Symmetric Extension Methods for M-Channel Linear-Phase Perfect-Reconstruction Filter Banks

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***Abstract*—The symmetric extension method has recently been shown to be an efficient way for subband processing of finite-length sequences. This paper presents an extension of this method to general linear-phase perfect-reconstruction filter banks. We derive constraints on the length and symmetry polarity of the permissible filter banks and propose a new design algorithm. In the algorithm, different symmetric sequences are formulated in a unified form based on the circular-symmetry framework. The length constraints in symmetrically extending the input sequence and windowing the subband sequences are investigated. The effect of shifting the input sequence is included. When the algorithm is applied to equal-length filter banks, we explicitly show that symmetric extension methods can always be constructed to replace the circular convolution approach.**

I. Introduction

Digital filter banks have become popular in the last few years as a method to channelize a signal into many subbands, use the subband contents to extract essential information, and then reconstruct the original signal. The spectral contents of the subband signals can be used in a number of communication, signal analysis, and enhancement applications. For example, in a data transmission system, the subband signals can be used to code the audio/video signals, whereas in a radar system, they might be used to null out a narrow-band interference adaptively. Fig. 1 illustrates a typical M-channel filter bank where the analysis filters $H_k(z)$ channelize the input signal $x(n)$ into M subband signals, which are downsampled by a factor M. At the receiving end, the M subband signals are decoded, interpolated, and recombined using a set of synthesis filters $F_k(z)$. The decimator, which decreases the sampling rate of the signal, and the expander, which increases the sampling rate of the signals, are denoted by blocks with down-arrows and up-arrows, respectively.

Due to the nonideal characteristics of $H_k(z)$ and $F_k(z)$ and the decimation and interpolation operations, the reconstructed signal $x'(n)$, in general, suffers from magnitude/phase distortions and aliasing errors. For infinite-length sequences, systems free from the above distortions are called perfect-reconstruction filter banks [1]–[7]. For finite-length sequences (e.g., the image rows and columns), besides the above errors, there is an additional error source due to the expansive effect around the borders. Given a sequence with N_x samples and a filter $H_k(z)$ with N_{h_k} taps, the linear convolution output, $v_k(n)$ in Fig. 1, will have $N_x + N_{h_k} - 1$ samples. This expansive effect is undesirable in data compression applications. A possible solution to obtain *nonexpansive* subband signals $v_k(n)$ is to truncate or window them. However, the truncating or windowing operations cause distortions around the borders in the form of false edges and ringing effects [11]. Several solutions for this problem have been suggested in [8]–[21], which are based on circular convolution [8], symmetric extension [11]–[18], time-variant filters [19], [20], and nonlinear-phase filters [21]. The circular convolution method is the simplest nonexpansive approach. The disadvantage is that it might create step discontinuities across the borders and is therefore sensitive to quantization errors at medium and low-bit rate coding applications [11], [12]. The symmetric extension method based on linear-phase filter banks was proposed as an improved approach [11]. Fig. 2 shows the basic idea behind the symmetric extension method.

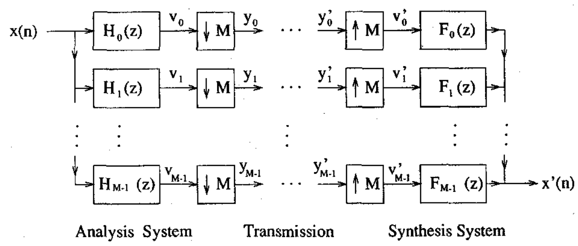

Fig. 1. M-channel analysis/synthesis systems

To maintain continuity across the borders, symmetric extension is first applied to the input sequence $x(n)$. The resulting symmetric sequence $\hat{x}(n)$ is used as the input to the analysis filter bank and circularly convolved. Suppose the analysis filters $h_k(n)$ are symmetric, then $\hat{v}_k(n)$ are also symmetric sequences. Hopefully, the symmetric properties will propagate through the decimators so that the subband sequences $\hat{y}_k(n)$ are also symmetric. Hence, only approximately half (depending on symmetry/antisymmetry and lengths of the sequences) of the samples in $\hat{y}_k(n)$ are retained and transmitted to the receiver, assuming the discarded samples can be retrieved by the *a priori* knowledge of the symmetry structures of $\hat{y}_k(n)$. If the number of samples in the sequences $z_k(n)$ (Fig. 2) do not exceed that of the original sequence $x(n)$, i.e. $\lambda \triangleq (\sum_{k=0}^{M-1} N_{z_k}) - N_x \leq 0$, the nonexpansive property is achieved.

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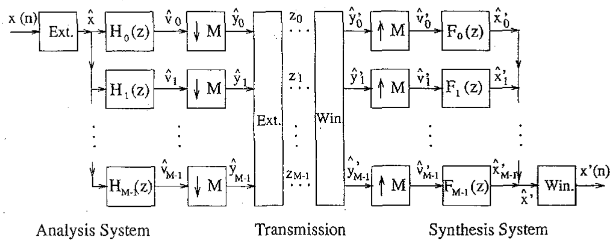

Fig. 2. Analysis/synthesis systems with symmetric extension and windowing

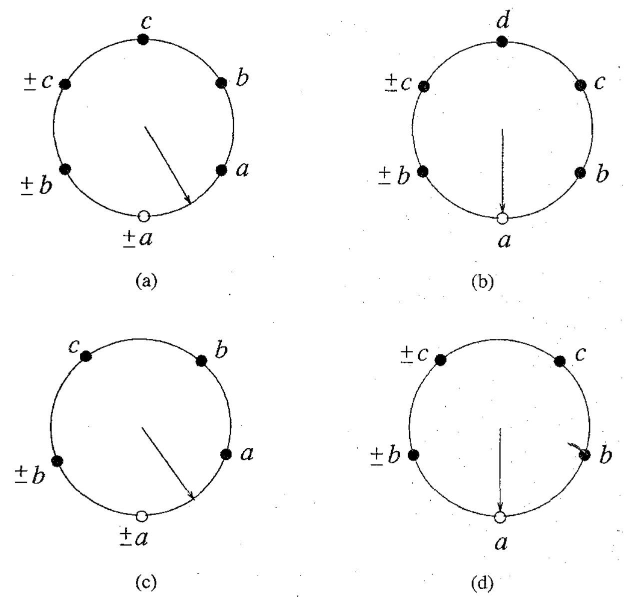

Fig. 3. Circular-formed symmetric sequences.

The above idea was first proposed by Smith and Eddins in two-channel case [11]. The filter banks are restricted to have odd orders. To relax the restriction, much effort has been devoted to this area. Different types of symmetry in finite-length and periodic sequences were shown and the results were extended to both even and odd-order two-channel filter banks in [13] and [15] and to M-channel filter banks in [16]–[18]. In [17], different types of symmetry are formulated using frequency domain representation. The influence of circular convolution and decimation are investigated. Based on the algorithm, symmetric extensions are constructed and lead the subband sequences to be symmetric. In [16] and [18], the symmetric extension methods are also generalized to M-channel cases. The nonexpansive constraints have been discussed. Fast transforms (DCT, DST) are used in [16] for the efficient implementation of symmetric convolutions.

In this paper, symmetric extension methods are further studied and extended to general linear-phase perfect-reconstruction filter banks. The constraints on the length and symmetry polarity of the permissible filter banks are derived and a new design algorithm is proposed. These constraints can be used as guidelines for the design of linear-phase filter banks. In the algorithm, different symmetric sequences are formulated in a unified form based on the circular-symmetry framework. The length constraints in symmetrically extending the input sequence and windowing the subband sequences are investigated. The effect of shifting the input sequence in [16] and [18] is included. The algorithm is also applied to equal-length filter banks and shows an important result that symmetric extension methods can always be built to replace the circular convolution approach and the corresponding symmetric extensions are shown in closed-form.

The organization of the paper is as follows. In Section II, different symmetries in finite-length and periodic sequences are discussed and circular-symmetry is introduced as a unified framework for them. With this framework, the symmetry conversion properties are formulated in closed-forms in Section III. The constraints for the subband sequences to be symmetric are shown. Section IV derives the nonexpansive constraints. In Section V, the algorithm and design examples are presented. The results are applied to linear phase filter banks with equal length. Finally, a summary is presented in Section VI.

Notation: Z, Z_{even} and Z_{odd} stand for the set of integers, even, and odd integers, respectively. $\frac{1}{2}Z$ denotes the integral multiples of $\frac{1}{2}$, i.e., $\{\cdots,-\frac{1}{2},0,\frac{1}{2},1,\cdots\}$. Similarly, we have notations: $\frac{1}{2}Z_{odd}$, $\frac{1}{2}Z_{even}$, MZ, etc. For a finite-length sequence $x(n)$, its periodic sequence is denoted as $\tilde{x}(n)$.

II. Symmetric Structures

As stated above, circular convolution instead of linear convolution is employed in symmetric extension methods. To show both the periodic and symmetric properties clearly, several symmetric sequences are placed on the circles in Fig. 3, where the unfilled dot refers to the first sample $x(0)$. From Fig. 3, the following characteristics can be observed:

1) $x(n)$ is either symmetric or anti-symmetric.
2) $x(n)$ has either even or odd number of samples.
3) The symmetry centers lie at either a sample ($\frac{1}{2}Z_{even}$) or midway between two samples ($\frac{1}{2}Z_{odd}$).
4) If c_x is a symmetry center, so is the opposite end of the diameter, $c_x + \frac{N_x}{2}$.

Despite the above differences, we will show that the symmetries can be converted from one to another in the filter banks. To address the different symmetries in [11]–[18] in a unified form, we introduce the following definition of "circular-symmetry."

Definition 2.1: A finite-length sequence $\{x(n), 0 \le n \le N_x-1\}$ is said to be circular-symmetric if its periodic sequence $\tilde{x}(n)$ satisfies

$$\tilde{x}(c_x + d) = J_x \cdot \tilde{x}(c_x - d) \qquad (1)$$

where $c_x \in [0, \frac{N_x-1}{2}]$; $c_x, d \in \frac{1}{2}Z; c_x \pm d \in Z$. c_x is the symmetry center, d is the distance between a sample and the center c_x. $J_x = \pm 1$ is the symmetry polarity. $x(n)$ is said to be symmetric or anti-symmetric for $J_x = 1$ or -1, respectively.

Since the period N_x can be an even or odd integer, and the symmetry center c_x can be an even or odd multiple of $\frac{1}{2}$, we have four types of symmetry, as shown in Table I. They exactly correspond to the four circular-symmetric sequences in Fig. 3.

Remarks: In the definition of circular symmetry, c_x is restricted to $[0, \frac{N_x-1}{2}]$. The symmetry center at the opposite end of the diameter $(c_x + \frac{N_x}{2})$ is uniquely determined by c_x

TABLE I
TYPES OF SYMMETRY IN FINITE-LENGTH AND PERIODIC SEQUENCES

Type	N_x	c_x
A	Z_{even}	$\frac{1}{2}Z_{odd}$
B	Z_{even}	$\frac{1}{2}Z_{even}$
C	Z_{odd}	$\frac{1}{2}Z_{odd}$
D	Z_{odd}	$\frac{1}{2}Z_{even}$

TABLE II
EXAMPLES OF SYMMETRY CONVERSION THROUGH CIRCULAR CONVOLUTION

n	0	1	2	3	4	5	6	7	Type	J	c
$x_1(n)$	1	2	3	4	4	3	2	1	A	1	3.5
$x_2(n)$	0	0	0	1	1	0	0	0	A	1	3.5
$y(n)$	7	5	3	2	3	5	7	8	B	1	3
$x_2'(n)$	0	0	0	−1	1	0	0	0	A	−1	3.5
$y'(n)$	1	1	1	0	−1	−1	−1	0	B	−1	3

and N_x. In Fig. 3(a), $\frac{1}{2}$ instead of $3\frac{1}{2}$ is used for c_x. For anti-symmetry B, C, and D, the values of the samples at the symmetry centers are zero, i.e., i) anti-sym. B, $x(c_x) = x(c_x + \frac{N_x}{2}) = 0$; ii) anti-sym. C, $x(c_x + \frac{N_x}{2}) = 0$; iii) anti-sym. D, $x(c_x) = 0$.

Symmetries of $\hat{x}(n)$ and $h_k(n)$: The input sequence $x(n)$ is extended to a symmetric sequence $\hat{x}(n)$. The symmetry center satisfies $c_{\hat{x}} \in \frac{1}{2}Z$, without loss of generality, it can be uniquely expressed as

$$c_{\hat{x}} \triangleq \frac{m_{\hat{x}}M + i_{\hat{x}}}{2}, ; m_{\hat{x}} \in Z,\ 0 \le i_{\hat{x}} \le M-1. \tag{2}$$

For the analysis filter $h_k(n)$ with linear-phase characteristic $h_k(n) = J_{h_k} \cdot h_k(N_{h_k} - 1 - n)$, the symmetry center is located at $c_{h_k} = \frac{N_{h_k}-1}{2}$. Let $N_{h_k} - 1 = m_{h_k}M + i_{h_k}$, the symmetry center is expressed as

$$c_{h_k} \triangleq \frac{m_{h_k}M + i_{h_k}}{2},\ m_{h_k} \in Z,\ 0 \le i_{h_k} \le M-1. \tag{3}$$

In the sections below, we derive the constraints on the parameters in (2) and (3). Section III derives the symmetric constraints for the subband sequences $\hat{y}_k(n)$ to be symmetric; Section IV shows the nonexpansive constraints for the subband sequences $\hat{y}_k(n)$ to be nonexpansive.

III. SYMMETRIC CONSTRAINTS

Using the circular symmetry framework, the symmetry conversions in circular convolution and decimation are formulated as follows. The proof of the circular convolution properties is shown in Appendix A. The decimation properties can be shown in a similar way. With these properties, we then derive the symmetric constraints in Proposition 3 that lead the subband sequences to be symmetric.

Proposition 3.1: Let $x_1(n)$ and $x_2(n)$ be two equal-length circular-symmetric sequences, the circular convolution output $y(n) = x_1(n) * x_2(n)$ is also a symmetric sequence

$$N_y = N_x,\ c_y = c_{x_1} + c_{x_2} + c_{x_1,x_2},\ J_y = J_{x_1} \cdot J_{x_2} \tag{4}$$

where

$$c_{x_1,x_2} \triangleq \begin{cases} 0, & 0 \le c_{x_1} + c_{x_2} \le \frac{N_x-1}{2} \\ -\frac{N_x}{2}, & \frac{N_x-1}{2} < c_{x_1} + c_{x_2} \le N_x - 1. \end{cases} \tag{5}$$

As indicated in Appendix A, the circular convolution output $y(n)$ has an infinite number of symmetric centers. The above c_{x_1,x_2} is to let us choose the symmetry center within $[0, \frac{N_y-1}{2}]$, i.e., on the first half-circle in Fig 3. Table II shows some examples of the symmetry conversions, where $y(n)$ and $y'(n)$ are the circular convolutions of $x_1(n)$ with $x_2(n)$ and $x_2'(n)$, respectively.

Proposition 3.2: Let $x(n)$ be a symmetric sequence of length $N_x \in MZ$, and $y(n)$ be the decimated sequence $y(n) = x(Mn)$. Then, $x(n)$ has a symmetry center satisfying $c_x \in \frac{1}{2}MZ$ is necessary and sufficient for $y(n)$ to be symmetric. The decimated output has the following properties

$$N_y = N_x/M,\ c_y = c_x/M, ; J_y = J_x. \tag{6}$$

Proposition 3.3: In an M-channel linear-phase filter bank, if the subband sequences $\hat{y}_k(n)$ are symmetric, then the filters $h_k(n)$ must satisfy $i_{h_k} = i_h$, i.e., all i_{h_k} are the same. The symmetric extensions are of the following forms

$$\begin{cases} N_{\hat{x}} \in MZ, \\ c_{\hat{x}} = \frac{1}{2}((m_{\hat{x}} + m_0)M - i_h), \end{cases} \tag{7}$$

where

$$m_0 \triangleq \begin{cases} 0, & i_h = 0, \\ 1, & i_h \ne 0. \end{cases} \tag{8}$$

Proof: The symmetry conversion properties in the analysis system can be derived from (4) and (6):

$$\text{a) } N_{\hat{y}_k} = \frac{N_{\hat{x}}}{M};\ \text{b) } c_{\hat{y}_k} = \frac{c_{\hat{x}} + c_{h_k} + c_{\hat{x},h_k}}{M} \tag{9}$$

where

$$c_{\hat{x},h_k} \triangleq \begin{cases} 0, & 0 \le c_{\hat{x}} + c_{h_k} \le \frac{N_{\hat{x}}-1}{2} \\ -\frac{N_{\hat{x}}}{2}, & \frac{N_{\hat{x}}-1}{2} < c_{\hat{x}} + c_{h_k} \le N_{\hat{x}} - 1. \end{cases} \tag{10}$$

Supposing the subband sequences $\hat{y}_k(n)$ are symmetric, we have

$$\text{a) } N_{\hat{y}_k} \in Z;\ \text{b) } c_{\hat{y}_k} = \frac{1}{2}m_{\hat{y}_k},\ m_{\hat{y}_k} \in Z. \tag{11}$$

The first constraint in (7) is easily shown from part a) of (9) and part a) of (11). Based on it, (10) can be expressed as

$$c_{\hat{x},h_k} = \frac{m_{\hat{x},h_k}M}{2} \tag{12}$$

where

$$m_{\hat{x},h_k} \triangleq \begin{cases} 0, & 0 \le c_{\hat{x}} + c_{h_k} \le \frac{N_{\hat{x}}-1}{2} \\ -\frac{N_{\hat{x}}}{M}, & \frac{N_{\hat{x}}-1}{2} < c_{\hat{x}} + c_{h_k} \le N_{\hat{x}} - 1. \end{cases} \tag{13}$$

Applying (2), (3), and (12) to (9)-b, the symmetry center $c_{\hat{y}_k}$ becomes

$$c_{\hat{y}_k} = \frac{(m_{\hat{x}} + m_{h_k} + m_{\hat{x},h_k})M + i_{\hat{x}} + i_{h_k}}{2M}. \tag{14}$$

Comparing it with (11)-b, we have

$$i_{\hat{x}} + i_{h_k} = m_0M,\ m_0 \in Z \tag{15}$$

TABLE III
CONSTRAINTS ON $N_{\hat{x}}$ AND N_x

Type	N_0
A	0+2ℓ
B	−2+2ℓ
C	−1+2ℓ
D	−1+2ℓ

$$m_{\hat{x}} + m_{h_k} + m_{\hat{x},h_k} + m_0 = m_{\hat{y}_k}. \quad (16)$$

If all $\hat{y}_k(n)$ are symmetric sequences, then (15) is imposed on all the analysis filters $h_k(n)$. Furthermore, noting $0 \le i_{\hat{x}} \le M-1$, $0 \le i_{h_k} \le M-1$, the filters are shown to satisfy $i_{h_k} = i_h$, $0 \le k \le M-1$, and m_0 is in the form of (8). Substituting (15) into (2), the second equation in (7) on $c_{\hat{x}}$ is derived. Q.E.D.

Most filter banks reported in the literature satisfy the property $i_{h_k} = i_h$, $0 \le k \le M-1$. For the case of two channels, it is shown in [4] that the lengths of linear-phase perfect-reconstruction filter banks have to be either both odd or both even. Consequently, the condition $i_{h_0} = i_{h_1}$ holds. It is also true for the paraunitary filter bank [22], since the filters in a paraunitary filter bank have the same lengths. An example of three-channel linear-phase filter bank is reported in [6], where the filter lengths are 56, 53, and 56, respectively. One can verify that $i_{h_k} = 1$, $0 \le k \le 2$ for this case.

Besides the length constraints $i_{h_k} = i_h$, the nonexpansive constraint in the next section will impose further restrictions on the length and symmetry polarity of the permissible filter banks.

IV. NONEXPANSIVE CONSTRAINTS

In the derivation of symmetric extension methods, the knowledge that the subband sequences are symmetric is not enough to guarantee the nonexpansive property. This can be seen in Fig. 3, where more than half of the samples are needed to represent the sequences with Symmetry B, C, and D. To find the nonexpansive constraints, we first derive the following two length constraints in the symmetrically extending the input sequence $x(n)$ and windowing the subband sequences $\hat{y}_k(n)$.

Constraints between $N_{\hat{x}}$ and N_x: Let $\hat{x}(n)$ be a circular-symmetric sequence extended from an input sequence $x(n)$. Then, $N_{\hat{x}}$ and N_x are constrained under

$$N_{\hat{x}} = 2N_x + N_0 \quad (17)$$

where N_0 is shown in Table III.

Table III can be readily justified from Fig. 3. For Type A symmetric extension, $N_{\hat{x}} = 2N_x + N_0 = 2N_x + (0 + 2\ell)$, where 0 indicates that the minimum length $N_{\hat{x}}$ is $2N_x$. For Symmetry B, the minimum N_0 is -2. The corresponding symmetric extension is shown in (18). Beyond the minimum length, we can also extend $x(n)$ to $\hat{x}(n)$ with longer lengths by padding even number (2ℓ) of samples. These samples can be chosen with certain freedom, provided that the resulting signal $\hat{x}(n)$ is symmetric and has no abrupt discontinuity around the borders. In (19), $x(n)$ is extended to Symmetry B with $N_0 = 0, 2\ell = 2$ and the simplest ways to duplicate the border sample $x(N_x - 1)$ to $\hat{x}(N_x)$ and $\hat{x}(N_x + 1)$ are

$$\hat{x}(n) = \begin{cases} x(n), & 0 \le n \le N_x - 1 \\ x(2N_x - n - 2), & N_x \le n \le 2N_x - 3 \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

$$\hat{x}(n) = \begin{cases} x(n), & 0 \le n \le N_x - 1 \\ x(N_x - 1), & N_x \le n \le N_x + 1 \\ x(2N_x - n), & N_x + 2 \le n \le 2N_x - 1 \\ 0, & \text{otherwise.} \end{cases} \quad (19)$$

It is worth mentioning that while padding a large number (2ℓ) of samples may lead the symmetric extension to be expansive, it is not always the case that the minimum value of N_0 ($2\ell = 0$) gives the perissible symmetric extensions. For example, as we will show in the next section, for filter banks with equal length, the above Symmetry B with ($N_0 = 0, 2\ell = 2$) in (19) is used to construct symmetric extensions instead of using (18), ($N_0 = -2, 2\ell = 0$). In the subsequent section, we examine the minimum storage requirement in representing the subband sequences and derive constraints for the permissible N_0 and the corresponding symmetric extension.

TABLE IV
MINIMUM STORAGE REQUIREMENT N_{z_k}

Type	$J_{\hat{y}_k}$	N_{z_k}	Type	$J_{\hat{y}_k}$	N_{z_k}
A	+1	$\frac{1}{2}N_{\hat{y}_k}$	C	+1	$\frac{1}{2}(N_{\hat{y}_k}+1)$
	−1	$\frac{1}{2}N_{\hat{y}_k}$		−1	$\frac{1}{2}(N_{\hat{y}_k}-1)$
B	+1	$\frac{1}{2}N_{\hat{y}_k}+1$	D	+1	$\frac{1}{2}(N_{\hat{y}_k}+1)$
	−1	$\frac{1}{2}N_{\hat{y}_k}-1$		−1	$\frac{1}{2}(N_{\hat{y}_k}-1)$

Minimum Storage Lengths N_{z_k}: Table IV shows the minimum storage requirement (N_{z_k}) in uniquely representing the subband sequences $\hat{y}_k(n)$ with different types of symmetry. Anti-symmetry B, C, and D have less storage requirements than their symmetric counterparts since the coefficients at the symmetry centers are zero. These zero-valued samples do not need to be stored or transmitted, provided that the receiving end knows *a priori* the symmetric structures in the subband sequences $\hat{y}_k(n)$ and the order in which windowed sequences $z_k(n)$ are sent. From Table IV, we can put the minimum storage requirements of $\hat{y}_k(n)$ in the following form:

$$N_{z_k} = \frac{1}{2}N_{\hat{y}_k} + \frac{1}{4}J_{\hat{y}_k}\{2 + (-1)^{2c_{\hat{y}_k}} \cdot (1 + (-1)^{N_{\hat{y}_k}})\}. \quad (20)$$

Proposition 4.1: For an M-channel linear-phase filter bank satisfying $i_{h_k} = i_h$, the nonexpansive symmetric extensions have the following forms:

$$\begin{cases} \lambda = \frac{1}{2}N_0 + \frac{1}{2}\sum_{k=0}^{M-1} J_{h_k}(1 + (-1)^{m_{\hat{x}} + m_{h_k} + m_0}) \le 0, \\ N_x \in \frac{1}{2}(MZ_{even} - N_0), \\ c_{\hat{x}} = \frac{1}{2}((m_{\hat{x}} + m_0)M - i_h), \end{cases} \quad (21)$$

$$\begin{cases} \lambda = \frac{1}{2}N_0 + \frac{1}{2}\sum_{k=0}^{M-1} J_{h_k} \le 0, \\ N_x \in \frac{1}{2}(MZ_{odd} - N_0), \\ c_{\hat{x}} = \frac{1}{2}((m_{\hat{x}} + m_0)M - i_h). \end{cases} \quad (22)$$

Proof: In a uniformly decimated analysis system, the subband sequences $\{\hat{y}_k(n), 0 \le k \le M-1\}$ have the same length. The proposition can be shown in the following two cases.

Case 1. $N_{\hat{y}_k} \in Z_{even}$, $0 \le k \le M-1$: Part a) of (9) indicates $N_{\hat{x}} = MN_{\hat{y}_k} \in MZ_{even}$. From (17), the second equation in (21) is shown. Since $N_{\hat{y}_k} \in Z_{even}$, the storage overhead is

$$\lambda = (\sum_{k=0}^{M-1} N_{z_k}) - N_x = \sum_{k=0}^{M-1}\{\frac{1}{2}N_{\hat{y}_k} + \frac{1}{2}J_{\hat{y}_k}(1+(-1)^{2c_{\hat{y}_k}})\} - N_x. \quad (23)$$

Noting that $N_{\hat{y}} = \frac{N_{\hat{x}}}{M} = \frac{2N_x+N_0}{M}$ and using part b) of (11) and (16), λ is simplified to

$$\lambda = \frac{1}{2}N_0 + \frac{1}{2}\sum_{k=0}^{M-1} J_{\hat{y}_k}(1+(-1)^{m_{\hat{x}}+m_{h_k}+m_{\hat{x},h_k}+m_0}). \quad (24)$$

In symmetric extension methods, it is desirable for $\hat{x}(n)$ to have smooth border transitions. Hence, $J_{\hat{x}} = +1$ is employed in symmetric extension. From the symmetry conversion properties in circular convolution and decimation, we have $J_{\hat{y}_k} = J_{\hat{x}}J_{h_k} = J_{h_k}$ in (4). Moreover, $N_{\hat{x}} \in MZ_{even}$ indicates that $m_{\hat{x},h_k}$ in (13) is an even integer. Applying these to (24), λ in (21) is shown.

Case 2. $N_{\hat{y}_k} \in Z_{odd}$, $0 \le k \le M-1$: Similarly, it is easy to show the second equation in (22). Noting $N_{\hat{y}_k} \in Z_{odd}$ in this case, we have $\lambda = \sum_{k=0}^{M-1}(\frac{1}{2}N_{\hat{y}_k} + \frac{1}{2}J_{\hat{y}_k}) - N_x$. Applying $N_{\hat{y}_k} = \frac{2N_x+N_0}{M}$, λ in (22) is shown. Q.E.D.

V. Algorithm and Examples

Given an input sequence $x(n)$ and the analysis filters $h_k(n)$, symmetric extensions can be constructed using the following algorithm:

1) Calculate J_{h_k}, m_{h_k} , i_{h_k}, m_0 and check the constraint $\{i_{h_k} = i_h, 0 \le k \le M-1\}$.
2) From the first constraints in (21) and (22), compute the permissible values for N_0 and $m_{\hat{x}}$.
3) From the second and third equations in (21) and (22), compute the permissible values for N_x and $c_{\hat{x}}$.
4) Compare $(c_{\hat{x}}, N_0)$ with Table 1, and 3 and determine the permissible symmetry types (A, B, C, D). Construct the symmetric extension with: *a*) $N_{\hat{x}} = 2N_x + N_0$; *b*) $c_{\hat{x}}$; *c*) $J_{\hat{x}} = 1$.
5) Construct the windowing scheme with: *a*) N_{z_k} in (20); *b*) $c_{\hat{y}_k}$ in (14); *c*) $J_{\hat{y}_k} = J_{h_k}$.

A. Two-Channel Case

For the two-channel case, the linear-phase paraunitary filter bank is degenerate [23]. There exist only two classes of linear-phase perfect-reconstruction filter banks that are nontrivial [4]. The first class is called the SAOO filter bank, where $h_0(n)$ and $h_1(n)$ are symmetric and anti-symmetric, respectively; both filters have odd orders. The second class is called the SSEE filter bank, where both $h_0(n)$ and $h_1(n)$ are symmetric and have even orders. For the SAOO filter bank presented in [4], $J_{h_0} = 1$, $J_{h_1} = -1$, $N_{h_0} - 1 = N_{h_1} - 1 = 21$, $m_{h_0} = m_{h_1} = 10$, $m_0 = 1$, $i_{h_0} = i_{h_1} = 1$. Since both (21) and (22) yield $\lambda = \frac{1}{2}N_0 \le 0$, they can be combined as $\begin{cases} N_x \in Z - \frac{1}{2}N_0 \\ c_{\hat{x}} = m_{\hat{x}} + \frac{1}{2}. \end{cases}$ Hence, $c_{\hat{x}} \in \frac{1}{2}Z_{odd}$, $N_0 \in Z_{even}$. Referring to Table I and Table III, only Type A symmetric extensions are permissible. Hence, $N_0 = 0$ and $\begin{cases} N_x \in Z, \\ c_{\hat{x}} \in \frac{1}{2}Z_{odd}. \end{cases}$

Example: For $N_x = 256$, the following two equivalent symmetric extensions (Type A, $N_0 = 0$) can be constructed with the center $c_{\hat{x}}$ at $\frac{1}{2}$ and $N_x - \frac{1}{2}$, respectively.

$$\hat{x}(n) = \begin{cases} x(0), & n = 0 \\ x(n-1), & 1 \le n \le N_x \\ x(2N_x - n), & N_x + 1 \le n \le 2N_x - 1 \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

$$\hat{x}(n) = \begin{cases} x(n), & 0 \le n \le N_x - 1 \\ x(2N_x - n - 1), & N_x \le n \le 2N_x - 1 \\ 0, & \text{otherwise.} \end{cases} \quad (26)$$

Equation (26) is actually the one proposed in [11] and [12].

B. Three-Channel Case

For the filter banks in Table III and Table IV in [6], $J_{h_0} = J_{h_1} = 1$, $J_{h_2} = -1$, $N_{h_0} - 1 = 55$, $N_{h_1} - 1 = 52$, $N_{h_2} - 1 = 55$, $m_{h_0} = 18$, $m_{h_1} = 17$, $m_{h_2} = 18$, $m_0 = 1$, $i_{h_0} = i_{h_1} = i_{h_2} = 1$. Substituting them into Proposition 4 yields

$$\begin{cases} \lambda = \frac{1}{2}(1 + (-1)^{m_{\hat{x}}} + N_0) \le 0, \\ N_x \in \frac{1}{2}(3Z_{even} - N_0), \\ c_{\hat{x}} = \frac{1}{2}(3m_{\hat{x}} + 2), \end{cases} \quad (27)$$

$$\begin{cases} \lambda = \frac{1}{2}(1 + N_0) \le 0, \\ N_x \in \frac{1}{2}(3Z_{odd} - N_0), \\ c_{\hat{x}} = \frac{1}{2}(3m_{\hat{x}} + 2). \end{cases} \quad (28)$$

Both equations indicate $N_0 \le 0$, i.e., all the 2ℓ in Table III should be 0. Setting $N_0 = 0$, $m_{\hat{x}} \in Z_{odd}$ in (27), we have $\lambda = 0$ and $N_x \in 3Z$. Similarly, setting $N_0 = -2$, $m_{\hat{x}} \in Z_{even}$ in (27) and $N_0 = -1$, $m_{\hat{x}} \in Z$ in (28), we have $N_x \in 3Z + 1$ and $N_x \in 3Z + 2$. Henceforth, for all $N_x \in Z$, nonexpansive symmetric extensions exist (provided that $N_{\hat{x}} = 2N_x + N_0 \ge \text{Max}(N_{h_0}, N_{h_1}, N_{h_2})$, since in circular convolution $N_{\hat{x}}$ should not be shorter than the filter lengths.)

Example: For $N_x = 256 \in 3Z + 1$, the following Type B symmetric extension ($c_{\hat{x}} = 1$, $N_0 = -2$) can be derived from (27).

$$\hat{x}(n) = \begin{cases} x(1), & n = 0 \\ x(n-1), & 1 \le n \le N_x \\ x(2N_x - n - 1), & N_x + 1 \le n \le 2N_x - 3 \\ 0, & \text{otherwise,} \end{cases} \quad (29)$$

C. Equal Length Filter Banks

Most of the symmetric extension methods presented in the literature use filter banks with equal length. In the following, we apply Proposition 4.1 to the case of equal-length filter banks and compare the results with the circular convolution approach.

Proposition 5.1: For equal-length filter banks, symmetric extensions exist for any input sequences with $N_x \in MZ$. In other words, symmetric extension methods can always be constructed to replace the circular convolution approach.

This proposition can be readily verified as follows. Since the filter banks are of equal length, hence $m_{h_k} = m_h$ and $i_{h_k} = i_h$. The storage overhead in (21) becomes $\lambda = \frac{1}{2}N_0 + \frac{1}{2}(1 + (-1)^{m_{\hat{x}}+m_h+m_0})\sum_{k=0}^{M-1} J_{h_k}$. By setting $m_{\hat{x}} + m_h + m_0 \in Z_{odd}$ and $N_0 = 0$, we can show nonexpansive symmetric extensions exist for any $N_x \in MZ$, and thus can be used to replace the circular convolution approach. The corresponding symmetric extensions are shown in closed-form as follows:

1) For $m_h + m_0 \in Z_{odd}$, set $m_{\hat{x}} = 0$ and let

$$\begin{cases} N_{\hat{x}} = 2N_x, \\ c_{\hat{x}} = \frac{1}{2}(m_0 M - i_h). \end{cases} \tag{30}$$

2) For $m_h + m_0 \in Z_{even}$, set $m_{\hat{x}} = 1$ and let

$$\begin{cases} N_{\hat{x}} = 2N_x, \\ c_{\hat{x}} = \frac{1}{2}((1 + m_0)M - i_h). \end{cases} \tag{31}$$

When $c_{\hat{x}} \in \frac{1}{2}Z_{even}$, symmetric extensions with Type B, $N_0 = 0$ are used. They are shift versions of the symmetric extension in (19).

VI. Conclusion

We have presented a new algorithm for designing symmetric extension methods for linear-phase filter banks. The circular-symmetry model has been shown to be a useful tool in integrating the different symmetries in a unified framework. Based on it, the symmetry conversion properties in the filter banks are shown. We have also investigated the length constraints in symmetrically extending the input sequence and in windowing the subband sequences. Symmetric and nonexpansive constraints are derived on the length and symmetry polarity of the permissible filter banks. The algorithm has also been applied to equal-length filter banks and shows that symmetric extension method can always be constructed to replace circular convolution methods and the corresponding symmetric extensions are shown in closed-form.

Appendix A

The circular convolution of $x_1(n)$ and $x_2(n)$ yields

$$\tilde{y}(n) = \sum_{l=0}^{N_x-1} \tilde{x}_1(l) \cdot \tilde{x}_2(n-l) \tag{32}$$

$$= \sum_{l=0}^{N_x-1} \tilde{x}_1(c_{x_1} + (l - c_{x_1})) \cdot \tilde{x}_2(c_{x_2} + (n - l - c_{x_2})). \tag{33}$$

Since $x_1(n)$ and $x_2(n)$ are symmetric, $\tilde{x}_1(c_{x_1} + d) = J_{x_1} \cdot \tilde{x}_1(c_{x_1} - d)$, $\tilde{x}_2(c_{x_2} + d) = J_{x_2} \cdot \tilde{x}_2(c_{x_2} - d)$, we have

$$\tilde{y}(n) = (J_{x_1}J_{x_2}) \cdot \sum_{l=0}^{N_x-1} \tilde{x}_1(2c_{x_1} - l) \cdot \tilde{x}_2(2c_{x_2} - n + l). \tag{34}$$

Due to the periodicity of $\tilde{x}_1(n)$ and $\tilde{x}_2(n)$, it follows that

$$\tilde{y}(n) = (J_{x_1}J_{x_2}) \cdot \sum_{l=0}^{N_x-1} \tilde{x}_1(l) \cdot \tilde{x}_2(2c_{x_1} + 2c_{x_2} - n - l). \tag{35}$$

Referring to the definition of circular convolution in (32), we have

$$\tilde{y}(n) = (J_{x_1}J_{x_2}) \cdot \tilde{y}(2c_{x_1} + 2c_{x_2} - n). \tag{36}$$

Since $\tilde{y}(n)$ is periodic, (36) can be written as

$$\tilde{y}(n) = (J_{x_1}J_{x_2}) \cdot \tilde{y}(2c_{x_1} + 2c_{x_2} - n + \rho N_x), \ \rho \in Z. \tag{37}$$

Substituting $n = c_{x_1} + c_{x_2} + \frac{1}{2}\rho N_x + d, \rho \in Z$ to the above equation, we can show that all members of the set of points $\{c_{x_1} + c_{x_2} + \frac{1}{2}\rho N_x, \rho \in Z\}$ are symmetry centers. The symmetry polarity satisfies $J_y = J_{x_1}J_{x_2}$. These symmetric centers are located at the opposite ends of the symmetric axis (diameter) in Fig. 3. In the definition of circular-symmetry, the symmetry center on the first half of the circle is used to determine the symmetry type, hence the symmetry center c_y in (4) is chosen. Q.E.D.

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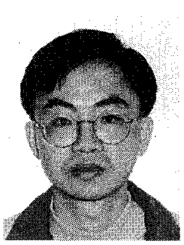

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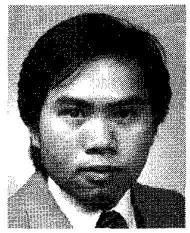

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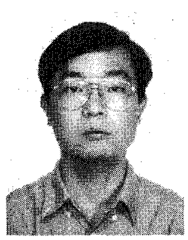

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