

## Memory is relevant in the symmetric phase of the minority game

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Minority game is a simple-minded econophysical model capturing the cooperative behavior among selfish players. Previous investigations, which were based on numerical simulations up to about 100 players for a certain parameter  $\alpha$  in the range  $0.1 \leq \alpha \leq 1$ , suggested that memory is irrelevant to the cooperative behavior of the minority game in the so-called symmetric phase. Here using a large scale numerical simulation up to about 3000 players in the parameter range  $0.01 \leq \alpha \leq 1$ , we show that the mean variance of the attendance in the minority game actually depends on the memory in the symmetric phase. We explain such dependence in the framework of crowd-anticrowd theory. Our findings conclude that one should not overlook the feedback mechanism buried under the correlation in the history time series in the study of minority game.

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### I. INTRODUCTION

Minority game (MG) [1,2] is the most studied econophysical model capturing the minority seeking behavior of independent selfish players. MG is a repeated game of  $N$  players, each picking one out of two alternatives independently in every time step based on the publicly posted minority choices of the previous  $M$  turns. Those correctly picking the minority choice are awarded one mark while the others are deducted one. The aim of each player is to maximize his/her own mark. To help players making their choices, each of them are assigned once and for all  $S$  deterministic strategies. Here, each strategy is a map from the set of all possible minority choices of the previous  $M$  turns, which we call the history, to the set of the two alternatives. The performance of each strategy is evaluated according to its virtual score, which is defined as the hypothetical mark it would get if the strategy were used throughout the game. Among the  $S$  assigned strategies, every player follows the suggestion of the one with the highest current virtual score to choose an alternative (in case of a tie, a player randomly uses one of his/her current best working strategies) [1,3].

The number of possible strategies in MG equals  $2^{2^M}$ . Nonetheless, out of these  $2^{2^M}$  strategies, only  $2^{M+1}$  of them are significantly different. These significantly different strategies formed the so-called reduced strategy space. In fact, for a fixed  $S$  and up to first order approximation, the dynamics of MG is robust with  $\alpha$ , which is the ratio of the reduced strategy space size  $2^{M+1}$  to the number of strategies at play  $NS$  [2,3].

The attendance of an alternative  $A(t)$  at turn  $t$  is defined as the number of players choosing that alternative in that turn. Since there is no prior bias in choosing the two alternatives in MG,  $\langle A(t) \rangle_{t,\xi} = N/2$  where the average is taken over time  $t$  and initially assigned strategies  $\xi$ . In contrast, the variance of attendance per player averaged over initially assigned strategies,  $\langle \sigma^2(A(t)) \rangle_{\xi} / N$ , is a more instructive quantity to study.

(We write  $\langle \sigma^2(A(t)) \rangle_{\xi}$  as  $\sigma^2$  and call it simply the variance of attendance from now on when confusion is not possible.) Numerical simulation shows that, over a wide range of parameters, the  $\sigma^2/N$  against  $\alpha$  curve is lower than the value in which all players make their choices randomly. This implies that these selfish and independent players maximize their own marks cooperatively. More importantly, a cusp is found at  $\alpha = \alpha_c$ ; this second order phase transition point  $\alpha_c$  divides the parameter space into the so-called symmetric ( $\alpha < \alpha_c$ ) and asymmetric ( $\alpha > \alpha_c$ ) phases [4].

The first numerical study on the effect of history concerning the dynamics and cooperative behavior of MG was carried out by Cavagna using  $N=101$ , who suggested that memory is irrelevant in MG. That is to say, the  $\sigma^2/N$  against  $\alpha$  curve is unaltered if we replace the history in each turn with a randomly and independently generated  $M$ -bit string, while keeping the virtual score calculation method unchanged [5]. (From now on, we denote the original MG and the one played using random history strings by  $MG_{\text{real}}$  and  $MG_{\text{rand}}$  respectively.) Later on, by exploring a wider range of parameters in their numerical simulations, Challet and Marsili pointed out that history is irrelevant if and only if  $\alpha < \alpha_c$ . As long as the occurrence of history is uniform,  $\sigma^2/N$  should not change when we replace the history with a randomly and independently generated time series of  $M$ -bit string [6]. Furthermore, Lee argued that although the time series of the attendance shows a strong periodic signal, this signal does not affect the volatility of the attendance [7]. To summarize, the current understanding is that memory plays no role on the volatility in the symmetric phase of MG.

In this paper, we perform a large scale numerical simulation of MG played using real and random histories with up to 3232 players for  $0.01 \leq \alpha \leq 10$ . Our simulation results show that memory is in fact relevant in the symmetric phase whenever  $\alpha \leq 0.2$ . Specifically, we discover that in such a low value of  $\alpha$ , the  $\sigma^2/N$  against  $\alpha$  curves corresponding to the games played using real and random histories split. We explain this split by crowd-anticrowd theory developed by Hart *et al.* [8,9] and argue that finite size effect prevents earlier investigations from revealing this discrepancy. Finally, we

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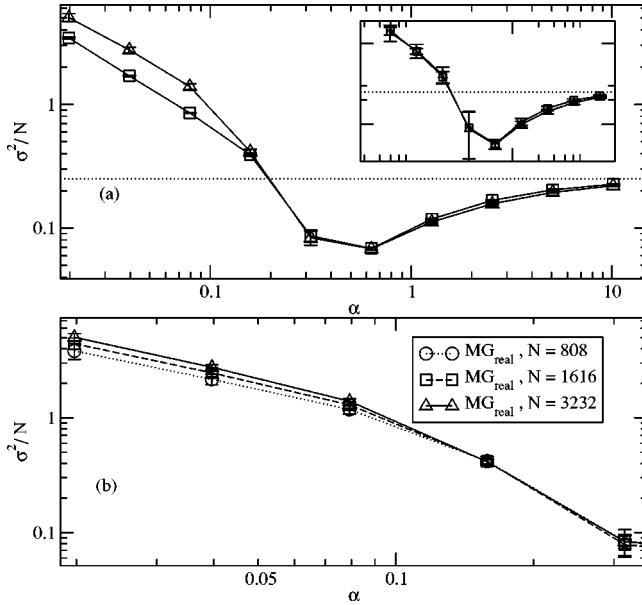


FIG. 1. A plot of  $\sigma^2/N$  against  $\alpha$  by varying  $M$  with  $S=2$ . Figure (a) compares the dependence of history for  $N=101$  (inset) and 3232 (main plot). We represent the  $\text{MG}_{\text{real}}$  and  $\text{MG}_{\text{rand}}$  data by triangles and squares, respectively. Figure (b) shows the dependence of  $\sigma^2/N$  on  $N$  for  $\text{MG}_{\text{real}}$ .

report two scaling relations in the symmetric phase of MG and explain their origin.

## II. NUMERICAL RESULTS

All the  $\sigma^2$ 's reported in our simulation are averaged over 1000 independent runs. And in each run, the variance of attendance is computed from the attendance of 15000 consecutive turns after the system equilibrates. Furthermore, we looked at the attendance time series over at least  $8 \times 10^6$  iterations in about 50 independent runs with various values of  $\alpha$  to verify that the system has indeed equilibrated. Our computation requires about 7 Gflops/yr of instructions.

In Fig. 1, we plot  $\sigma^2/N$  against  $\alpha$  for different values of  $N$  for both the MG played using real and random histories. By putting  $N=101$ , we successfully reproduce the two  $\sigma^2/N$  against  $\alpha$  curves for  $\text{MG}_{\text{real}}$  and  $\text{MG}_{\text{rand}}$  obtained by Challet and Marsili in Ref. [6] in the range of  $0.1 \lesssim \alpha \lesssim 10$ . In fact, these two curves coincide in the symmetric phase. Surprisingly, by increasing the number of players  $N$ ,  $\sigma^2/N$  for  $\text{MG}_{\text{real}}$  is consistently higher than that of  $\text{MG}_{\text{rand}}$  for  $\alpha \lesssim 0.2$ . Furthermore, Fig. 1(b) shows that for a fixed  $\alpha \lesssim 0.2$ ,  $\sigma^2/N$  for  $\text{MG}_{\text{real}}$  increases as  $N$  increases. In contrast, we find that the  $\sigma^2/N$  against  $\alpha$  curve for  $\text{MG}_{\text{rand}}$  is independent of  $N$ . Thus, the discrepancy between the two volatilities increases with  $N$  when  $\alpha \lesssim 0.2$ . (Although the values of  $N$  used in all curves reported in this paper are in the form  $2^k \times 101$  for some integer  $k$ , our numerical simulation shows that the same conclusions are reached for the case of odd  $N$ .)

From the above discussions,  $\text{MG}_{\text{rand}}$  follows the scaling relation

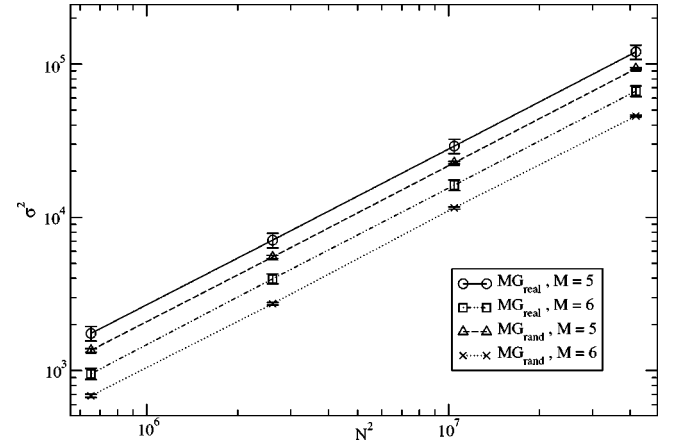


FIG. 2. A plot of  $\sigma^2$  against  $N^2$ . Each curve is drawn by joining  $\sigma^2$  for different  $N^2$  at a fixed  $M$  with  $S=2$ .

$$\frac{\sigma_{\text{rand}}^2}{N} \sim f(\alpha, S) \quad (1)$$

in the symmetric phase ( $\alpha < \alpha_c$ ), where  $\sigma_{\text{rand}}^2$  is the variance of the attendance for  $\text{MG}_{\text{rand}}$  and  $f$  is a scaling function depending only on  $\alpha$  and  $S$ . Moreover, Fig. 2 shows that, for a fixed memory size  $M$ , the value of  $\sigma^2/N^2$  is independent of  $N$  in the symmetric phase. That is to say,

$$\frac{\sigma_{\text{real}}^2}{N^2} \sim g_{\text{real}}(M, S) > g_{\text{rand}}(M, S) \sim \frac{\sigma_{\text{rand}}^2}{N^2} \quad (2)$$

provided that  $\alpha < \alpha_c$ , where  $\sigma_{\text{real}}^2$  is the variance of attendance for  $\text{MG}_{\text{real}}$  and  $g_i$ 's are scaling functions depending on  $M$  and  $S$  only.

## III. THE CROWD-ANTICROWD EXPLANATION

The above numerical findings can be explained by the crowd-anticrowd theory proposed by Hart *et al.* using the notion of reduced strategy space [8,9]. Recall that decisions made by two distinct strategies in a reduced strategy space are either mutually anti-correlated or uncorrelated when averaged over all possible history strings. Besides, the dynamics is very close to the original MG if all strategies are picked from the reduced strategy space [3]. In this formalism, the crowd-anticrowd theory states that every pair of anti-correlated strategies contributes independently to the variance of attendance  $\sigma^2$  [8,9].

To understand the difference in volatility between  $\text{MG}_{\text{real}}$  and  $\text{MG}_{\text{rand}}$ , we first review the periodic dynamics observed in the symmetric phase of  $\text{MG}_{\text{real}}$  [6,8–12]. First, a prominent period  $2^{M+1}$  peak in the Fourier transform of the minority choice time series for  $\text{MG}_{\text{real}}$  is found. We call this phenomenon “period  $2^{M+1}$  dynamics.” In addition, a conspicuous period two peak in the Fourier transform of the time series of the minority choice conditioned on an arbitrary but fixed history string in  $\text{MG}_{\text{real}}$  is also observed. We refer this as the “period two dynamics” in our subsequent discussions. Note that the above two dynamics are also observed by replacing the minority choice with attendance.

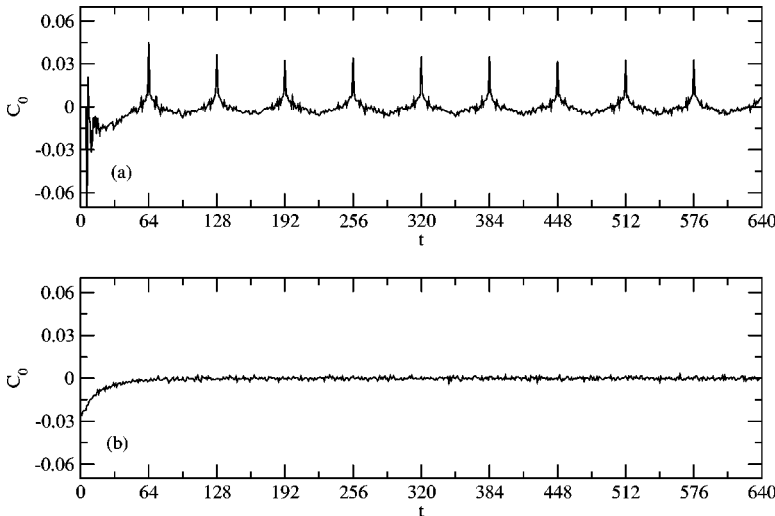


FIG. 3. The autocorrelation function  $C_0$  of the minority choice time series averaged over 50 runs for (a)  $MG_{\text{real}}$  and (b)  $MG_{\text{rand}}$ . The parameters used in both plots are  $N=1616$ ,  $M=5$  and  $S=2$ .

For  $MG_{\text{real}}$  in the symmetric phase, the number of strategies at play is much larger than the reduced strategy space size. So, it is likely that each strategy in a reduced strategy space is assigned to more than one player in this phase. Initially, for a given history string  $\mu$ , every alternative has equal chance to win. And the virtual score of a strategy  $\beta$  that has correctly predicted the winning alternative is increased by one while that of its anti-correlated strategy  $\bar{\beta}$  is decreased by one. Since the  $M$ -bit history string in  $MG_{\text{real}}$  provides complete information of the winning choices in the previous  $M$  turns, players will prefer to use strategy  $\beta$  to  $\bar{\beta}$  at the time when the same history string appears next. As more players are using the strategy  $\beta$  in the symmetric phase, these players are less likely to guess the minority choice correctly. In  $MG_{\text{real}}$ , the history strings of two consecutive turns are highly correlated. Actually, one can convert the history string in turn  $t$  to that of turn  $(t+1)$  by deleting the  $(t-1-M)$ th turn minority choice from one end of the former string and then appending the minority choice in turn  $t$  to the other end. Using these observations, Challet and Marsili studied the dynamics of  $MG_{\text{real}}$  by means of a de Bruijn graph [6]. A consequence of their analysis is that the history strings in the symmetric phase from the  $(2^M k+1)$ th to the  $[2^M(k+1)]$ th turn is likely to form a de Bruijn sequence for any natural number  $k$ . In other words, for any natural number  $k$ , an arbitrarily given history string  $\mu$  is likely to appear exactly once between  $(2^M k+1)$ th and  $[2^M(k+1)]$ th turns. Besides, the history  $\mu$  is likely to appear exactly twice between  $(2^{M+1} k+1)$ th and  $[2^{M+1}(k+1)]$ th turns—one of the turn at which a particular alternative wins and the other turn at which the same alternative loses. As a result, it is highly probable that the virtual scores of all strategies in the  $(2^{M+1} k+1)$ th turn agree. Since the decision of a player depends on the difference between the virtual scores of his/her strategies, the dynamics of the game  $MG_{\text{real}}$  has a strong tendency to “reset” itself once every  $2^{M+1}$  turns. This is the origin of the period two and period  $2^{M+1}$  dynamics [6,8–12]. Because of the above two constraints on the history time series, the time series of the minority choice as well as that of the virtual score of a strategy in  $MG_{\text{real}}$  are not random walks.

From the above discussions, we know that the existence of period  $2^{M+1}$  dynamics is closely related to the following three facts that are unique for  $MG_{\text{real}}$ : the  $M$ -bit history string gives complete information on the winning alternatives in the previous  $M$  turns, the virtual score of a strategy is updated according to the current history string, and history strings of two consecutive turns are highly correlated. In contrast, for  $MG_{\text{rand}}$ , the history string does not correlate with the winning alternatives, the virtual score is updated according to the historical winning alternatives rather than the randomly generated history string, and the randomly generated history strings in two distinct turns are uncorrelated. Consequently, although  $MG_{\text{rand}}$  has a uniformly distributed history, it does not have a mechanism to ensure the game to “reset” itself once every  $2^{M+1}$  turns. Thus there is no reason for its history time series to follow a period  $2^{M+1}$  dynamics. Indeed, the autocorrelation on the minority choice time series in Fig. 3 confirms the absence of period  $2^{M+1}$  dynamics in  $MG_{\text{rand}}$  although Fig. 4 shows that  $MG_{\text{rand}}$  still exhibits period two dynamics. Our finding is consistent with Lee’s observation that the history occurrence probability density function for  $MG_{\text{real}}$  and  $MG_{\text{rand}}$  are different [7]. To summarize, for a

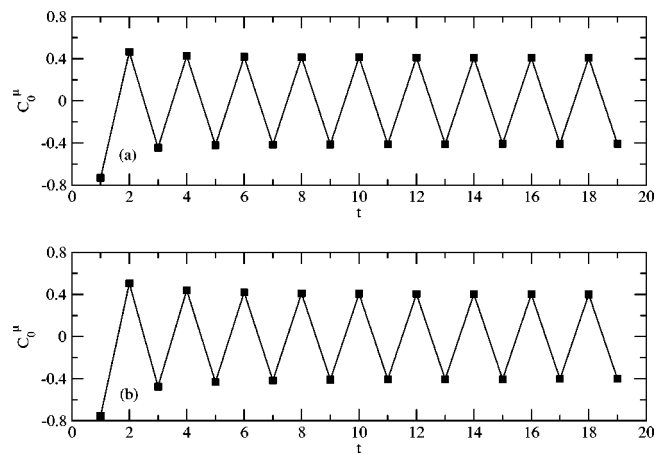


FIG. 4. The autocorrelation function  $C_0^k$  of the minority choice time series conditioned on a given history string  $\mu$  averaged over 50 runs for (a)  $MG_{\text{real}}$  and (b)  $MG_{\text{rand}}$ . The parameters used are the same as that in Fig. 3.

sufficiently small  $\alpha$ , the entropy of the minority choice time series for  $\text{MG}_{\text{rand}}$  is higher than that of  $\text{MG}_{\text{real}}$ .

We have gathered enough information to explain the discrepancy in the  $\sigma^2/N$  against  $\alpha$  curves between  $\text{MG}_{\text{real}}$  and  $\text{MG}_{\text{rand}}$ . Recall that period two dynamics is observed in the symmetric phase of both games. This is because a significant number of players are using a particular strategy. And according to the crowd-anticrowd theory, this results in a high volatility in the symmetric phase in both games [8,9]. Further recall that the game  $\text{MG}_{\text{real}}$  is very likely to “reset” itself once every  $2^{M+1}$  turns leading to the period  $2^{M+1}$  dynamics in the symmetric phase. In contrast, there is no mechanism to “reset” the game  $\text{MG}_{\text{rand}}$  once a while for  $\alpha \ll \alpha_c$ . Therefore, the absolute deviation of the virtual score difference between two distinct strategies for  $\text{MG}_{\text{real}}$  is less than that for  $\text{MG}_{\text{rand}}$  whenever  $\alpha \ll \alpha_c$ . Consequently, a player in  $\text{MG}_{\text{rand}}$  is more likely to stick to a strategy. Hence, players in  $\text{MG}_{\text{rand}}$  cooperate slightly better than those in  $\text{MG}_{\text{real}}$  in the symmetric phase. This explains why for any given sufficiently small  $\alpha$ , the variance of attendance per player  $\sigma^2/N$  for  $\text{MG}_{\text{real}}$  is higher than that of  $\text{MG}_{\text{rand}}$ .

Why do the  $\sigma^2/N$  against  $\alpha$  curves for  $\text{MG}_{\text{real}}$  and  $\text{MG}_{\text{rand}}$  coincide in the symmetric phase in the simulations of Cava-gna [5], Challet and Marsili [6] together with Lee [7]? We believe this is due to the finite size effect as they picked  $N = 101$  and  $S = 2$  in their simulations. Due to the small number of players and the small value of  $M$  involved, the fluctuation between different runs overwhelms the discrepancy between  $\text{MG}_{\text{real}}$  and  $\text{MG}_{\text{rand}}$ .

After discussing the reason why  $\sigma_{\text{real}}^2$  and  $\sigma_{\text{rand}}^2$  differ, we move on to explain the two scaling relations (1) and (2). Researchers have argued that  $\sigma^2$  is well approximated by a function of  $\alpha$  only [3,10]. Indeed, when  $\alpha \gtrsim 0.1$  and hence the number of strategies at play is at most ten times the reduced strategy space size, the above approximation is reasonably good for  $\text{MG}_{\text{real}}$  and  $\text{MG}_{\text{rand}}$ . Nevertheless, our simulation shows that this approximation breaks down in  $\text{MG}_{\text{real}}$  when  $\alpha \lesssim 0.1$ . On the other hand, using mean field approximation, Manuca *et al.* showed that

$$\sigma^2 \approx \frac{N}{4} + \frac{N^2}{3(2^M)} \chi^2(S) \quad (3)$$

where  $\chi$  is a slow varying function of  $S$  provided that  $\alpha \ll \alpha_c$  [11]. Since the correlation between history strings in successive turns is ignored in this derivation, the scaling relation (1) is applicable to  $\text{MG}_{\text{rand}}$  in the regime of  $\alpha \lesssim 0.1$ . In contrast, our simulation shows that this relation is not applicable to  $\text{MG}_{\text{real}}$  in the same regime.

Since we have already argued that  $\sigma_{\text{real}}^2(M, S)/N \gtrsim \sigma_{\text{rand}}^2(M, S)/N$  in the symmetric phase, in order to prove

the validity of relation (2), it suffices to show that  $\sigma_{\text{real}}^2(M, S)$  and  $\sigma_{\text{rand}}^2(M, S)$  both scale as  $N^2$ . According to the crowd-anticrowd theory, for a sufficiently small  $\alpha$ , all players in  $\text{MG}_{\text{real}}$  hold strategies whose virtual scores are similar. Therefore, most players will not stick to a particular strategy when making their choices. Besides, the existence of period two dynamics in the symmetric phase implies that both  $\sigma_{\text{real}}^2$  and  $\sigma_{\text{rand}}^2$  scales as  $N^2$  [8–11]. Thus relation (2) holds.

Finally, we briefly explain why  $\sigma^2$  decreases as  $M$  increases. Since the average time between successive appearances of a given history increases exponentially with  $M$ , the absolute deviation of virtual score difference between two distinct strategies increases. Therefore, the period two dynamics is weakened. So,  $\sigma^2$  for  $\text{MG}_{\text{real}}$  and  $\text{MG}_{\text{rand}}$  decreases as  $M$  increases.

#### IV. OUTLOOK

In this paper, we report that history is relevant in determining the mean variance of attendance per player for  $\text{MG}$  in the symmetric phase when  $\alpha \lesssim 0.2$  by an extensive numerical simulation using real and random histories. We explain our finding using crowd-anticrowd theory. Although all graphs shown in this paper are drawn by fixing the number of strategies per player  $S$  to 2 and by letting the strategies to be drawn from the full strategy space, our conclusions apply equally well to the case of  $S > 2$  as well as the case of drawing strategies from the reduced strategy space. Our findings show that the feedback mechanism buried under the correlation in the history time series is important in the study of volatility in the minority game.

It is instructive to study the relevance of history in the parameter region of  $0.2 \lesssim \alpha \ll \alpha_c$ . Our numerical simulation suggests that memory is irrelevant in this regime and hence the symmetric phase of  $\text{MG}$  can be subdivided into two phases. Further investigation is needed to test our hypothesis.

Finally, it would be nice if one could analytically solve the dynamics of the game in the symmetric phase. Since the difference in  $\sigma^2/N$  for  $\text{MG}_{\text{real}}$  and  $\text{MG}_{\text{rand}}$  in the symmetric phase originates from the period  $2^{M+1}$  dynamics, any such attempt must take the periodic dynamics of the minority choice time series into account—something that all attempts using replica trick [13,14] and generating functional method [15,16] so far have not been very successful to incorporate.

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