Statistical analysis of magnetic-field spectra

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We have calculated and statistically analyzed the magnetic-field spectrum (the B spectrum) at fixed electron Fermi energy for two quantum dot systems with classically chaotic shape. This problem arises naturally in transport measurements where the incoming electron has a fixed energy while one tunes the magnetic field to obtain resonance conductance patterns. The B spectrum, defined as the collection of values $\{B_i\}$ at which conductance $g(B_i)$ takes extremal values, is determined by a quadratic eigenvalue equation, in distinct difference to the usual linear eigenvalue problem satisfied by the energy levels. We found that the lower part of the B spectrum satisfies the distribution belonging to the Gaussian unitary ensemble, while the higher part obeys a Poisson-like behavior. We also found that the B spectrum fluctuations of the chaotic system are consistent with the results we obtained from random matrices. [S0163-1829(98)03543-7]

Due to recent advances in controlled crystal growth and lithographic techniques, it is now possible to fabricate various ''artificial atoms'' or quantum dots the sizes of which are so small such that transport is in the ballistic regime.[1] Among the many interesting phenomena associated with quantum ballistic transport, it was proposed[2] that these quantum dots could be used to examine the theoretical notion of quantum chaos.[3–5] More recently, Taylor *et al.* fabricated an electronic Sinai billiard[6] and investigated fluctuations of the resistance as an external magnetic field is tuned. An important discovery of this work is the apparent fractal resistance fluctuations,[7] which may be discussed on the basis of a semiclassical theory.[8]

The problem of quantum chaos is a very interesting theoretical physics issue, and our theoretical understanding of it has been advanced by random matrix theory (RMT), which classifies the statistics of the eigenspectrum of a quantum system according to the Wigner-Dyson ensembles.[9] It is well established that, for a classically nonchaotic system such as a particle confined to move inside a rectangular box, the normalized nearest-neighbor energy spacings $\{s\}$ obey a Poisson distribution[10] $P(s)=e^{-s}$. On the other hand, for a classically chaotic system such as a spinless particle confined to move inside a stadium shaped box,[11,12] the spacings follow the Wigner distribution that belongs to the Gaussian orthogonal ensemble (GOE) in the language of RMT.[13] Furthermore, when time-reversal invariance is broken, say, by applying a magnetic field, the system is described[14] by the Gaussian unitary ensemble (GUE), where $P_2(s) \approx (32s^2/\pi^2)e^{-(4s^2/\pi)}$.

In order to study certain aspects of quantum chaos using quantum *transport* techniques,[2,15,6] one must deal with open systems where a scattering problem of charge carriers by some peculiar boundary must be solved. Experimentally it is quite difficult to study conduction as a function of the *incoming electron energy* in a quantitatively accurate fashion,[16] although measurements on a tunnel junctions made of Al nanoparticles have recently be made[17] and its relation to quantum chaos discussed.[18] Using a two-dimensional (2D) electron gas fabricated with compound semiconductors, experiments usually measure conductance as a function of external magnetic fields,[2,15,6,19] $g(B)$, at a fixed electron Fermi energy E_o. When a quantum dot is weakly coupled to the external leads, the magnetoconductance $g(B)$ may show resonancelike behavior as the magnetic field B is varied,[5] if the measurement is indeed in a regime that probes the internal electronic states.[17] This behavior may be understood as follows. As B is varied, the energy levels $\{\epsilon_i\}$ of the scattering states labeled by indices $i=1,2,\ldots$ in the quantum dot change with it: $\epsilon_i=\epsilon_i(B)$. These levels are well separated since the the dot region is weakly coupled to the leads. For an incoming electron with a fixed Fermi energy E_o, each time when the internal state energy $\epsilon_i(B)$ is tuned to be equal to E_o as B is varied, a resonance peak occurs in $g(B)$ due to a junction resonance. This was indeed observed in numerical simulations[5] by solving the quantum scattering problem. Clearly this behavior should be observable if the Coulomb blockade effects are small, which happens when the system has a large capacitance, thus small charging energy. We shall assume this to be the case.

Hence in this junction resonance regime, it is interesting to define a B spectrum as the collection of values of the special magnetic fields $B=\{B_i\}$ at which $g(B_i)$ is peaked.[5] It is important to ask the following questions: what are the statistical properties of this B spectrum? How do its statistical properties change with the system shape? These are also useful questions to answer because it is increasingly possible to directly probe the internal electronic states of an isolated quantum dot, as demonstrated by the experiment reported in Ref. 17.

Motivated by these questions, in this paper we report our studies on *closed* quantum dot systems where these questions can be answered clearly. Simons, Szafer, and Altshuler[4] have investigated the correlations of slopes of the energy *levels* as a function of an external parameter such as the magnetic field B. We, however, emphasize that in their studies, the focus is

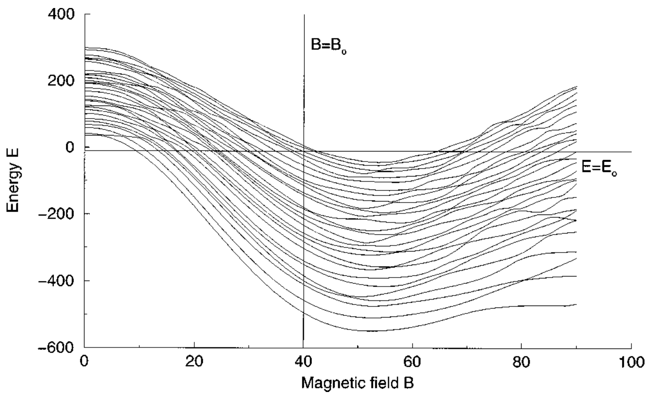

FIG. 1. Typical curves of the energy levels as a function of the magnetic field, $E=E(B)$, for the Sinai-like billiard. The vertical solid line at $B=B_o$ intersects the spectrum and the intersections give the usual energy levels $E=E(B_o)$. The B spectrum is the collection of intersections of the horizontal solid line at a fixed energy $E=E_o$.

on how a particular energy level E_i varies with the external parameter B. Our investigation, on the other hand, is completely different as it does not focus on any energy level: the energy E_o is a fixed parameter in the Schrödinger equation by the equilibrium Fermi energy of the system. Rather, we investigate the statistical properties of the set $\{B_i\}$, which makes E_o an eigenvalue of the Hamiltonian. Indeed, we note that fixing B and studying energy levels $\{E_i\}$, or fixing E_o and studying the B spectrum $\{B_i\}$, are two different problems. The former is about the energy eigenvalue spectra and its relation to an external parameter B, while the latter is relevant for transport situations where the incoming electron has energy E_o, which is fixed by the electron reservoir and cannot change, while one tunes B to special values $\{B_i\}$ where conductance $g(B_i)$ is peaked. To our knowledge the statistical ensemble that is satisfied by the B spectrum has not previously been determined. As we shall see below, this B spectrum is determined by a *quadratic* eigenvalue problem, in distinct contrast to the usual energy spectrum at a fixed B, which is a linear eigenvalue problem. When a quantum dot is weakly coupled with the external leads, our answers are valid since in the weak coupling regime the levels are well separated and statistical properties should not change.[20,21]

To make the problem at hand clearer, the inset of Fig. 3 shows the two-dimensional quantum dot systems we have studied: a Sinai-like billiard and a stadium-shaped quantum dot. These systems are classically chaotic systems. Experimentally transport measurements have been reported through these quantum dots, which were fabricated using the split-gate technology.[15,6,2] Figure 1 shows the energy levels of a Sinai-like billiard as a function of magnetic field B. For a given $B=B_o$, the levels satisfy GUE as mentioned above. These are the intersections of the line $B=B_o$ (the vertical solid line) with the spectrum. However, for a fixed energy $E=E_o$, we are interested in the intersections of the horizontal solid line with the spectrum, which defines the B spectrum. From the curves of Fig. 1 it is clearly not obvious what statistics the B spectrum will satisfy.

In the presence of a magnetic field, the single-particle Schrödinger equation can be written as

$$\{-\nabla^2-e_0+b^2A_0^2+2bi\vec{A}_0\cdot\nabla\}\psi=0, \tag{1}$$

where $\vec{A}=B\vec{A}_0=Bx\hat{j}$ is the vector potential, B is the magnetic field, $E=\hbar^2e_0/2m$ is the energy, $b=eB/\hbar c$, and $i^2=-1$. Discretizing the spatial derivative, Eq. (1) can be cast into a matrix form,

$$(M_1+ibM_2+b^2M_3)\psi=e_0\psi, \tag{2}$$

where M_1, M_2, M_3 are real matrices and M_3 is also diagonal. From Eq. (2) it is clear that for a fixed magnetic field the solution of all the allowed energies forms the usual linear eigenvalue problem, which has been studied intensively. However, for a fixed energy e_0, the solution of all the allowed magnetic fields B, the B spectrum, forms a quadratic eigenvalue problem.

There are two methods to find the B spectrum. The first method is by brute force: one directly calculates the energy eigenvalues E for a given magnetic field B and traces out the curves of E versus B. These curves may cross over each other. One then calculates, from this E versus B curve, the set of magnetic fields B for a fixed energy E_o. Figure 1 shows the E versus B curves for the first 40 eigenstates obtained using the Lanczos eigenvalue technique[22] for the Sinai-like billiard system. However, this method gets time consuming very quickly if higher and higher states are needed.

The second method is to transform the quadratic eigenvalue problem Eq. (2) into a usual linear eigenvalue problem.[23] Let $\Psi(t)=e^{ibt}\psi$ and define $N_1=M_3^{-1}(M_1-e_0)$, $N_2=M_3^{-1}M_2$, so that Eq. (2) becomes $-I\Psi''+N_2\Psi'+N_1\Psi=0$ where $\Psi'=ib\Psi$ is the first derivative with respect to the parameter t, and I is the unit matrix. With $\Psi'=\Phi$ $(\Phi'=ib\Phi)$, we have $\Phi=\Psi'=ib\Psi$ and $N_2\Phi+N_1\Psi=I\Phi'=ib\Phi$, or

$$\begin{pmatrix} 0 & I \\ N_1 & N_2 \end{pmatrix}\begin{pmatrix} \Psi \\ \Phi \end{pmatrix}=ib\begin{pmatrix} \Psi \\ \Phi \end{pmatrix}. \tag{3}$$

This is a linear eigenvalue problem for b. The matrix in Eq. (3) is, however, not Hermitian, hence its eigenvalues may or may not be real. To obtain the physical solution, we look for all imaginary eigenvalues so that b is real. We verified that the results coincide with those obtained with the conventional Lanczos technique described above. In this way we have calculated the B spectrum from Eq. (3) for the two chaotic systems (inset of Fig. 3), for different energies e_0. The confining potential is assumed to be hard wall.

For the Sinai-like billiard, we considered a 4.7×4.3 μm rectangular quantum dot with three hard disks inside the dot. The radius of the disks were fixed at 0.9, 0.8, and 0.5 μm with their centers randomly chosen but without the disks overlapping each other. Randomly changing positions of the disks allows us to generate different configurations for ensemble averaging to obtain reasonable statistics. The number of physical solutions N obtained from Eq. (3) depends on the fixed energy e_0, with N increasing with the value of the energy e_0. For instance, when e_0 is fixed at 30 meV, we have $N\sim1433$ physical solutions for the B spectrum out of a total of about 3000 eigenvalues. We found that the statistics of the B spectrum behaves differently for the lower and

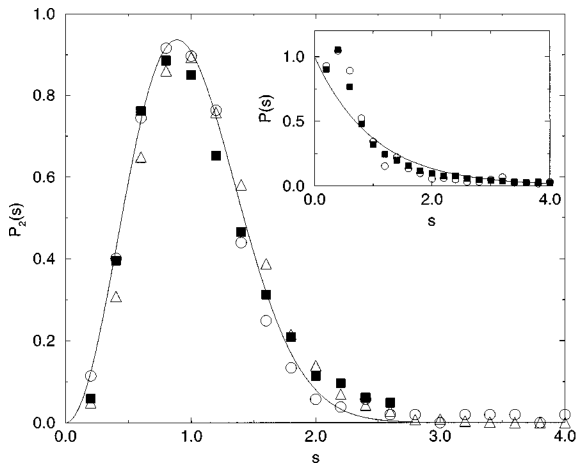

FIG. 2. The distribution function of the nearest-neighbor B-level spacings obtained from the lower part of the B spectrum. The solid line is the analytical formula for GUE $P_2(s)$. Solid squares, ensemble-averaged data for 20 independent Sinai-like billiards; open circles, for a stadium-shaped quantum dot; triangles, for random matrices averaged over 13 configurations. Inset: the distribution function obtained from the higher part of the B spectrum of the same systems (symbols) and the solid line is the Poisson distribution $P(s)$. The energy is fixed at $e_0=30$ meV.

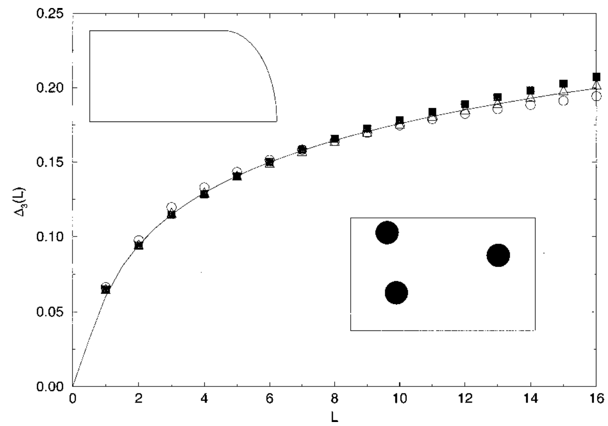

FIG. 3. The spectral rigidity $\Delta_3(L)$, where L is the number of the B levels involved in computing Δ_3. The solid line is the analytical expression for GUE statistics, taken from Ref. 18. Solid squares, ensemble averaged data for 20 independent Sinai-like billiards; open circles, for a stadium-shaped quantum dot; triangles, from the random matrices averaged over 13 configurations. Inset: sketch of the chaotic structures studied here, a Sinai-like billiard with three hard disks confined inside a rectangle; and a stadium-shaped quantum dot.

higher part of the spectrum, respectively. The lowest[24] ~ 300 B levels give GUE statistics and the highest ~ 800 B levels give a Poisson-like behavior. Figure 2 plots the distribution function obtained from our numerical data of the nearest-neighbor B level spacings for the Sinai-like billiards. With the ensemble average of 20 different configurations, the distribution determined from the lower part of the B spectrum agrees well with GUE statistics (solid line) $P_2(s)$ discussed above. On the other hand, the higher part of the spectrum, shown in the inset of Fig. 2, has a Poisson-like behavior.

Another often used measure in studying level statistics is the spectral rigidity Δ_3, defined as[9,25] the mean-square deviation of the best local fit straight line to the staircase cumulative spectral density over a normalized energy scale. This quantity measures longer range correlations of the level spectrum and often provides a more critical test of the level statistics. To compute Δ_3 we followed a scheme presented in Ref. 26, and the numerical data are compared with the analytical formula from random matrix theory.[27] Figure 3 shows the Δ_3 analysis of the lower part of the B spectrum. It is clear that the data are in very good agreement with the GUE statistics.[28]

To test the statistical properties of the B spectrum further, we studied another chaotic system, namely a stadium-shaped quantum dot (inset of Fig. 3). The distribution function and Δ_3 for its B spectrum are included in Figs. 2 and 3. Here we did not use an ensemble average and the data is for one system only. The general trend is the same as for the Sinai-like billiards, and still clearly shows the two distinct behaviors for different parts of the B spectrum, namely a GUE behavior of the lower part and a Poisson-like behavior for the higher part. Finally, we have verified that the same statistical behavior is observed for both the Sinai-like billiard and the stadium-shape billiard with many different values of the fixed energy e_0, and conclude that the lower part of the B spectrum of both systems satisfies GUE statistics.

It is not difficult to understand that the higher part of the B spectrum should behave differently. The higher part corresponds to larger values of the magnetic fields, which are known to destroy chaos.[29,3] For the particular sizes of the Sinai-like billiard and the stadium-shaped dot, the magnetic-field B_c that roughly separates the low-lying and high-lying part of the B spectrum is about $B_c \sim 3$ to 4 T. The classical cyclotron radius of the electron at and above this field strength is quite small compared with the system size. Hence the electron ''skips'' along the wall of the confining potential or makes circular motion inside the quantum dot, thereby reducing the effect of chaotic scattering by the geometry.[30] Nevertheless, the field range up to B_c is quite wide and it should be possible to investigate the B spectrum experimentally using resonant magneto-conductance measurements for systems having weak coupling to the leads.

While the quadratic eigenvalue problems reported above were for billiard systems where the continuum Schrödinger equation is discretized to obtain the matrix equation (3), we also studied a similar quadratic eigenvalue problem using random matrices to replace the matrices M_1, M_2, and M_3 in Eq. (2). Two requirements must be satisfied: first, the matrices M_1, M_2, and M_3 must be real; second, M_1 must be symmetric, and M_2 antisymmetric so that the Hamiltonian be Hermitian. The random matrices were set up in the standard fashion: for a $N\times N$ matrix, M_1 has $N(N+1)/2$ independent matrix elements and M_2 has $N(N-1)/2$, where the matrix elements are Gaussian random numbers. We then diagonalized Eq. (3) using the same numerical methods discussed above and found all the physical solutions. We chose $N=1500$ so that the matrix that must be diagonalized is 3000×3000. Typically we obtained about 700 physical B levels for various energies e_0. The results are included in Figs. 2 and 3. It is clear that the B-level statistics from the random matrices is completely consistent with the GUE statistics. We may conclude that B spectra coming from the quantum billiard systems and from random matrices are still consistent with each other and a universality can still be established for this quadratic eigenvalue problem using the random matrices.

In summary, we have numerically investigated the statistical properties of the magnetic-field spectra (the B spectrum), which is determined by a quadratic eigenvalue problem. This spectrum is defined by the allowed magnetic fields for an electron moving in a quantum dot with its fixed Fermi energy. This problem arises for systems with weak coupling to the leads in which the scattering states are well separated in energy. For two different chaotic billiards, e.g., two-dimensional quantum dots in the shape of a Sinai-like billiard and a stadium billiard, the B spectra have distinctly different statistical behavior at the lower and higher parts of the spectra. In particular, the lower part is well described by the GUE statistics while the higher part is Poisson like. We found that the same quadratic eigenvalue problem can be studied using random matrices as well, and the eigenvalues from the random matrices have precisely the same behavior as those of the billiards. Thus the notion of universality classes using the random matrix theory can be carried over to this new problem. While our numerical data provided clear evidence of the statistical properties of the present quadratic eigenvalue problem, it is not at all obvious *a priori* that such statistical properties are controlled by the GUE universality. Further work is needed to provide an analytical understanding.

Experimentally the statistical properties of the B spectra could be examined for quantum dot systems which couple weakly to the external leads and thus the transmission is controlled by junction resonances. The weak coupling could be provided by adding constrictions at the connections of the leads with the quantum dot for which only the lowest few subbands of the leads can propagate. In typical experimental situation on submicron structures the single-particle level spacing is around 0.05 meV, thus they can be measured if the temperature is kept to less than 500 mK. We thus conclude that for a magnetotransport measurement in the junction resonance regime, the special magnetic-field strengths $\{B_i\}$ at which $g(B_i)$ takes extremal values, should satisfy GUE statistics.

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