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# Supercurrent determined from the Aharonov-Bohm effect in mesoscopic superconducting rings

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We have solved the Bogoliubov–de Gennes equation for a clean, one-dimensional mesoscopic superconducting ring threaded by a magnetic flux $\Phi$. We show that the superfluid velocity is driven directly by $\Phi$ while the relative motion of the pair of electrons is independent of $\Phi$. Meanwhile, the fluxoid quantization is obtained straightforwardly. More importantly, we have also calculated the supercurrent numerically and self-consistently and find it is periodic in $\Phi$ with the period $\Phi_s \equiv hc/2e$ for $\Phi_s \leqslant \Phi_d = (mv_d L/\hbar\pi)\Phi_s$ and with the period $\Phi_0 \equiv hc/e$ for $\Phi_d < \Phi_s$, which arises from mesoscopic effects.

In classical electromagnetism the electromagnetic field is described by the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$. The scalar potential $\phi(\boldsymbol{r})$ and the vector potential $\boldsymbol{A}(\boldsymbol{r})$ are introduced just for convenience. The equation of motion for an electron is determined by the Lorentz force and does not depend directly on the scalar and vector potentials. Actually, there are an infinite number of ways in choosing these potentials which give the same $\boldsymbol{E}$ and $\boldsymbol{B}$, and physically observable quantities are independent of these different gauges. In quantum theory, however, these potentials appear explicitly in the Schrödinger equation and their effects appear explicitly in physical quantities. As early as 1959, it was shown by Aharonov and Bohm[1] (AB) that the standard Schrödinger-equation analysis of the scattering of an electron by a thin impenetrable solenoid implies the remarkable result that such a particle is deflected even when classical forces are absent. One of the most well-known demonstrations of this AB effect is the persistent equilibrium current occurring in isolated mesoscopic normal-metal rings pierced by AB flux $\Phi = \oint A \cdot dl$.[2–5] In superconductors, electrons form Cooper pairs due to the effectively attractive interaction. The resistance of superconductors disappears completely below a critical temperature $Tc$. As a result, electrical currents in superconducting rings, once set up, can circulate persistently. It is then anticipated that the AB effect could manifest itself more clearly in supercurrents in mesoscopic superconducting rings. In addition, some novel behaviors of the persistent current via AB and pair-breaking effects could be expected. In this paper we will address these points by solving the Bogoliubov–de Gennes equation.

Throughout our work we only consider the case that the magnetic flux $\Phi$ is confined in an impenetrable solenoid which threads the ring axially and that electrons always move in a magnetic-field-free space. The situation is strictly of Aharonov-Bohm type and does not involve the Meissner effect. For a one-dimensional (1D) ring with the circumference $L$, the degree of freedom of an electron can be expressed in terms of the spatial variable $x = L\theta/2\pi$ instead of the azimuthal angle $\theta$, so that $x$ varies between 0 and $L$. The Bogoliubov–de Gennes (BdG) equation[6] has the form of two Schrödinger equations for electron and hole wave functions $u(x)$ and $v(x)$, coupled by the pairing potential $\Delta(x)$,

$$\begin{pmatrix} \mathscr{H}(x) & \Delta(x) \\ \Delta^*(x) & -\mathscr{H}^*(x) \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = E \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \qquad (1)$$

where the 1D Hamiltonian in the presence of the vector potential is

$$\mathscr{H}(x) = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx} - \frac{eA(x)}{c}\right)^2 + V(x) - E_F, \qquad (2)$$

with $V(x) = 0$ for the clean limit. The excitation energy $E$ is measured relative to the Fermi energy $E_F$.

For the ring geometry, the periodic boundary conditions lead to the usual quantization of the excitation energy. In this paper, however, we choose a gauge where the vector potential does not appear explicitly in the Hamiltonian and the current operators but enters the calculation via the flux-modified boundary conditions

$$u(x+L) = u(x)e^{+i2\pi\Phi/\Phi_0}, \qquad (3)$$

$$v(x+L) = v(x)e^{-i2\pi\Phi/\Phi_0}, \qquad (4)$$

$$\Delta(x+L) = \Delta(x)e^{i4\pi\Phi/\Phi_0}, \qquad (5)$$

where $\Phi_0 \equiv hc/e$ is the usual flux quantum. The basic idea of this transformation was originated by Byers and Yang[7] and Bloch,[8] here we employ it to solve the BdG equation in the presence of the vector potential. If the electron wave vector is $k$ and the wave vector for the collective drift motion (superfluid motion) of the paired electrons is $q$, the initial $k$ and $-k$ pairing is generalized to pair the states $(k+q)$ and $(-k+q)$. Following Ref. 6, we can write the pairing potential $\Delta(x)$ as

$$\Delta(x) = \Delta e^{2iqx}, \qquad (6)$$

where $\Delta$ is real and generally $q$ dependent. The solution of the BdG equation is then found,[6]

$$E_{k,q} = (\hbar k)\left(\frac{\hbar q}{m}\right) + \sqrt{E_A^2 + \Delta^2}, \qquad (7)$$

$$u(x) = u_k e^{i(k+q)x}, \qquad (8)$$

$$v(x) = v_k e^{i(k-q)x}, \qquad (9)$$

 

where

$$E_A = \frac{\hbar^2 k^2}{2m} + \frac{\hbar^2 q^2}{2m} - E_F, \tag{10}$$

and for the electronlike branch the two coherence factors of the BCS theory $|u_k|^2$ and $|v_k|^2$ are

$$|u_k|^2 = \frac{\sqrt{1+(\Delta/E_A)^2} + 1}{2\sqrt{1+(\Delta/E_A)^2}}, \tag{11}$$

$$|v_k|^2 = \frac{\sqrt{1+(\Delta/E_A)^2} - 1}{2\sqrt{1+(\Delta/E_A)^2}}, \tag{12}$$

with $|u_k|^2 + |v_k|^2 = 1$. The boundary condition Eqs. (3) and (4) on the solution of the BdG equation leads to

$$k = \frac{\pi}{L} n_k, \tag{13}$$

$$q = \frac{\pi}{L}\left(n_q + \frac{\Phi}{\Phi_s}\right), \tag{14}$$

where $n_k$, $n_q = 0, \pm 1, \pm 2, \ldots$ and $n_k + n_q = even$, and $\Phi_s = \Phi_0/2 \equiv hc/2e$ is the superconducting flux quantum. The magnetic flux enters into the wave vector $q$ for the collective drift motion but not into the wave vector $k$ for the relative motion of the paired electrons. In fact, if we regard a pair of electrons as a composite particle with the effective charge $e^* = 2e$ and the effective mass $m^* = 2m$, $\Delta(x)$ can be viewed as an effective macroscopic wave function describing the motion of this composite particle and the AB phase is gained via the process that the particle moves one turn around the ring. Imposing the boundary condition Eq. (5) on Eq. (6), one is able to find Eq. (14) again. It is interesting to note that the so-called fluxoid quantization

$$\Phi' = \frac{c}{e^*} \oint \left(m^* v_s - \frac{e^* A}{c}\right) \cdot dl$$

$$= n_q \Phi_s, \tag{15}$$

which is required by the general Bohr-Sommerfeld quantum condition,[9] is naturally recovered, where $v_s = \hbar(2q)/m^*$ is the superfluid velocity.

Once the eigenvalue problem is solved, one can calculate the equilibrium electrical current, which is given by[6]

$$I_Q = \frac{2\hbar e}{m} \sum_k \mathrm{Im}\{f(E_{k,q}) u^*(x) \nabla u(x)$$

$$- [1 - f(E_{k,q})] v^*(x) \nabla v(x)\}, \tag{16}$$

where the prefactor of 2 accounts for both spin directions, and $f(E_{k,q})$ is the Fermi distribution function, i.e.,

$$f(E_{k,q}) = \frac{1}{1 + e^{E_{k,q}/k_B T}}. \tag{17}$$

Substituting Eqs. (8) and (9) into Eq. (16), one yields

$$I_Q = 2e\left(\frac{\hbar q}{m}\right) \sum_k \{f(E_{k,q})|u_k|^2 + [1 - f(E_{k,q})]|v_k|^2\}$$

$$+ 2e \sum_k f(E_{k,q})\left(\frac{\hbar k}{m}\right). \tag{18}$$

So far, we have developed the formalism of the supercurrent in terms of the excitation spectrum of a 1D mesoscopic superconducting ring in the presence of the AB flux.[12] From Eqs. (3) and (4), the physical quantities such as the supercurrent are obviously periodic in $\Phi$ with the period $\Phi_0$. Notice that the periodicity of the supercurrent given by Eq. (18) can be understood in the way that one changes $n_q$ whenever $\Phi$ exceeds integer multiples of $\Phi_0$; once $\Phi$ exceeds $\Phi_0$ but less than $(3/2)\Phi_0$, $n_q$ should change to $n_q - 2, \ldots$ . Therefore, it is sufficient to consider only the range $\Phi \in [0, \Phi_0]$ because the flux can always be reduced by subtracting an integer times $\Phi_0$ so that it lies in the above range. It should be emphasized that the constraint $n_k + n_q = even$ must be satisfied when one calculates $I_Q$ for different $n_q$, which also implies that although $\Phi_s$ is the minimum period for the periodic condition of $\Delta(x)$ the period of the total supercurrent is generally $\Phi_0$. This result seems to be quite different from the intuitive understanding based only on the Ginzburg-Landau theory.[6,9,13] Moreover, due to quantum-size effects in 1D mesoscopic rings, both $k$ and $q$ are significantly quantized. Consequently, the superfluid velocity in the absence of the magnetic flux $\Phi$ can only take a discrete value and only a few of the superfluid velocities are below the Landau depairing velocity,[14] $v_d = \Delta/p_F$, where $p_F = \hbar k_F$ is the Fermi momentum. Of course, the magnetic flux plays the role of modulating continuously the magnitude of the superfluid velocity and each discrete value of the superfluid velocity evolves into a branch.

Strictly, the flux $\Phi$ which drives the electrical current should be the sum of the external flux $\Phi_{ext}$ and the flux $\Phi_I$ produced by the current itself. For simplicity, as usual, the self-inductance of the ring is assumed to be so small that the corrections to the flux due to the self-inductance can be neglected.[5] In Fig. 1, we plot the supercurrent as a function of the magnetic flux for different values of $\Delta/E_F$ with $\Delta$ as a fixed value. (The values of the Fermi level $n_F = k_F L/\pi = 400$ are taken, which gives rise to a reasonable Fermi energy $E_F \simeq 60$ meV for the ring with the circumference $L = 1000$ nm.[10,11]) Figures 1(a) and 1(b) show that if the characteristic parameter $\Phi_d/\Phi_s = m v_d L/\hbar\pi = (n_F/2)(\Delta/E_F) < 1$, the supercurrent curve at zero temperature exhibits a linear $\Phi$ dependence followed by a drastic and discontinuous drop when the magnetic flux reaches $\Phi_d$. This feature results from the fact that above $\Phi_d$, zero-energy excitations occur at $k = -k_F$, which leads to a negative contribution to the supercurrent. Because of the discreteness of the wave vector $k$, the variation of the supercurrent at $\Phi_d$ is significantly different from that for an infinite 1D superconductor[11] where the supercurrent changes continuously at $v_d$. This implies indirectly that the gapless superconductivity does not occur in 1D mesoscopic superconducting rings even without calculating the pairing potential $\Delta$ self-consistently. It is interesting to note that if $\Phi_d/\Phi_s \geqslant 1$, the supercurrent is periodic in $\Phi$ with the period $\Phi_s$, which

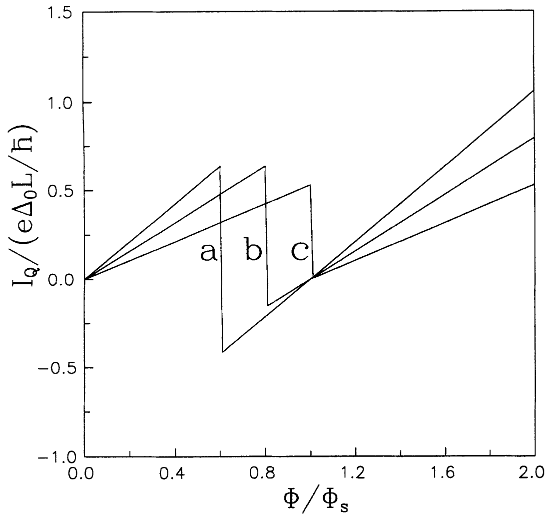


FIG. 1. Supercurrent versus the magnetic flux $\Phi/\Phi_s$ at zero temperature $T=0$ for different values of $\Delta_0/E_F$: (a) $3.0\times10^{-3}$; (b) $4.0\times10^{-3}$; (c) $6.0\times10^{-3}$.

is attributed to the fact that contributions to $I_Q$ from the *even*-$n_k$ summation and *odd*-$n_k$ summation are almost the same.

At this stage, we wish to emphasize that the BdG equation (1) should in fact be solved self-consistently by computing the pairing potential $\Delta(x)$ from the set of $u$'s and $v$'s,

$$\Delta(x)=|g|\sum_k v^*(x)u(x)[1-2f(E_{k,q})] \qquad (19)$$

where $|g|$ is the attractive matrix element. We have performed the numerical calculations for $\Delta$ and $I_Q$. In Fig. 2,

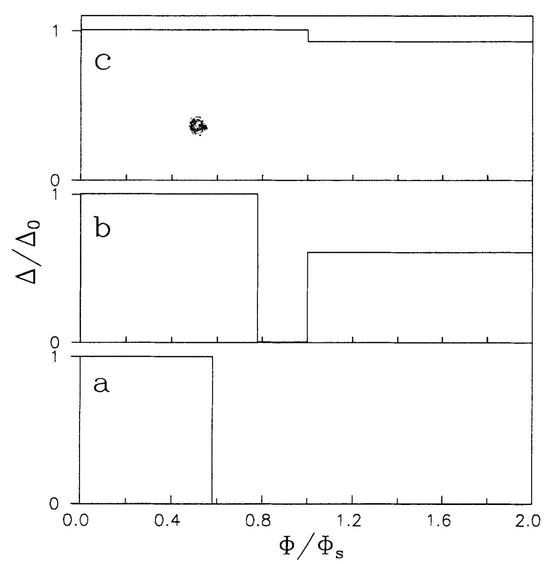


FIG. 2. Self-consistently numerical calculation of the normalized pairing potential $\Delta/\Delta_0$ versus the magnetic flux $\Phi/\Phi_s$ at $T=0$. The values of $\Delta_0/E_F$ correspond to (a), (b), and (c) in Fig. 1.

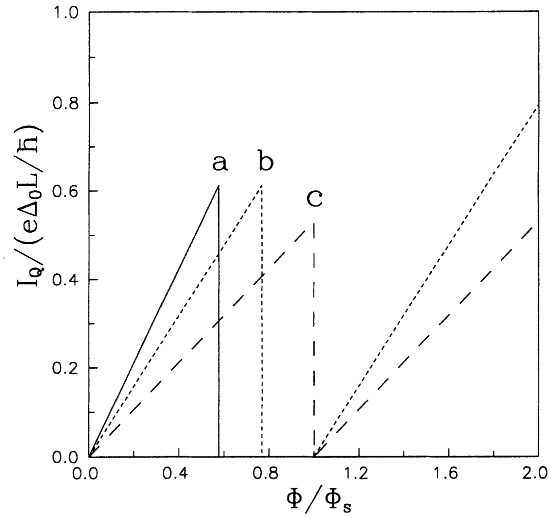


FIG. 3. Supercurrent versus the magnetic flux $\Phi/\Phi_s$ at $T=0$, calculated using the self-consistent pairing potential $\Delta$.

we depict the entire dependence of $\Delta(x)$ on the magnetic flux for different values of $\Delta_0/E_F$ (equivalent to $\Phi_d$) with $\Delta_0$ as the pairing potential in the absence of the superfluid velocity.[15] Obviously, the typical behaviors of $\Delta$ depend on the magnitude of $\Phi_d$. (i) For $0<\Phi_d<\Phi_c$ [Fig. 2(a)] with $\Phi_c$ as another characteristic system parameter ($\approx 0.69$ for the given $n_F$, $L$, and Debye frequency $\omega_D$), if $\Phi<\Phi_d$ all excitation energies, $E_{k,q}$, are greater than zero, and $f(E_{k,q})$ vanishes at zero temperature, which makes the pairing potential $\Delta$ unaffected. As $\Phi$ reaches $\Phi_d$, the states near $k=-k_F$ oppose the contribution to the sum in Eq. (19) from other states. Moreover, the large density of quasiparticle states of 1D mesoscopic systems near $k=-k_F$ makes their contribution outweigh that from other states, so that at $T=0$ K Eq. (19) has no nonzero solution for all $\Phi>\Phi_d$. This result differs from that for 3D bulk superconductors where there is a small region of $v_s$ above $v_d$ for which the pairing potential $\Delta$ remains nonzero.[16,17] (ii) For $\Phi_c\leqslant\Phi_d<\Phi_s$ [Fig. 2(b)], $\Delta$ exhibits similar behavior to that in Fig. 2(a) as $\Phi<\Phi_s$, but it becomes another $\Phi$-independent constant smaller than $\Delta_0$ for any $\Phi\in[\Phi_s,\Phi_0]$. (iii) For $\Phi_d\geqslant\Phi_s$ [Fig. 2(c)], Eq. (19) always has a nonzero solution for any $\Phi\in[0,\Phi_0]$. Due to quantum-size effects and the difference between the $k$ summations in $[0,\Phi_s]$ and $[\Phi_s,\Phi_0]$, there exists the discrepancy of the magnitude of $\Delta$ between the two ranges. Corresponding to the above three cases, the supercurrent in (i) and (ii) is in marked contrast with the previous non-self-consistent calculation. In detail, (i) as shown in Fig. 3(a), the supercurrent increases linearly with $\Phi$ up to $\Phi_d$ and then drops abruptly to zero due to the depairing effect, which seems to be novel and is significantly different from that for 1D normal-metal rings where the persistent current increases linearly with $\Phi$ in the whole period $\Phi_0\equiv hc/e$;[5] (ii) the supercurrent increases initially with $\Phi$ and then drops abruptly to zero at $\Phi_d$; however, the supercurrent increases again with $\Phi$ from $\Phi_s$ to $\Phi_0$ because the pairing potential is basically unaffected in this region [Fig. 3(b)]; (iii) the supercurrent is periodic in $\Phi$ with the period $\Phi_s$ and is similar to that in normal-metal rings [Fig. 3(c)]. Moreover, we also find that a maximum supercurrent $I_M$ occurs at $\Phi_d$ for the case $\Phi_d \leqslant \Phi_s$ and $I_M/(e\Delta_0 L/\hbar)$ is a universal constant ($\approx 2/\pi$), which can be easily understood in consideration of the fact that all Cooper pairs drift with the depairing velocity $v_d$. However, we must note that, for $\Phi_c < \Phi_d \leqslant \Phi_s$, there is no above property for $I_M$ in the region $[\Phi_s, \Phi_0]$, because the possible depairing velocity of the present case is larger than $\Delta_0/p_F$, i.e., $(\Delta_0 + \delta\Delta)/p_F$ with $\delta\Delta$ being also dependent on $E_F$. Similarly, for the case $\Phi_d > \Phi_s$, the above maximum supercurrent $I_M$ is unable to be reached because the maximum superfluid velocity within the period $\Phi_s$ is less than $v_d$. We expect some of these interesting features, particularly the periodicity of the supercurrent, could be examined by some well-designed experiments, and may have some potential applications in device designing.

Finally, we wish to point out that, although the system considered here is different from the superconducting–normal-metal–superconducting (SNS) sandwich considered by Bardeen and Johnson,[18] the behaviors of the supercurrent are quite similar. In particular, at $T=0$K, both the supercurrent for $\Phi_d \geqslant \Phi_s$ in the present AB case and that in the SNS sandwich are piecewise periodic functions of the phase (or AB flux). In each periodic region, they vary linearly with the phase (or AB flux) and there are discontinuous jumps when one period is over. It is believed that the underlying essential physics of such similarities is the quantum effect through the generation of relative phases which accumulate on the wave function (or "macroscopic" wave function) of a particle (or Cooper pair) moving through a multiply-connected force-free region (or Josephson junction).

In conclusion, we have solved the Bogoliubov–de Gennes equation for a clean, one-dimensional mesoscopic superconducting ring threaded by a magnetic flux $\Phi$. We show that the superfluid velocity is driven directly by $\Phi$ while the relative motion of the pair of electrons is independent of $\Phi$. Meanwhile, the fluxoid quantization is obtained straightforwardly. More importantly, we have also calculated the supercurrent numerically and self-consistently, and find that it is periodic in $\Phi$ with the period $\Phi_s$ for $\Phi_s \leqslant \Phi_d$ while with the period $\Phi_0$ for $\Phi_d < \Phi_s$.

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