E_n(s)g(s)$ according to Theorem 1 are given by

$$\|G - L_n g\|_\infty \leq \frac{2}{(\beta n^{2/3})^2} = \frac{1}{2(3^{2/3})n^{4/3}} \tag{15}$$

and

$$\|G - L_n g\|_2 \leq \frac{1}{3\sqrt{\pi} n}\sqrt{1 - \frac{1}{84n^2}}. \tag{16}$$

The achieved errors for $n = 1, 2, 3, 4$ are summarized in Table I (figures in parentheses are the calculated bounds according to (15) and (16)).

It can be observed that the bounds are about a factor of two higher than the actual errors. The close agreement between the true errors and their bounds depends very much on the upper bound estimation of $|g(j\omega)|$. This is one of those examples where such estimation is good. When the approximation order becomes higher, it is also expected that the true error and its bound will become closer (in a relative sense) since the upper bound of $|g(j\omega)|$ is more and more accurate at high frequencies.

III. CONCLUSION

In this note, we presented a detailed analysis on the commonly used Laguerre formula for approximating delay systems of the form $e^{-sT}g(s)$. It was shown that the approximation scheme converges in L_∞ and L_2 when certain mild conditions on the rational part $g(s)$ are satisfied. When $g(s)$ is of relative degree greater than three, maximum rate of convergence can be achieved at $\mathcal{O}(n^{-2})$. Easily computable error bounds are also constructed for the approximants.

APPENDIX A
PROOF OF INEQUALITIES IN PROPOSITION 1

Consider with $\phi \equiv \phi_n(\omega T) \geq 0$ and

$$f(\phi) := \frac{1}{36}\phi^6 - (8 + 2\phi^2 + (2\phi^2 - 8)\cos\phi - 8\phi\sin\phi)$$

we have

$$
\begin{aligned}
f^{(1)}(\phi) &= \frac{1}{6}\phi^5 + 2\phi^2\sin\phi + 4\phi\cos\phi - 4\phi \\
f^{(2)}(\phi) &= \frac{5}{6}\phi^4 + (2\phi^2 + 4)\cos\phi - 4 \\
f^{(3)}(\phi) &= \frac{10}{3}\phi^3 - (2\phi^2 + 4)\sin\phi + 4\phi\cos\phi \\
f^{(4)}(\phi) &= 2\phi(5\phi - \phi\cos\phi - 4\sin\phi) \\
&= 2\phi(\phi(1 - \cos\phi) + 4(\phi - \sin\phi)).
\end{aligned}
$$

Since

$$f(0) = f^{(1)}(0) = f^{(2)}(0) = f^{(3)}(0) = f^{(4)}(0) = 0$$

while $f^{(4)}(\phi) > 0$ for $\phi > 0$. Hence, for $\phi > 0$

$$f^{(3)}(\phi) > 0, \qquad f^{(2)}(\phi) > 0, \qquad f^{(1)}(\phi) > 0, \qquad f(\phi) > 0.$$

The following inequalities thus hold for $\phi \geq 0$

$$\frac{8 + 2\phi^2 + (2\phi^2 - 8)\cos\phi - 8\phi\sin\phi}{4 + \phi^2} \leq \frac{\phi^6}{36(4 + \phi^2)} \leq \frac{\phi^6}{144}$$

which implies

$$\sqrt{\frac{8 + 2\phi^2 + (2\phi^2 - 8)\cos\phi - 8\phi\sin\phi}{4 + \phi^2}} \leq \frac{\phi^3}{6\sqrt{4 + \phi^2}} \leq \frac{\phi^3}{12}$$

and this is the required result. □

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An Eigenstructure Assignment Algorithm for the Design of Fault Detection Filters

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***Abstract*—In this paper, we present an algorithm for the construction of detection spaces in a fault detection filter by direct eigenstructure assignment. The algorithm permits great flexibility and simplifies the process of designing fault detection filters.**

I. INTRODUCTION

In recent years there has been growing interest in the application of model-based fault detection theory in the aerospace, chemical, and automotive industries. Among the various approaches that have been proposed, the fault detection filter (FDF) has received considerable attention in the literature; detection filters are designed so that the output error residual vector has directional characteristics that can be easily associated with a set of faults. Each additive fault event vector is associated with a detection space, spanned by eigenvectors which are colinear in output space. This property guarantees a fixed-direction output residual, and permits straight forward isolation of each fault. The detection filter design process consists of assigning the eigenstructure of the detection space associated with each fault to achieve the desired directional properties. The construction of the detection space has been studied by many researchers [1]–[5] and is a critical aspect in the design and implementation of FDF's. The subspace orthogonal to the detection space is termed the completion space; the design of the completion space is not of interest in the present note, although it will be shown that the algorithm introduced here can be used to iteratively assign the desired eigenstructure until no freedom is left (and therefore the completion space is the null space).

As the detection space associated with a fault becomes of larger dimension, the number of detection-equivalent faults (which are defined in [3]) is increased, and the isolation of the fault event vector becomes increasingly difficult. Therefore, for most practical applications, one- or two-dimensional detection spaces are the most common (and desirable) occurrence. While the construction of a one-dimensional detection space is a well understood procedure (consisting of the assignment of one eigen pair), there is a lack of simple algorithms for the design of two- (or greater) dimensional detection spaces.

In this paper, we present an algorithm that permits the construction of two-dimensional detection spaces in a fault detection filter by direct eigenstructure assignment. The algorithm greatly simplifies the process of designing fault detection filters with respect to other constructive algorithms proposed in [1]–[4]. Further, the procedure presented in this note can be applied iteratively to accommodate detection and isolation of multiple faults.

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II. A NEW FORMULATION OF THE FAULT DETECTION FILTER

In this section a new formulation for the detection filter is proposed; the novelty in our approach lies in the ability to generate a closed-form expression for the detection filter, leading to a greatly simplified design procedure and to a more intuitive interpretation. In this paper, the formulation is used mainly to derive a very simple algorithm for the construction of a two-dimensional detection space by direct eigenstructure assignment.

A. Preliminaries

Consider a linear, time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + f_i n(t) \tag{2.1}$$

$$y(t) = Cx(t) \tag{2.2}$$

where $x \in R^n$ is a state vector, $u \in R^p$ is a control vector, $y \in R^m$ is a measurement (or sensor) output vector, and A, B, C are real matrices of compatible dimensions. For simplicity, we shall omit the argument (t) from here on.

In the remainder of this paper we shall assume that the pair (A, C) is observable and that faults may be modeled by an additive term in (2.1): where $f_i \in R^n$ is defined as an actuator fault event vector, and $n(t)$ is a scalar function which represents the evolution of the fault. It is well known that the fault model mentioned above applies to actuator and some component faults. Further, it has recently been shown [5], [6] that this model can also represent all sensor faults providing these are suitably modeled.

B. Fault Detection Space and Order

Consider the system described by (2.1)–(2.2). A detection filter takes the form

$$\dot{\hat{x}} = A\hat{x} + Bu + D(y - \hat{y}) \tag{2.3}$$

$$\hat{y} = C\hat{x} \tag{2.4}$$