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Local Stability of Limit Cycles for Time-Delay Relay-Feedback Systems

Chong Lin, Qing-Guo Wang, Tong Heng Lee, and James Lam

***Abstract*—This brief is concerned with the local stability of limit cycles for linear systems under relay feedback, for the cases where the linear system includes a time-delay in its dynamics and the relay can possess asymmetric hysteresis. The limit cycle considered can be asymmetric, have more than two switchings a period, zero output derivatives at the switching instants. It shows that if a certain constructed matrix is Schur stable, then, the local stability of the considered limit cycle is guaranteed. The effectiveness of the presented results is illustrated by a numerical example.**

***Index Terms*—Hysteresis, limit cycles, local stability, relay-feedback systems, time delay.**

I. Introduction

Relay-feedback systems have been widely employed in a rather broad range of settings for many decades. One of the important particularity of relay-feedback systems, as well as many other nonlinear systems, is that periodic motions may occur in the trajectories. These periodic orbits are often termed limit cycles if they are isolated and have a limiting nature that attracts and/or repels nearby trajectories. The limit cycle property is very useful in modern control applications such as automatic tuning of controllers and identification [2], [3], [15]. This activates the intensive investigation for limit cycle behaviors. The involved study consists in establishing their existence, determining their frequency and form, investigating their stability and so on. For single-input single-output (SISO) systems, the existence problem was investigated early by describing the function method [4], [13]. Exact methods are reported recently in [1] to determine limit cycles with two switches a period. This type of periodic orbits is revisited and investigated further in [14] for delay-free systems. Another work [8] presents a sufficient condition for the existence of a symmetric stable limit cycle with chattering. For evaluating limit cycle periods and characteristics for multi-input multi-output (MIMO) systems, see [11], [12].

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Another important analysis topic is the stability of limit cycles. This includes mainly *local* stability and *global* stability of limit cycles. The local stability ensures that all nearby trajectories converge asymptotically to the limit cycle as time tends to infinity while the global stability means that such converge is ensured for all trajectories. Some classical techniques such as phase–plane approach are employed in [7], [13]. Exact methods have also been reported. See [1], [6], [8]–[10] and the references therein. Astrom [1] gives elegant criteria for the local stability of limit cycles by considering the linear approximation of the Poincare map. Johansson *et al.* [8], [9] emphasize the fast switches and present local stability results for limit cycles with sliding motion. In [5], the method of linear matrix inequalities is used to compute a local stability bound. Another discussion for the local stability is given in [10]. For the global stability of limit cycles, a recent paper [6] obtains sufficient conditions in terms of a set of linear matrix inequalities by finding the so-called surface Lyapunov function of Poincare maps. As seen, in the stability analysis, a limit cycle is always assumed to exist *a priori*.

In this brief, we consider the local stability of limit cycles for a time-delay relay-feedback system with the relay containing asymmetric hysteresis. The relay is not required merely to switch two times a period and the assumed limit cycle is not confined to be symmetric. Besides, it is not required that the trajectory of the limit cycle is nontangent with the switching planes at the switching instants. From an engineering point of view, time-delay systems are of considerable interest (most industrial processes have time delay). Theoretically, the nonzero time delay ensures a system trajectory evolves uniquely at the intersecting points. Also, the nonzero time delay makes it possible to relax the nontangent condition of the trajectory of the limit cycle at the traversing points. This relies on the continuity at the intersecting points, and intuitively it is the "overshoot" effect. For delay-free systems, the nontangent condition at traversing points has to be assumed, like the case considered in [1], [8], and [10]. Such a condition makes the local stability analysis simpler. We will includes this case for delay-free systems in Remark 3.1.

This brief is organized as follows. Section II is the problem formulation. Section III presents a sufficient condition for the local stability of a limit cycle with two switchings a period. Section IV gives the extension result for limit cycles with more than two switchings a period. An illustrative example is also given. This brief is concluded in Section V.

II. Problem Formulation and Preliminaries

In this brief, the following notations are adopted.

$\mathbb{R}$	Field of real numbers.
$\mathbb{R}^n$	n-dimensional real Euclidean space.
I	Identity matrix.
A^{-1}	Inverse of matrix A.
$\lambda(A)$, $\rho(A)$	Eigenvalues, spectral radius of square matrix A.
$\in, \forall, \sum$	Belong to, for all, sum, respectively.
$\lvert\cdot\rvert$, $\lVert\cdot\rVert$	Absolute value (or modulus), spectural norm, respectively.
$f(t_-)$	$= \lim_{\epsilon\to 0^+} f(t-\epsilon)$.
$\prod_{i=1}^k A_i$	$= A_k A_{k-1}\cdots A_1$.
$m!$	$= m(m-1)\cdots 2\cdots 1$ for nonnegative integer m.
$O(t^k)$	Infinitesimal of order t^k.

Consider a SISO plant described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + bu(t-\tau)\\ y(t) &= cx(t)\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, and $u(t-\tau) \in \mathbb{R}$ are the state, output, and control input, respectively; A, b, c are constant real matrices or vectors with appropriate dimensions; $\tau > 0$ stands for the time delay. The plant is under relay feedback, as shown in (2), at the bottom of the page, where $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ stand for the hysteresis; u_α, $u_\beta \in \mathbb{R}$ and $u_\alpha \neq u_\beta$. Due to time delay $\tau > 0$, we specify the initial function $u(\tilde{t})$ for $\tilde{t} \in [-\tau, 0]$ as

$$u(\tilde{t}) \equiv \begin{cases} u_\beta, & \text{if } y(0) > \alpha \\ u_\alpha, & \text{if } y(0) \leq \alpha. \end{cases} \tag{3}$$

We call (1)–(3) a relay-feedback system and denote it by Σ.

We see for system Σ, if $\tau > 0$ then, the existence and uniqueness of trajectories is guaranteed. If $\tau = 0$, the existence and uniqueness is always guaranteed if $\alpha > \beta$; if $\alpha = \beta$, the existence of solutions is discussed in [6]. Note that with the definition of relay as above or in [6], even if existence is guaranteed, uniqueness is not.

We define the switching planes as

$$\mathcal{S}_\alpha := \{\xi \in \mathbb{R}^n : c\xi = \alpha\} \tag{4}$$

$$\mathcal{S}_\beta := \{\xi \in \mathbb{R}^n : c\xi = \beta\}. \tag{5}$$

Let $\mathcal{S}_{+\alpha} := \{\xi \in \mathbb{R}^n : c\xi > \alpha\}$ and $\mathcal{S}_{-\alpha} := \{\xi \in \mathbb{R}^n : c\xi < \alpha\}$, and let $\mathcal{S}_{+\beta}$ and $\mathcal{S}_{-\beta}$ be defined similarly. Starting at time $t = 0$ with $y(0) > \alpha$ (respectively, $y(0) \leq \alpha$), if a trajectory of system Σ intersects $\mathcal{S}_\alpha$ (respectively, $\mathcal{S}_\beta$) at x_α (respectively, x_β) from $\mathcal{S}_{+\alpha}$ (respectively, $\mathcal{S}_{-\beta}$), we call the state x_α (respectively, x_β) an *intersecting point*. The time corresponding to the intersecting point is called *intersecting instant*. It should be stressed that in our convention, if a trajectory intersects $\mathcal{S}_\alpha$ (respectively, $\mathcal{S}_\beta$) at x_α (respectively, x_β) from $\mathcal{S}_{-\alpha}$ (respectively, $\mathcal{S}_{+\beta}$), the state x_α (respectively, x_β) is not an intersecting point and the corresponding time is not intersecting instant, since such intersecting does not cause any switch in $u(t)$. If a trajectory not only intersects but also traverses $\mathcal{S}_\alpha$ (respectively, $\mathcal{S}_\beta$) at x_α (respectively, x_β) from $\mathcal{S}_{+\alpha}$ (respectively, $\mathcal{S}_{-\beta}$) to $\mathcal{S}_{-\alpha}$ (respectively, $\mathcal{S}_{+\beta}$), we call such an intersecting point x_α (respectively, x_β) a *traversing point*. The time corresponding to the traversing point is called *traversing instant*. It should be noted that for $\tau > 0$, at traversing instant, the relay $u(t-\tau)$ remains u_β (or u_α) for a time duration of τ after which it changes to u_α (or u_β).

III. Local Stability of Limit Cycles

In the local stability analysis for limit cycles of system Σ, we assume that there exists a limit cycle x^* of the following form.

Form 1: The limit cycle x^* makes the relay switch twice a period with traversing points $x^*_\alpha \in \mathcal{S}_\alpha$ and $x^*_\beta \in \mathcal{S}_\beta$. The period is $(\tau + h_\alpha) + (\tau + h_\beta)$ with $h_\alpha > 0$ and $h_\beta > 0$, where $\tau + h_\alpha$ (respectively, $\tau + h_\beta$) is the time for x^* to move from x^*_α to x^*_β (respectively, from x^*_β to x^*_α).

For illustration, see Fig. 1 where $x^*(t)$ denotes the system solution corresponding to the limit cycle x^*. As for determining the existence and the period of a limit cycle of the above form with $h_\alpha = h_\beta$, a numerical method is stated in [1], and, for $\tau = 0$, the result is further developed in [14]. For determining a limit cycle in Form 1, the following is a straightforward necessary condition.

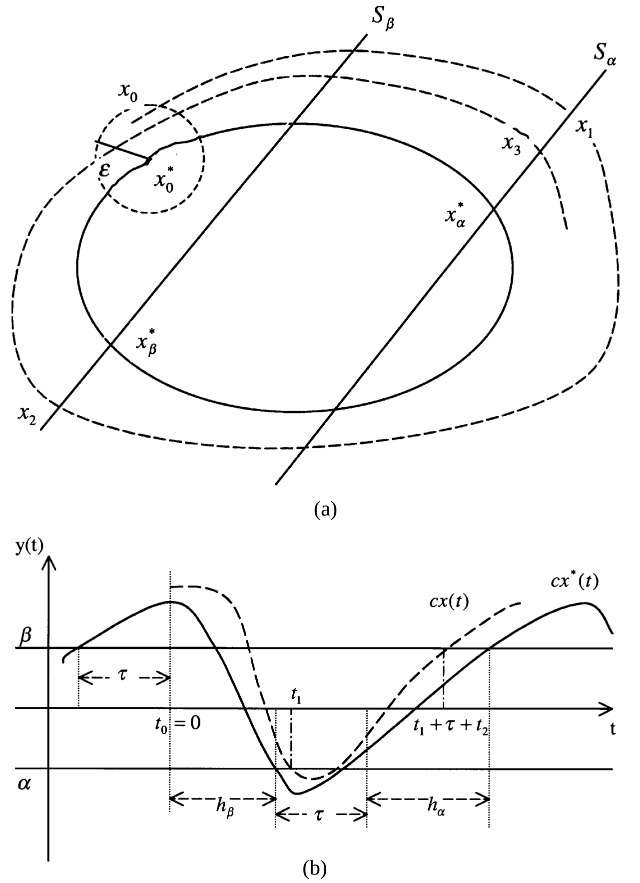

Fig. 1. (a) The trajectories of $x^*(t)$ and $x(t)$ starting from $x_0 \in \mathcal{R}_\epsilon$. (b) The trajectories of $cx^*(t)$ (solid) and $cx(t)$ (dashed).

Proposition 3.1: Assume that A has no roots in the imaginary axis. If there is a limit cycle in Form 1, then h_α and h_β satisfy the following:

$$\begin{aligned} \alpha =& c\left(I - e^{A(2\tau+h_\alpha+h_\beta)}\right)^{-1} \\ &\times \left(\int_{h_\beta}^{h_\alpha+\tau+h_\beta} e^{As} b u_\alpha ds + \int_{h_\alpha+\tau+h_\beta}^{2\tau+h_\alpha+h_\beta} e^{As} b u_\beta ds \right. \\ &\left. + \int_0^{h_\beta} e^{As} b u_\beta ds\right) \\ \beta =& c\left(I - e^{A(2\tau+h_\alpha+h_\beta)}\right)^{-1} \\ &\times \left(\int_{h_\alpha}^{h_\alpha+\tau+h_\beta} e^{As} b u_\beta ds + \int_{h_\alpha+\tau+h_\beta}^{2\tau+h_\alpha+h_\beta} e^{As} b u_\alpha ds \right. \\ &\left. + \int_0^{h_\alpha} e^{As} b u_\alpha ds\right) \end{aligned} \tag{6}$$

and x^*_α and x^*_β are given by

$$\begin{aligned} x^*_\alpha =& \left(I - e^{A(2\tau+h_\alpha+h_\beta)}\right)^{-1} \\ &\times \left(\int_{h_\beta}^{h_\alpha+\tau+h_\beta} e^{As} b u_\alpha ds + \int_{h_\alpha+\tau+h_\beta}^{2\tau+h_\alpha+h_\beta} e^{As} b u_\beta ds \right. \\ &\left. + \int_0^{h_\beta} e^{As} b u_\beta ds\right) \end{aligned}$$

$$u(t) = \begin{cases} u_\beta, & \text{if } y(t) > \beta, \text{ or } y(t) > \alpha \text{ and } u(t_-) = u_\beta, \\ u_\alpha, & \text{if } y(t) < \alpha, \text{ or } y(t) < \beta \text{ and } u(t_-) = u_\alpha \\ u_\beta \text{ or } u_\alpha, & \text{if } y(t) = \alpha \text{ and } u(t_-) = u_\beta \text{ or } y(t) = \beta \text{ and } u(t_-) = u_\alpha \end{cases} \tag{2}$$

$$x_\beta^* = \left(I - e^{A(2\tau+h_\alpha+h_\beta)}\right)^{-1} \times \left(\int_{h_\alpha}^{h_\alpha+\tau+h_\beta} e^{As} bu_\beta ds + \int_{h_\alpha+\tau+h_\beta}^{2\tau+h_\alpha+h_\beta} e^{As} bu_\alpha ds + \int_0^{h_\alpha} e^{As} bu_\alpha ds\right). \tag{7}$$

Proof: By assumption, we see that $I - e^{At}$ is invertible for $t \neq 0$. The desired result follows easily from the expressions of the solution corresponding to the limit cycle. □

Without loss of generality, we set $t_0 = 0$ corresponding to the time instant when the trajectory of x^* makes the relay switch from u_α to u_β, see Fig. 1. We define

$$\begin{aligned}\mathcal{R}_\epsilon &:= \{\xi \in R^n : \|\xi - x_0^*\| \leq \epsilon\} \\ &= \{\xi \in R^n : \xi = x_0^* + \Delta, \Delta \in R^n, \|\Delta\| \leq \epsilon\}.\end{aligned} \tag{8}$$

Since $cx_0^* > \alpha$, let a scalar ϵ_1 satisfy

$$0 < \epsilon_1 < \|c\|^{-1}(cx_0^* - \alpha). \tag{9}$$

Then, from (3), $u(-\tau) = u_\beta$ holds for any trajectory starting from $\mathcal{R}_{\epsilon_1}$.

To achieve our stability result, we need to establish some lemmas first. Let

$$\mathcal{N} = \{0, 1, \ldots, n-1\}. \tag{10}$$

The first lemma specifies two integers $n_\alpha, n_\beta \in \mathcal{N}$, which will be used in the development.

Lemma 3.1: For the limit cycle x^* in Form 1, there exist two even integers $n_\alpha, n_\beta \in \mathcal{N}$ such that

$$\begin{aligned} cA^{i+1}x_\alpha^* + cA^i bu_\beta &= 0, \qquad i = 0, 1, \ldots, n_\alpha - 1 \\ cA^{n_\alpha+1}x_\alpha^* + cA^{n_\alpha} bu_\beta &< 0 \end{aligned} \tag{11}$$

$$\begin{aligned} cA^{j+1}x_\beta^* + cA^j bu_\alpha &= 0, \qquad j = 0, 1, \ldots, n_\beta - 1 \\ cA^{n_\beta+1} + x_\beta^* cA^{n_\beta} bu_\alpha &> 0. \end{aligned} \tag{12}$$

Proof: See Appendix. □

It is seen that if $n_\alpha = n_\beta = 0$, then, the trajectory of the limit cycle is nontangent with the switching planes, $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$, at the traversing points. The conditions in Lemma 3.1 ensures that the vector fields point in the "right" direction on both sides of the switching planes, e.g., the trajectory of the limit cycle traverses the switching planes. Here, for convenience, we introduce some quantities for later use. For $t \in \mathbb{R}$, define

$$\begin{aligned} F_\alpha(t) &:= (e^{At} - I)x_\alpha^* + \int_0^t e^{As} bu_\beta ds \\ f_\alpha(t) &:= cF_\alpha(t) \\ F_\beta(t) &:= (e^{At} - I)x_\beta^* + \int_0^t e^{As} bu_\alpha ds \\ f_\beta(t) &:= cF_\beta(t). \end{aligned} \tag{13}$$

By defining

$$\begin{aligned} \left.\frac{f_\alpha(t)}{t^{n_\alpha+1}}\right|_{t=0} &:= \lim_{t\to 0} \frac{f_\alpha(t)}{t^{n_\alpha+1}} \\ &= \frac{1}{(n_\alpha+1)!}\left(cA^{n_\alpha+1}x_\alpha^* + cA^{n_\alpha} bu_\beta\right) < 0 \\ \left.\frac{f_\beta(t)}{t^{n_\beta+1}}\right|_{t=0} &:= \lim_{t\to 0} \frac{f_\beta(t)}{t^{n_\alpha+1}} \\ &= \frac{1}{(n_\beta+1)!}\left(cA^{n_\beta+1}x_\beta^* + cA^{n_\beta} bu_\alpha\right) > 0 \end{aligned}$$

there exist two scalars $r_\alpha^* > 0$ and $r_\beta^* > 0$ such that $f_\alpha(t)/t^{n_\alpha+1} < 0$ and $f_\beta(t)/t^{n_\alpha+1} > 0$ are continuous on $t \in [-r_\alpha^*, r_\alpha^*]$ and $t \in [-r_\beta^*, r_\beta^*]$, respectively. Let

$$r_{\min} = \min\left\{h_\alpha, h_\beta, r_\alpha^*, r_\beta^*\right\}. \tag{14}$$

We denote

$$\mathcal{S}_{(\epsilon, x_\alpha^*)} := \{\xi \in \mathcal{S}_\alpha : \|\xi - x_\alpha^*\| \leq \epsilon\} \tag{15}$$

$$\mathcal{S}_{(\epsilon, x_\beta^*)} := \left\{\xi \in \mathcal{S}_\beta : \left\|\xi - x_\beta^*\right\| \leq \epsilon\right\}. \tag{16}$$

Now, we analyze the trajectory starting from a nearby point to x_0^*. By continuity, if $\|x_0 - x_0^*\|$ is small enough, then the trajectory of $x(t)$ starting from x_0 will traverse $\mathcal{S}_\alpha$ at a nearby point to x_α^*. Besides, the time taken by the trajectory to move to the traversing point is close to h_β. To study the local stability of x^*, we need to verify the occurrence of successive switchings. The next lemma is useful, which characterizes a fixed scalar $\epsilon_{\delta_0} > 0$ such that any trajectory evolving from traversing points in $\mathcal{S}_{(\epsilon_{\delta_0}, x_\alpha^*)}$ (or $\mathcal{S}_{(\epsilon_{\delta_0}, x_\beta^*)}$) will traverse $\mathcal{S}_\beta$ (or $\mathcal{S}_\alpha$).

Lemma 3.2: For any $\delta_0 \in (0, r_{\min}]$, there exists a scalar $\epsilon_{\delta_0} > 0$ such that the trajectory evolving from any traversing point in $\mathcal{S}_{(\epsilon_{\delta_0}, x_\alpha^*)}$ (or $\mathcal{S}_{(\epsilon_{\delta_0}, x_\beta^*)}$) (Here, set the traversing instant to be zero.) will traverse $\mathcal{S}_\beta$ (or $\mathcal{S}_\alpha$), and the traversing instant $\tau + t_{\text{trav}}$ satisfies $|t_{\text{trav}} - h_\alpha| < \delta_0$ (or $|t_{\text{trav}} - h_\beta| < \delta_0$).

Proof: See Appendix. □

Let the first traversing point be $x(t_1) \in \mathcal{S}_\alpha$. Then, $\|x(t_1) - x_\alpha^*\|$ can be made arbitrarily small by choosing x_0 close to x_0^*. The next lemma concerns the second traversing point.

Lemma 3.3: There exists $\epsilon_2 \in (0, \epsilon_1]$ such that any trajectory starting from $\mathcal{R}_{\epsilon_2}$ will traverse $\mathcal{S}_\beta$ after the first traversing instant t_1, and the second traversing point $x(t_1 + \tau + t_2)$ satisfies

$$x(t_1 + \tau + t_2) - x_\beta^* = \left(I - \frac{F_\beta(t_2 - h_\alpha)c}{f_\beta(t_2 - h_\alpha)}\right) \times e^{A(\tau+t_2)}(x(t_1) - x_\alpha^*) \tag{17}$$

where $\tau + t_2$ with $t_2 > 0$ is the time duration for $x(t)$ to move from $x(t_1)$ to $x(t_1 + \tau + t_2)$.

Proof: See Appendix. □

To specify a local stability region $\mathcal{R}_\epsilon$, we need the following lemma as well.

Lemma 3.4: Given a positive integer p, suppose that A_i, $\Theta_{ij} \in \mathcal{R}^{\backslash\times\backslash}$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots)$ and $\rho(A_1 A_2 \cdots A_p) < 1$. Then, there exists $\theta_0 > 0$ such that for all Θ_{ij} satisfying $\|\Theta_{ij} - A_i\| \leq \theta_0$, it holds $\|\prod_{j=1}^k (\Theta_{1j}\Theta_{2j}\cdots\Theta_{pj})\| \to 0$ as $k \to \infty$.

Proof: Since $\rho(A_1 A_2 \cdots A_p) < 1$, there is a scalar $\theta > 0$ such that for all $\Theta_i \in R^{n\times n}$ satisfying $\|\Theta_i\| \leq \theta$, it holds $\|\prod_{j=1}^k (A_1 A_2 \cdots A_p + \Theta_j)\| \to 0$ as $k \to \infty$. For this $\theta > 0$, there exists $\theta_0 > 0$ such that if $\|\Theta_{ij} - A_i\| \leq \theta_0$, then the matrix $\Theta_{1j}\Theta_{2j}\cdots\Theta_{pj}$ is expressed as

$$\Theta_{1j}\Theta_{2j}\cdots\Theta_{pj} = A_1 A_2 \cdots A_p + \Omega_j$$

where Ω_j satisfies $\|\Omega_j\| \leq \theta$. This proves the lemma. □

With the above lemmas in hands, we are now in a position to present the main result.

Theorem 3.1: The limit cycle x^* in Form 1 is locally stable if

$$\rho(W_1 W_2) < 1 \tag{18}$$

where

$$\begin{aligned} W_1 &= \left(I - \frac{A^{n_\alpha}(Ax_\alpha^* + bu_\beta)c}{cA^{n_\alpha}(Ax_\alpha^* + bu_\beta)}\right) e^{A(\tau+h_\beta)} \\ W_2 &= \left(I - \frac{A^{n_\beta}(Ax_\beta^* + bu_\alpha)c}{cA^{n_\beta}(Ax_\beta^* + bu_\alpha)}\right) e^{A(\tau+h_\alpha)}. \end{aligned} \tag{19}$$

Here, n_α and n_β are even integers as given in Lemma 3.1.

Proof: See Appendix. □

Theorem 3.1 presents a criterion to check the local stability of the limit cycle x^*. The idea is to find a scalar $\epsilon > 0$ such that any trajectory starting from $\mathcal{R}_\epsilon$ of the form (8) will tend asymptotically to $x^*(t)$ and make the relay switch consecutively. Since $\rho(W_2 W_1) = \rho(W_1 W_2)$, then $\rho(W_2 W_1) < 1$ is an alternative sufficient condition. This can be verified by letting $t_0 = 0$ correspond to a time instant when the relay switches from u_β to u_α.

Remark 3.1: We should make it clear that our results in this brief are for the case $\tau > 0$. If $\tau = 0$, the technique developed here is not applicable due to possible occurrence of multiple trajectories at traversing instant. For $\tau = 0$, if $n_\alpha = n_\beta = 0$, then Theorem 3.1 still works. Indeed, $n_\alpha = n_\beta = 0$ implies that the limit cycle is nontangent with the switching planes at traversing instant, like the case considered in [1], [8], [10]. However, the methods used in [1], [8], and [10] are not applicable to deal with the local stability of limit cycles in Form 1.

IV. Extension

In this section, we give an extension result for the local stability of limit cycles with $2q(q \geq 1)$ switchings a period. The limit cycle considered is as follows.

Form 2: The limit cycle x^* makes the relay switch $2q$ times a period with traversing points $x^*_{\alpha i} \in \mathcal{S}_\alpha$ and $x^*_{\beta i} \in \mathcal{S}_\beta$ $(i = 1, 2, \ldots, q)$. The period is $\sum_{i=1}^{q}(\tau + h_{\alpha i} + \tau + h_{\beta i})$ with $h_{\alpha i} > 0$ and $h_{\beta i} > 0$ $(i = 1, 2, \ldots, q)$, where $\tau + h_{\alpha i}$ (respectively, $\tau + h_{\beta i}$) is the time duration for x^* to move from $x^*_{\alpha i}$ to $x^*_{\beta i}$ (respectively, from $x^*_{\beta i}$ to $x^*_{\alpha(i+1)}$). Note that $x^*_{\alpha(q+1)} = x^*_{\alpha 1}$.

Similar to Lemma 3.1, there exist $2q$ even integers $n_{\alpha l}, n_{\beta l} \in \mathcal{N}$, $l = 1, 2, \ldots, q$, such that the following holds for all $l = 1, 2, \ldots, q$:

$$\begin{aligned} cA^{i+1}x^*_{\alpha l} + cA^i bu_\beta &= 0, \qquad i = 0, 1, \ldots, n_{\alpha l} - 1, \\ cA^{n_{\alpha l}+1}x^*_{\alpha l} + cA^{n_{\alpha l}} bu_\beta &< 0 \\ cA^{i+1}x^*_{\beta l} + cA^i bu_\alpha &= 0, \qquad i = 0, 1, \ldots, n_{\beta l} - 1, \\ cA^{n_{\beta l}+1}x^*_{\beta l} + cA^{n_{\beta l}} bu_\alpha &> 0. \end{aligned} \tag{20}$$

The extended stability result in this section is as follows.

Theorem 4.1: The limit cycle in Form 2 is locally stable if for some $k \in \{1, 2, \ldots, 2q\}$, it holds

$$\rho(W_k W_{k-1} \cdots W_1 W_{2q} W_{2q-1} \cdots W_{k+1}) < 1 \tag{21}$$

where, for $l = 1, 2, \ldots, q$

$$\begin{aligned} W_{2l-1} &= \left(I - \frac{A^{n_{\alpha l}}(Ax^*_{\alpha l} + bu_\beta)c}{cA^{n_{\alpha l}}(Ax^*_{\alpha l} + bu_\beta)}\right) e^{A(\tau + h_{\beta(l-1)})} \\ W_{2l} &= \left(I - \frac{A^{n_{\beta l}}(Ax^*_{\beta l} + bu_\alpha)c}{cA^{n_{\beta l}}(Ax^*_{\beta l} + bu_\alpha)}\right) e^{A(\tau + h_{\alpha l})}. \end{aligned} \tag{22}$$

Here, $h_{\beta 0} = h_{\beta q}$.

Proof: The proof follows a similar line to that of Theorem 3.1 and, thus, is omitted here. □

Finally, we give a numerical example to illustrate the use of our results.

Example 4.1: Consider system Σ with

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \\ b &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ c &= [1 \quad 0 \quad 0] \\ \tau &= 0.1 \\ \alpha &= -0.1 \\ \beta &= 0.2 \\ u_\alpha &= 2 \\ u_\beta &= -1. \end{aligned}$$

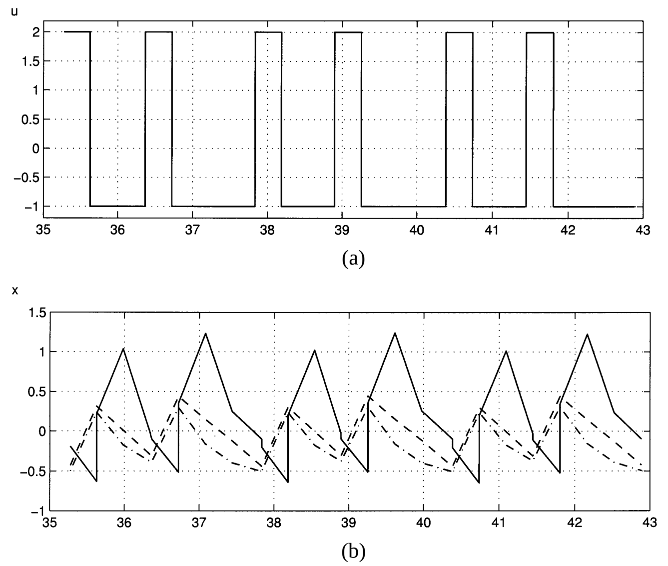

Fig. 2. (a) The control $u(t)$; (b) The trajectories of $x^*(t) = [x^*_1(t) \quad x^*_2(t) \quad x^*_3(t)]^T$ ($x^*_1(t)$ solid, $x^*_2(t)$ dash-dot, $x^*_3(t)$ dashed).

We check that A is not Hurwitz, but the system has a limit cycle as shown in Fig. 2. The limit cycle meets Form 2 with $q = 2$. The period and the four traversing points are computed to be

$$\begin{aligned} h_{\alpha 1} &= 0.25 \\ h_{\alpha 2} &= 0.25 \\ h_{\beta 1} &= 0.65 \\ h_{\beta 2} &= 1.05 \\ x_{\alpha 1} &= \begin{bmatrix} -0.1 \\ -0.5 \\ -0.43 \end{bmatrix} \\ x_{\alpha 2} &= \begin{bmatrix} -0.1 \\ -0.39 \\ -0.3 \end{bmatrix} \\ x_{\beta 1} &= \begin{bmatrix} 0.2 \\ 0.23 \\ 0.3 \end{bmatrix} \\ x_{\beta 2} &= \begin{bmatrix} 0.2 \\ 0.3 \\ 0.44 \end{bmatrix}. \end{aligned}$$

It is easy to obtain from (20) that $n_{\alpha 1} = n_{\alpha 2} = n_{\beta 1} = n_{\beta 2} = 0$. We further compute from (22) that $\lambda(W_4 W_3 W_2 W_1) = \{0, 0.0055, 0.0743\}$, giving $\rho(W_4 W_3 W_2 W_1) < 1$. Hence, we conclude from Theorem 4.1 that the limit cycle is locally stable.

V. Conclusion

This brief studies the local stability of limit cycles for time-delay relay-feedback systems. The considered limit cycle is not confined to be symmetric, and its trajectory is not required to be nontangent with the switching planes at the traversing instants. Sufficient conditions are established based on the state-space method. It is noted that the stability analysis in this brief is based on a small starting region $\mathcal{R}_\epsilon$. How to verify the stability within a large starting region (or even the whole space) deserves a study. The extensions of our results to MIMO systems is also very important for future research work.

APPENDIX

Proof of Lemma 3.1

We take the proof of (12) for example. For (11), the proof is similar.

Let the instant $t = t_\beta$ correspond to $x^*(t_\beta) = x^*_\beta$. For a sufficiently small $\delta > 0$, we have the following expansion of $x^*(t)$ in $t \in [t_\beta - \delta, t_\beta + \delta]$

$$x^*(t) = x^*(t_\beta) + \sum_{i=0}^{n_\beta} \frac{1}{(i+1)!} A^i (Ax^*_\beta + bu_\alpha)(t - t_\beta)^{i+1} + O(t - t_\beta)^{n_\beta + 2}$$

where $n_\beta \geq 0$ is an integer such that

$$\begin{aligned} cA^{i+1}x^*_\beta + cA^i bu_\alpha &= 0, \qquad i = 0, 1, \ldots, n_\beta - 1 \\ cA^{n_\beta+1}x^*_\beta + cA^{n_\beta} bu_\alpha &\neq 0. \end{aligned}$$

From the Cayley–Hamilton Theorem, it is easy to get that $n_\beta \in \mathcal{N}$. Since $cx^*_\beta = \beta$, $cx^*(t) > \beta$ for $t \in (t_\beta, t_\beta + \delta]$ and $cx^*(t) < \beta$ for $t \in [t_\beta - \delta, t_\beta)$, we have

$$\begin{aligned} &\frac{1}{(n_\beta+1)!} cA^{n_\beta}(Ax^*_\beta + bu_\alpha)(t - t_\beta)^{n_\beta+1} \\ &\quad + cO(t - t_\beta)^{n_\beta+2} > 0, \qquad t \in (t_\beta, t_\beta + \delta] \\ &\frac{1}{(n_\beta+1)!} cA^{n_\beta}(Ax^*_\beta + bu_\alpha)(t - t_\beta)^{n_\beta+1} \\ &\quad + cO(t - t_\beta)^{n_\beta+2} < 0, \qquad t \in [t_\beta - \delta, t_\beta). \end{aligned}$$

Letting $t \to t_\beta$ from both sides, we see that n_β must be even and the following holds:

$$cA^{n_\beta+1}x^*_\beta + cA^{n_\beta} bu_\alpha > 0.$$

This proves the lemma. □

Proof of Lemma 3.2

Firstly, consider the trajectory of $x(t)$ evolving from traversing points in $\mathcal{S}_\alpha$. The trajectory of $x^*(t)$ is governed by

$$\begin{aligned} x^*(t) &= e^{At}x^*_\alpha + \int_0^t e^{A(t-s)} bu_\beta ds \qquad \forall\, t \in [0, \tau] \\ x^*(t) &= e^{At}x^*_\alpha + \int_0^\tau e^{A(t-s)} bu_\beta ds \\ &\quad + \int_0^{t-\tau} e^{A(t-\tau-s)} bu_\alpha ds \qquad \forall\, t \in \quad [\tau, \tau + h_\alpha]. \end{aligned}$$

For some δ_1, satisfying $0 < \delta_1 < \min\{h_\alpha, \tau, \delta_0\}$, it holds that

$$\begin{aligned} cx^*(t) &< \beta \qquad \forall\, t \in [0, \tau + h_\alpha) \\ cx^*(t) &> \beta \qquad \forall\, t \in (\tau + h_\alpha, \tau + h_\alpha + \delta_1], \\ cx^*(\tau + h_\alpha) &= \beta. \end{aligned}$$

The trajectory of $x(t)$ evolving from $x^*_\alpha + \Delta \in \mathcal{S}_\alpha$ with small $\|\Delta\|$ is governed by

$$\begin{aligned} x(t) &= e^{At}(x^*_\alpha + \Delta) + \int_0^t e^{A(t-s)} bu_\beta ds \qquad \forall\, t \in [0, \tau] \\ x(t) &= e^{At}(x^*_\alpha + \Delta) + \int_0^\tau e^{A(t-s)} bu_\beta ds \\ &\quad + \int_0^{t-\tau} e^{A(t-\tau-s)} bu_\alpha ds \qquad \forall\, t \in [\tau, \tau + \delta_2] \\ cx(t) &< \beta \qquad \forall\, t \in [0, \tau + \delta_2], \end{aligned}$$

where $\delta_2 > 0$ is a sufficiently small scalar. By continuity, it can be shown that there exists $\epsilon_{\delta_0 1}$ such that the trajectory of $x(t)$ evolving from $x^*_\alpha + \Delta \in \mathcal{S}_{(\epsilon_{\delta_0 1}, x^*_\alpha)}$ will traverse $\mathcal{S}_\beta$. Moreover, the traversing instant $\tau + t_{\rm trav}$ satisfies $|t_{\rm trav} - h_\alpha| < \delta_1$, and thus, $|t_{\rm trav} - h_\alpha| < \delta_0$.

Next, consider traversing points in $\mathcal{S}_\beta$. Similarly, for the given $\delta_0 > 0$, there exists $\epsilon_{\delta_0 2}$ such that any trajectory evolving from $\mathcal{S}_{(\epsilon_{\delta_0 2}, x^*_\beta)}$ will traverse $\mathcal{S}_\alpha$, and the traversing instant $\tau + t_{\rm trav}$ satisfies $|t_{\rm trav} - h_\beta| < \delta_0$. The result follows immediately by letting $\epsilon_{\delta_0} = \min\{\epsilon_{\delta_0 1},\ \epsilon_{\delta_0 2}\}$. □

Proof of Lemma 3.3

If $\|\Delta_1\|$ is small, the trajectory of $x(t)$ evolving from $x(t_1) = \Delta_1 + x^*_\alpha \in \mathcal{S}_\alpha$ will traverse $\mathcal{S}_\beta$, and the time duration $\tau + t_2$ can be made approaching $\tau + h_\alpha$. Thus, there exists ϵ_2 satisfying $0 < \epsilon_2 \leq \epsilon_1$ such that the trajectory of $x(t)$ starting from $\mathcal{R}_{\epsilon_2}$ will make the time duration $\tau + t_2$ satisfy $|t_2 - h_\alpha| < r_{\min}$. Since $cx^*_\beta = cx(t_1 + \tau + t_2) = \beta$, where

$$\begin{aligned} x^*_\beta &= e^{A(\tau+h_\alpha)}x^*_\alpha + \int_0^\tau e^{A(\tau+h_\alpha-s)} bu_\beta ds \\ &\quad + \int_0^{h_\alpha} e^{A(h_\alpha-s)} bu_\alpha ds \\ x(t_1 + \tau + t_2) &= e^{A(\tau+t_2)}x(t_1) + \int_0^\tau e^{A(\tau+t_2-s)} bu_\beta ds \\ &\quad + \int_0^{t_2} e^{A(t_2-s)} bu_\alpha ds \end{aligned} \tag{23}$$

after some manipulations, we have

$$ce^{A(\tau+t_2)}(x(t_1) - x^*_\alpha) + f_\beta(t_2 - h_\alpha) = 0.$$

Noting that $(t_2 - h_\alpha)^{n_\beta+1} f_\beta^{-1}(t_2 - h_\alpha)$ is well-defined for $|t_2 - h_\alpha| \leq r_{\min}$, we arrive at

$$(t_2 - h_\alpha)^{n_\beta+1} = -\frac{(t_2 - h_\alpha)^{n_\beta+1} ce^{A(\tau+t_2)}}{f_\beta(t_2 - h_\alpha)}(x(t_1) - x^*_\alpha). \tag{24}$$

Using (23), we obtain

$$\begin{aligned} x(t_1 + \tau + t_2) - x^*_\beta &= e^{A(\tau+t_2)}(x(t_1) - x^*_\alpha) \\ &\quad + (e^{A(t_2-h_\alpha)} - I)x^*_\beta \\ &\quad + \int_0^{t_2-h_\alpha} e^{As} bu_\alpha ds \\ &= e^{A(\tau+t_2)}(x(t_1) - x^*_\alpha) \\ &\quad + \frac{F_\beta(t_2 - h_\alpha)}{(t_2 - h_\alpha)^{n_\beta+1}}(t_2 - h_\alpha)^{n_\beta+1} \\ &= \left(I - \frac{F_\beta(t_2 - h_\alpha)c}{f_\beta(t_2 - h_\alpha)}\right) \\ &\quad \times e^{A(\tau+t_2)}(x(t_1) - x^*_\alpha). \end{aligned}$$

This completes the proof. □

Proof of Theorem 3.1

Suppose $\rho(W_1W_2) < 1$. By virtue of Lemma 3.4, there exists a scalar $\theta_0 > 0$ such that for all $\Theta_{ij} \in R^{n\times n}$ $(i = 1, 2;\ j = 1, 2, \ldots)$ satisfying $\|\Theta_{ij} - W_i\| \leq \theta_0$, it holds $\|\prod_{j=1}^k(\Theta_{1j}\Theta_{2j})\| \to 0$ as $k \to \infty$. In other words, there is a positive integer N_0 such that for all $\Theta_{ij} \in R^{n\times n}$ satisfying $\|\Theta_{ij} - W_i\| \leq \theta_0$, it holds that

$$\left\|\prod_{j=1}^{N_0+k}(\Theta_{1j}\Theta_{2j})\right\| < 1 \qquad \forall\, k = 0, 1, 2, \ldots. \tag{25}$$

For $j = 1, 2, \ldots$, let

$$\begin{aligned} W(\delta_{1j}, x^*_\alpha) &= \left(I - \frac{F_\alpha(\delta_{1j})c}{f_\alpha(\delta_{1j})}\right) e^{A(\tau+h_\beta+\delta_{1j})} \\ W(\delta_{2j}, x^*_\beta) &= \left(I - \frac{F_\beta(\delta_{2j})c}{f_\beta(\delta_{2j})}\right) e^{A(\tau+h_\alpha+\delta_{2j})} \end{aligned} \tag{26}$$

where $\delta_{1j}, \delta_{2j} \in \mathbb{R}$. It is seen that $W(\delta_{1j}, x_\alpha^*) \to W_1$ and $W(\delta_{2j}, x_\beta^*) \to W_2$ as $\delta_{ij} \to 0$ for $i = 1, 2$, and $j = 1, 2, \ldots$. Thus, for the above $\theta_0 > 0$, there exists $\delta_0 > 0$ such that

$$\begin{aligned} \|W(\delta_{1j}, x_\alpha^*) - W_1\| \le& \theta_0, \\ \forall\, |\delta_{1j}| \le& \delta_0, \qquad j = 1, 2, \ldots \\ \|W(\delta_{2j}, x_\beta^*) - W_2\| \le& \theta_0 \\ \forall\, |\delta_{2j}| \le& \delta_0, \qquad j = 1, 2, \ldots. \end{aligned} \tag{27}$$

Let $\delta_{\min} = \min\{\delta_0, r_{\min}\}$ where $r_{\min}$ is as in (14). For this, $\delta_{\min} > 0$, and by Lemma 3.2, there exists a fixed scalar $\epsilon_{\min} > 0$ such that any trajectory evolving from traversing points in $\mathcal{S}_{(\epsilon_{\min}, x_\alpha^*)}$ (or $\mathcal{S}_{(\epsilon_{\min}, x_\beta^*)}$) will traverse $\mathcal{S}_\beta$ (or $\mathcal{S}_\alpha$) by spending time duration $\tau + t_{\text{trav}}$, where t_{trav} satisfies $|t_{\text{trav}} - h_\alpha| < \delta_{\min}$ (or $|t_{\text{trav}} - h_\beta| < \delta_{\min}$).

Now, let

$$\begin{aligned} w =& \max\{\|W_1\|, \|W_2\|\} \\ \bar{\epsilon}_{\min} =& \min\left\{\epsilon_{\min}, \frac{\epsilon_{\min}}{(w+\theta_0)^{2N_0}}\right\}. \end{aligned} \tag{28}$$

Then, there exists a scalar $\epsilon \in (0, \epsilon_1]$ such that any trajectory starting from $x_0 \in \mathcal{R}_\epsilon$ will traverse $\mathcal{S}_\alpha$, and moreover, the first traversing point x_1 satisfies $\|x_1 - x_\alpha^*\| \le \bar{\epsilon}_{\min} \le \epsilon_{\min}$. We show next that with this $\epsilon > 0$, $\mathcal{R}_\epsilon$ is a locally stable region. This is two folded, i.e., any trajectory starting from $\mathcal{R}_\epsilon$ will make the relay switch consecutively, and converge asymptotically to the limit cycle x^*. In what follows, if the ith $(i \ge 2)$ traversing occurs, we then denote the traversing point by x_i and the time duration for the trajectory to move from x_{i-1} to x_i by $\tau + t_i$.

Since $\|x_1 - x_\alpha^*\| \le \bar{\epsilon}_{\min} \le \epsilon_{\min}$, the second traversing will occur at $\mathcal{S}_\beta$. By virtue of Lemma 3.3 and the above analysis, x_2 and $\tau + t_2$ satisfy the following: [see (17)]

$$|t_2 - h_\alpha| \le \delta_{\min} \tag{29}$$

$$\begin{aligned} x_2 - x_\beta^* =& \left(I - \frac{F_\beta(t_2 - h_\alpha)c}{f_\beta(t_2 - h_\alpha)}\right) e^{A(\tau + t_2)}(x_1 - x_\alpha^*) \\ =& W(t_2 - h_\alpha, x_\beta^*)(x_1 - x_\alpha^*). \end{aligned} \tag{30}$$

From (29), we see that (27) holds, yielding $\|W(t_2 - h_\alpha, x_\beta^*)\| \le w + \theta_0$. Thus, (30) gives

$$\|x_2 - x_\beta^*\| \le (w + \theta_0)\bar{\epsilon}_{\min} \le \epsilon_{\min} \tag{31}$$

which implies that the third traversing will occur at $\mathcal{S}_\alpha$. Continue the process. At the $(2N_0 + 1)$th traversing point, there holds

$$\begin{aligned} |t_{2N_0+1} - h_\beta| \le& \delta_{\min} \\ x_{2N_0+1} - x_\alpha^* =& \left(I - \frac{F_\alpha(t_{2N_0+1} - h_\beta)c}{f_\alpha(t_{2N_0+1} - h_\beta)}\right) \\ &\times e^{A(\tau + t_{2N_0+1})}(x_{2N_0} - x_\beta^*) \\ =& W(t_{2N_0+1} - h_\beta, x_\alpha^*)(x_{2N_0} - x_\beta^*), \\ \|x_{2N_0+1} - x_\alpha^*\| \le& (w+\theta_0)^{2N_0}\bar{\epsilon}_{\min} \le \epsilon_{\min}. \end{aligned}$$

This implies that the $(2N_0 + 2)$th traversing will occur at $\mathcal{S}_\beta$, and thus

$$\begin{aligned} |t_{2N_0+2} - h_\alpha| \le& \delta_{\min} \\ x_{2N_0+2} - x_\beta^* =& W(t_{2N_0+2} - h_\alpha, x_\beta^*)(x_{2N_0+1} - x_\alpha^*) \\ =& W(t_{2N_0+2} - h_\alpha, x_\beta^*) \\ &\times \left(\prod_{j=1}^{N_0} W(t_{2j+1} - h_\beta, x_\alpha^*) \right. \\ &\left. \times W(t_{2j} - h_\alpha, x_\beta^*)\right)(x_1 - x_\alpha^*). \end{aligned}$$

Taking into account (25), it is easy to see that

$$\left\| \prod_{j=1}^{N_0} W(t_{2j+1} - h_\beta, x_\alpha^*) W(t_{2j} - h_\alpha, x_\beta^*) \right\| < 1$$

which leads to

$$\begin{aligned} \left\|x_{2N_0+2} - x_\beta^*\right\| \le& \left\|W(t_{2N_0+2} - h_\alpha, x_\beta^*)\right\| \|x_1 - x_\alpha^*\| \\ \le& (w+\theta_0)\bar{\epsilon}_{\min} \le \epsilon_{\min}. \end{aligned}$$

This indicates that the $(2N_0+3)$th traversing will occur at $\mathcal{S}_\alpha$. Continuing the process and noting (25) conclude that for any integer $k \ge 1$, the $(2N_0 + 2k)$th and the $(2N_0 + 2k + 1)$th traversing will occur at $\mathcal{S}_\beta$ and $\mathcal{S}_\alpha$, respectively. This shows that the relay will switch consecutively.

To the end of the proof, since

$$x_{2k+1} - x_\alpha^* = \left(\prod_{j=1}^{k} W(t_{2j+1} - h_\beta, x_\alpha^*) W(t_{2j} - h_\alpha, x_\beta^*)\right)(x_1 - x_\alpha^*)$$

$$\begin{aligned} \|W(t_{2j} - h_\alpha, x_\beta^*) - W_2\| \le& \theta_0 \\ \|W(t_{2j+1} - h_\beta, x_\alpha^*) - W_1\| \le& \theta_0 \end{aligned}$$

using Lemma 3.4 again and from the statement in the very beginning of the proof, we have $\|x_{2k+1} - x_\alpha^*\| \to 0$ as $k \to \infty$. This completes the proof of the theorem. □

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