as $a = 1, m_r = 1, m_l = 1, b_r = 0.5, b_l = -0.5$. The objective is to control the system state x to follow a desired trajectory $y_r(t) = 4\sin(2t)$. First, we choose the dead-zone inverse $v(t) = \hat{\mathrm{DI}}(u_d(t))$ as in (11) and the filters

$$\dot{\xi}_0 = A_0\xi_0 + ky + \chi \quad \dot{\xi}_1 = A_0\xi_1 + Y_1 e_2$$
$$\dot{\eta} = A_0\eta + e_2 u \tag{58}$$

$$\hat{\omega}_2 = \frac{p + k_1}{p^2 + k_1 p + k_2} I_4[\hat{\omega}] \tag{59}$$

where $Y_1 = x^2 \quad k = [k_1, k_2]^T = [1, 3]^T$

$$A_0 = \begin{bmatrix} -k_1 & 1 \\ -1_2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 0 \end{bmatrix}. \tag{60}$$

Then we apply our control design to the plant. In the simulations, taking $c_1 = c_2 = 2, \Gamma_a = 0.1, \gamma_2 = 0.2, \Gamma_\theta = [0.1, 0.1, 0.1, 0.1]^T, e_0 = 1, \delta_1 = 0.02$ and the initial parameters $\hat{a}(0) = 1.5, \hat{D}(0) = 0.4, \hat{\theta}(0) = [1, 1, 0.4, -0.4]^T$. The initial state is chosen as $x(0) = 0.4$. The tracking error and the controller output $v(t)$ are shown in Figs. 4 and 5. Clearly, the simulation results verify our theoretical findings and show the effectiveness of our control scheme.

VI. Conclusion

This note presents an output feedback backstepping adaptive controller design scheme for a class of uncertain nonlinear single-input–single-output system preceded by uncertain dead-zone actuator nonlinearity. We propose a new smooth adaptive inverse to compensate the effect of the unknown dead-zone. Such an inverse can avoid possible chattering phenomenon which may be caused by nonsmooth inverse. The inverse function is employed in the backstepping controller design. For the design and implementation of the controller, no knowledge is assumed on the unknown system parameters. Besides showing stability, we also give an explicit bound on the L_2 performance of the tracking error in terms of design parameters. Simulation results illustrates the effectiveness of our schemes.

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H_∞ Control for Networked Systems With Random Communication Delays

Fuwen Yang, Zidong Wang, Y. S. Hung, and Mahbub Gani

***Abstract*—This note is concerned with a new controller design problem for networked systems with random communication delays. Two kinds of random delays are simultaneously considered: i) from the controller to the plant, and ii) from the sensor to the controller, via a limited bandwidth communication channel. The random delays are modeled as a linear function of the stochastic variable satisfying Bernoulli random binary distribution. The observer-based controller is designed to exponentially stabilize the networked system in the sense of mean square, and also achieve the prescribed H_∞ disturbance attenuation level. The addressed controller design problem is transformed to an auxiliary convex optimization problem, which can be solved by a linear matrix inequality (LMI) approach. An illustrative example is provided to show the applicability of the proposed method.**

***Index Terms*—H_∞ control, linear matrix inequalities (LMIs), networked systems, random communication delays, stochastic stability.**

I. Introduction

Recent advances in network technology have led to more and more control systems whose feedback control loop is based on a network. This kind of control systems are called networked control systems (NCSs) [7], [10], [23]. The network itself is a dynamic system and induces possible delays via network communication due to limited

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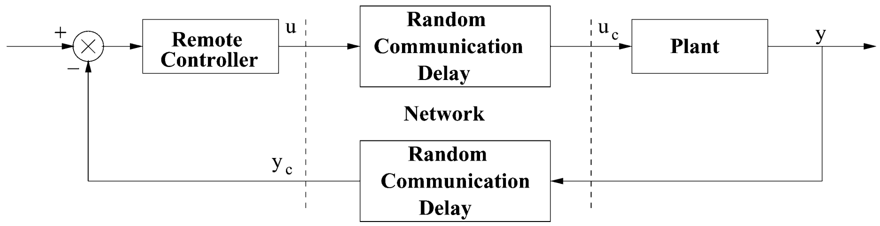

Fig. 1. Structure of a networked control system with communication delays.

bandwidth. A realistic networked control system design should take the communication delays into account, since the delays are widely known to degrade the performance of the control system [3], [4]. Therefore, in the past decade, the control problem of networked systems with time-delays has received increasing attention; see, e.g., [5], [9], [12], [13], [16], [18], and the references therein.

Since network delays are usually random and time-varying by nature, the existing control methods for *deterministic* time-delays cannot be directly used [21]. Recently, there have been significant research efforts on the control problems for networked systems with random delays, where the random network delays have been modeled in various ways in terms of the probability and characteristics of sources and destinations. For example, in [8], the delay value has been treated as an unknown variable but with known statistical properties, and the probabilistic delay averaging approach has been employed to determine the optimal control, independent of the delay value. In [13] the time delays have been assumed to be varying in a random fashion and have statistically mutually independent transfer-to-transfer probability distribution. In [16], the random communication delays have been considered as white in nature with known probability distributions. It should be mentioned that, among others, the *binary random* delay has gained considerable research interests because of its simplicity and practicality in describing network-induced delays, where the binary switching sequence is viewed as a Bernoulli distributed white sequence taking on values of 0 and 1; see, e.g., [11], [14], [19], [20], and [22].

Although networked control systems with random time-delays have been studied for a number of years, there are still some interesting problems that deserve further research. For example, most literature, including our previous works in [20], [21], has dealt with the sensor-to-controller delays only. Another important type of delays, controller-to-actuator delays, have not been fully investigated. Also, in the presence of random network delays, the H_∞ disturbance rejection performance has not received enough attention.

It is, therefore, the intention of this note to study the H_∞ control problem for a class of networked control systems with both sensor-to-controller delays and controller-to-actuator delays. These random delays are modeled as a linear function of the stochastic variable satisfying Bernoulli random binary distribution. An observer-based controller is designed such that the closed-loop networked control system is stochastically exponentially stable, and the prescribed H_∞ disturbance attenuation performance is achieved. An LMI approach is developed to tackle the addressed problem, which can be solved conveniently by Matlab LMI toolbox.

Notation: The notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x. $\mathrm{Prob}\{\cdot\}$ means the occurrence probability of the event "$\cdot$." If A is a matrix, $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the largest (respectively, smallest) eigenvalue of A. $l_2[0,\infty)$ is the space of square integrable vectors, and $\mathbb{I}^+$ is the set of positive integer. In symmetric block matrices, "$*$" is used as an ellipsis for terms induced by symmetry.

II. Problem Formulation and Preliminaries

Consider the networked control system with random communication delays shown in Fig. 1.

The plant is assumed to be of the form

$$\begin{cases} x_{k+1} = Ax_k + B_2 u_{c,k} + B_1 w_k \\ z_k = Dx_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_{c,k} \in \mathbb{R}^m$ is the control input, $z_k \in \mathbb{R}^r$ is the controlled output, $w_k \in \mathbb{R}^q$ is the disturbance input belonging to $l_2[0,\infty)$, A, B_1, B_2, and D are known real matrices with appropriate dimensions. The measurement with randomly varying communication delays is described by

$$\begin{cases} y_k = Cx_k \\ y_{c,k} = (1-\delta)y_k + \delta y_{k-1} \end{cases} \tag{2}$$

where the stochastic variable $\delta \in \mathbb{R}$ is a Bernoulli distributed white sequence with

$$\mathrm{Prob}\{\delta = 1\} = \mathbb{E}\{\delta\} := \bar{\delta} \tag{3}$$

$$\mathrm{Prob}\{\delta = 0\} = 1 - \mathbb{E}\{\delta\} := 1 - \bar{\delta} \tag{4}$$

and $y_{c,k} \in \mathbb{R}^p$ is the measured output, $y_k \in \mathbb{R}^p$ is the output, and C is a known real matrix with appropriate dimension.

In this note, we propose a dynamic observer-based control scheme for (1) described by

$$\text{Observer}: \begin{cases} \hat{x}_{k+1} = A\hat{x}_k + B_2 u_{c,k} + L(y_{c,k} - \bar{y}_{c,k}) \\ \bar{y}_{c,k} = (1-\bar{\delta})C\hat{x}_k + \bar{\delta}C\hat{x}_{k-1} \end{cases} \tag{5}$$

$$\text{Controller}: \begin{cases} u_k = K\hat{x}_k \\ u_{c,k} = (1-\beta)u_k + \beta u_{k-1} \end{cases} \tag{6}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the state estimate of the system (1), $\bar{y}_{c,k} \in \mathbb{R}^p$ is the observer output, $L \in \mathbb{R}^{n \times p}$ and $K \in \mathbb{R}^{m \times n}$ are the observer gain and controller gain, respectively. The stochastic variable $\beta \in \mathbb{R}$, mutually independent of δ, is also a Bernoulli distributed white sequence with expected value $\bar{\beta}$.

Remark 1: This measurement mode (2) was introduced in [14], and has been used in [20] and [22]. The output y_k produced at a time k is sent to the observer via a communication channel and arrives at the time $k + \tau_d$. If the sampling period is long compared with τ_d, there is no need to consider the influence of the delay, i.e., $y_{c,k} = y_k$. If τ_d is longer than one sampling period and shorter than two sampling periods, then the measurement $y_{c,k} = y_{k-1}$. It can be easily seen that, at kth sampling time, the actual system output takes the value y_{k-1} with probability $\bar{\delta}$, and the value y_k with probability $1 - \bar{\delta}$. Obviously, if the binary stochastic variable δ takes the value 1 consecutively at different sample times, long time delays would occur.

Remark 2: It is worth mentioning that the control scheme (6) introduces the random communication delay as well. The random delay mode is similar to that in the measurement but with a different switching probability, since the control and measurement are carried out through the same communication channel with limited bandwidth. With simultaneous presence of binary random delays in both the control and measurement structures, the observer design problem becomes much more involved. On the other hand, all the system matrices (A, B_1, B_2, C, D) are assumed to be known, which greatly simplifies our analysis. When there exist parameter uncertainties, the corresponding robust H_∞ control problem with random communication delays becomes more important, and the results will appear in the near future.

Defining the estimation error by

$$e_k := x_k - \hat{x}_k \tag{7}$$

we obtain the closed-loop system, as shown in (8) at the bottom of the page, by substituting (2), (5) and (6) into (1) and (7). We rewrite (8) in a compact form as follows:

$$\eta_{k+1} = (\bar{A} + \tilde{A})\eta_k + \bar{B}w_k \tag{9}$$

where $\bar{A}, \tilde{A}$, and $\bar{B}$ are defined at the bottom of the page.

Since the closed-loop system (9) contains both stochastic quantities β and δ, it is actually a stochastic parameter system, and we need to introduce the notion of stochastic stability in the mean-square sense for the problem formulation.

Definition 1: The closed-loop system is said to be *exponentially mean-square stable* if with $w_k = 0$, there exist constants $\alpha > 0$ and $\tau \in (0, 1)$ such that

$$\mathbb{E}\{\|\eta_k\|^2\} \le \alpha\tau^k \mathbb{E}\{\|\eta_0\|^2\} \quad \text{for all } \eta_0 \in \mathbb{R}^n, \quad k \in \mathbb{I}^+. \tag{10}$$

With this definition, our objective is to design the controller (6) for the system (1) such that, for all addressed random communication delays, the closed-loop system (9) is exponentially mean-square stable, and the H_∞ performance constraint is satisfied. In other words, we aim to design a controller such that the closed-loop system satisfies the following requirements Q1) and Q2) simultaneously.

Q1) The closed-loop system (9) is exponentially mean-square stable.

Q2) Under the zero-initial condition, the controlled output z satisfies

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|z_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|w_k\|^2\} \tag{11}$$

for all nonzero w_k, where $\gamma > 0$ is a prescribed scalar.

III. Stability Analysis

In this section, we will investigate the stability conditions for the closed-loop system (9). The following lemma will be needed in our derivation.

Lemma 1: Let $V(\eta_k)$ be a Lyapunov functional. If there exist real scalars $\lambda \ge 0, \mu > 0, \nu > 0$, and $0 < \psi < 1$ such that

$$\mu\|\eta_k\|^2 \le V(\eta_k) \le \nu\|\eta_k\|^2 \tag{12}$$

and

$$\mathbb{E}\{V(\eta_{k+1}) \,|\, \eta_k\} - V(\eta_k) \le \lambda - \psi V(\eta_k) \tag{13}$$

then the sequence η_k satisfies

$$\mathbb{E}\{\|\eta_k\|^2\} \le \frac{\nu}{\mu}\|\eta_0\|^2(1-\psi)^k + \frac{\lambda}{\mu\psi}. \tag{14}$$

Proof: The proof is similar to [17, Th. 2]. ■

The following theorem shows the closed-loop system (9) is exponentially stable in the mean-square sense if an LMI is feasible.

Theorem 1: Given the controller gain matrix K and the observer gain matrix L. The closed-loop system (9) is exponentially mean-square stable if there exist positive–definite matrices P_1, S_1, P_2, and S_2 satisfying (15), as shown at the bottom of the page, where $\alpha_1 = [(1-\bar{\beta})\bar{\beta}]^{1/2}, \alpha_2 = [(1-\bar{\delta})\bar{\delta}]^{1/2}$.

$$\begin{cases} x_{k+1} = [A + (1-\bar{\beta})B_2K]x_k - (1-\bar{\beta})B_2Ke_k + \bar{\beta}B_2Kx_{k-1} - \bar{\beta}B_2Ke_{k-1} - (\beta-\bar{\beta})B_2Kx_k \\ \qquad +(\beta-\bar{\beta})B_2Ke_k + (\beta-\bar{\beta})B_2Kx_{k-1} - (\beta-\bar{\beta})B_2Ke_{k-1} + B_1w_k \\ e_{k+1} = [A - (1-\bar{\delta})LC]e_k - \bar{\delta}LCe_{k-1} + (\delta-\bar{\delta})LCx_k - (\delta-\bar{\delta})LCx_{k-1} + B_1w_k \end{cases} \tag{8}$$

$$\eta_k = \begin{bmatrix} x_k \\ e_k \\ x_{k-1} \\ e_{k-1} \end{bmatrix} \quad \bar{A} = \begin{bmatrix} A + (1-\bar{\beta})B_2K & -(1-\bar{\beta})B_2K & \bar{\beta}B_2K & -\bar{\beta}B_2K \\ 0 & A - (1-\bar{\delta})LC & 0 & -\bar{\delta}LC \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} -(\beta-\bar{\beta})B_2K & (\beta-\bar{\beta})B_2K & (\beta-\bar{\beta})B_2K & -(\beta-\bar{\beta})B_2K \\ (\delta-\bar{\delta})LC & 0 & -(\delta-\bar{\delta})LC & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} B_1 \\ B_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P_2 - P_1 & * & * & * & * & * & * & * \\ 0 & S_2 - S_1 & * & * & * & * & * & * \\ 0 & 0 & -P_2 & * & * & * & * & * \\ 0 & 0 & 0 & -S_2 & * & * & * & * \\ A + (1-\bar{\beta})B_2K & -(1-\bar{\beta})B_2K & \bar{\beta}B_2K & -\bar{\beta}B_2K & -P_1^{-1} & * & * & * \\ 0 & A - (1-\bar{\delta})LC & 0 & -\bar{\delta}LC & 0 & -S_1^{-1} & * & * \\ B_2K & -B_2K & -B_2K & B_2K & 0 & 0 & -\alpha_1^{-2}P_1^{-1} & * \\ LC & 0 & -LC & 0 & 0 & 0 & 0 & -\alpha_2^{-2}S_1^{-1} \end{bmatrix} < 0 \tag{15}$$

Proof: Define a Lyapunov functional

$$V_k = x_k^T P_1 x_k + x_{k-1}^T P_2 x_{k-1} + e_k^T S_1 e_k + e_{k-1}^T S_2 e_{k-1} \tag{16}$$

where P_1, P_2, S_1, and S_2 are positive–definite matrices. It follows from (8) that

$$\begin{aligned}
&\mathbb{E}\{V_{k+1}|x_k,\dots,x_0,e_k,\dots,e_0\} - V_k \\
&\quad = \mathbb{E}\left\{x_{k+1}^T P_1 x_{k+1} + e_{k+1}^T S_1 e_{k+1}\right\} \\
&\qquad + x_k^T P_2 x_k + e_k^T S_2 e_k - x_k^T P_1 x_k \\
&\qquad - x_{k-1}^T P_2 x_{k-1} - e_k^T S_1 e_k - e_{k-1}^T S_2 e_{k-1} \\
&\quad = \{[A+(1-\bar\beta)B_2K]x_k - (1-\bar\beta)B_2Ke_k \\
&\qquad + \bar\beta B_2Kx_{k-1} - \bar\beta B_2Ke_{k-1}\}^T P_1\{[A+(1-\bar\beta)B_2K]x_k \\
&\qquad - (1-\bar\beta)B_2Ke_k + \bar\beta B_2Kx_{k-1} - \bar\beta B_2Ke_{k-1}\} \\
&\qquad + \{[A-(1-\bar\delta)LC]e_k - \bar\delta LCe_{k-1}\}^T S_1 \\
&\qquad \cdot \{[A-(1-\bar\delta)LC]e_k - \bar\delta LCe_{k-1}\} \\
&\qquad + \mathbb{E}\{(\beta-\bar\beta)^2\}[B_2Kx_k \\
&\qquad - B_2Ke_k - B_2Kx_{k-1} + B_2Ke_{k-1}]^T P_1 \\
&\qquad \cdot [B_2Kx_k - B_2Ke_k - B_2Kx_{k-1} + B_2Ke_{k-1}] \\
&\qquad + \mathbb{E}\{(\delta-\bar\delta)^2\}[LCx_k - LCx_{k-1}]^T \\
&\qquad \cdot S_1[LCx_k - LCx_{k-1}] + x_k^T P_2 x_k + e_k^T S_2 e_k \\
&\qquad - x_k^T P_1 x_k - x_{k-1}^T P_2 x_{k-1} - e_k^T S_1 e_k - e_{k-1}^T S_2 e_{k-1}.
\end{aligned} \tag{17}$$

Noting that $\mathbb{E}\{(\beta-\bar\beta)^2\} = (1-\bar\beta)\bar\beta$ and $\mathbb{E}\{(\delta-\bar\delta)^2\} = (1-\bar\delta)\bar\delta$, we have

$$\begin{aligned}
&\mathbb{E}\{V_{k+1}|x_k,\dots,x_0,e_k,\dots,e_0\} - V_k \\
&\quad = \{[A+(1-\bar\beta)B_2K]x_k - (1-\bar\beta)B_2Ke_k \\
&\qquad + \bar\beta B_2Kx_{k-1} - \bar\beta B_2Ke_{k-1}\}^T \\
&\qquad \times P_1\{[A+(1-\bar\beta)B_2K]x_k \\
&\qquad - (1-\bar\beta)B_2Ke_k + \bar\beta B_2Kx_{k-1} - \bar\beta B_2Ke_{k-1}\} \\
&\qquad + \{[A-(1-\bar\delta)LC]e_k - \bar\delta LCe_{k-1}\}^T S_1 \\
&\qquad \cdot \{[A-(1-\bar\delta)LC]e_k - \bar\delta LCe_{k-1}\} \\
&\qquad + (1-\bar\beta)\bar\beta[B_2Kx_k - B_2Ke_k - B_2Kx_{k-1} \\
&\qquad + B_2Ke_{k-1}]^T P_1 \\
&\qquad \cdot [B_2Kx_k - B_2Ke_k - B_2Kx_{k-1} + B_2Ke_{k-1}] \\
&\qquad + (1-\bar\delta)\bar\delta[LCx_k - LCx_{k-1}]^T S_1[LCx_k - LCx_{k-1}] \\
&\qquad + x_k^T P_2 x_k + e_k^T S_2 e_k - x_k^T P_1 x_k \\
&\qquad - x_{k-1}^T P_2 x_{k-1} - e_k^T S_1 e_k - e_{k-1}^T S_2 e_{k-1} \\
&\quad = \eta_k^T \Lambda \eta_k
\end{aligned} \tag{18}$$

where (19), as shown at the bottom of the page, holds. By Schur complement, (15) implies that $\Lambda < 0$. We know from (15) that

$$\begin{aligned}
&\mathbb{E}\{V_{k+1}|x_k,\dots,x_0,e_k,\dots,e_0\} - V_k \\
&\qquad = \eta_k^T\Lambda\eta_k \le -\lambda_{\min}(-\Lambda)\eta_k^T\eta_k < -\alpha\eta_k^T\eta_k
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
&0 < \alpha < \min\{\lambda_{\min}(-\Lambda), \sigma\} \\
&\qquad \sigma := \max\{\lambda_{\max}(P_1), \lambda_{\max}(S_1), \lambda_{\max}(P_2), \lambda_{\max}(S_2)\}.
\end{aligned} \tag{21}$$

From (20), we have

$$\begin{aligned}
&\mathbb{E}\{V_{k+1}|x_k,\dots,x_0,e_k,\dots,e_0\} \\
&\qquad - V_k < -\alpha\eta_k^T\eta_k < -\frac{\alpha}{\sigma}V_k := -\psi V_k.
\end{aligned} \tag{22}$$

Therefore, by Definition 1, it can be verified from Lemma 1 that the closed-loop system (9) is exponentially mean-square stable. This completes the proof. ■

IV. H_∞ Controller Design

Different from the standard H_∞ performance formulation, we shall use the expression (11) to describe the H_∞ performance of the stochastic system, where the expectation operator is utilized on both the controlled output and the disturbance input, see [1] for more details.

Theorem 2: Given a scalar $\gamma > 0$. The system (9) is exponentially mean-square stable and the H_∞-norm constraint (11) is achieved for all nonzero w_k, if there exist positive definite matrices P_1, S_1, P_2 and S_2, and real matrices K and L satisfying (23), as shown at the bottom of the next page.

Proof: It is obvious that (23) implies (15), hence it follows from Theorem 1 that the system (9) is exponentially mean-square stable.

Next, for any nonzero w_k, it follows from (8) and (18) that

$$\begin{aligned}
&\mathbb{E}\{V_{k+1}\} - \mathbb{E}\{V_k\} + \mathbb{E}\left\{z_k^T z_k\right\} - \gamma^2\mathbb{E}\left\{w_k^T w_k\right\} \\
&\quad = \mathbb{E}\left\{\begin{bmatrix}\eta_k \\ w_k\end{bmatrix}^T\right. \\
&\qquad \times \begin{bmatrix} \Lambda + \tilde D^T\tilde D & B_3^T P_1 B_1 + B_4^T S_1 B_1 \\ B_1^T P_1 B_3 + B_1^T S_1 B_4 & B_1^T(P_1+S_1)B_1 - \gamma^2 I\end{bmatrix} \\
&\qquad \left. \times \begin{bmatrix}\eta_k \\ w_k\end{bmatrix}\right\}
\end{aligned} \tag{24}$$

where

$$\tilde D = [D \quad 0 \quad 0 \quad 0] \tag{25}$$

$$\begin{aligned} B_3 = [&A+(1-\bar\beta)B_2K \\ &-(1-\bar\beta)B_2K \quad \bar\beta B_2K \quad -\bar\beta B_2K] \end{aligned} \tag{26}$$

$$B_4 = [0 \quad A-(1-\bar\delta)LC \quad 0 \quad -\bar\delta LC]. \tag{27}$$

$$\begin{aligned}
\Lambda = &\begin{bmatrix} A+(1-\bar\beta)B_2K & -(1-\bar\beta)B_2K & \bar\beta B_2K & -\bar\beta B_2K \\ 0 & A-(1-\bar\delta)LC & 0 & -\bar\delta LC \\ B_2K & -B_2K & -B_2K & B_2K \\ LC & 0 & -LC & 0 \end{bmatrix}^T \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & S_1 & 0 & 0 \\ 0 & 0 & (1-\bar\beta)\bar\beta P_1 & 0 \\ 0 & 0 & 0 & (1-\bar\delta)\bar\delta S_1 \end{bmatrix} \\
&\cdot \begin{bmatrix} A+(1-\bar\beta)B_2K & -(1-\bar\beta)B_2K & \bar\beta B_2K & -\bar\beta B_2K \\ 0 & A-(1-\bar\delta)LC & 0 & -\bar\delta LC \\ B_2K & -B_2K & -B_2K & B_2K \\ LC & 0 & -LC & 0 \end{bmatrix} + \begin{bmatrix} P_2-P_1 & 0 & 0 & 0 \\ 0 & S_2-S_1 & 0 & 0 \\ 0 & 0 & -P_2 & 0 \\ 0 & 0 & 0 & -S_2 \end{bmatrix}.
\end{aligned} \tag{19}$$

By Schur complement, (23) implies that

$$\begin{bmatrix} \Lambda + \tilde{D}^T \tilde{D} & B_3^T P_1 B_1 + B_4^T S_1 B_1 \\ B_1^T P_1 B_3 + B_1^T S_1 B_4 & B_1^T (P_1 + S_1) B_1 - \gamma^2 I \end{bmatrix} < 0. \tag{28}$$

Thus, we have

$$\mathbb{E}\{V_{k+1}\} - \mathbb{E}\{V_k\} + \mathbb{E}\left\{z_k^T z_k\right\} - \gamma^2 \mathbb{E}\left\{w_k^T w_k\right\} < 0. \tag{29}$$

Now, summing up (29) from 0 to ∞ with respect to k yields

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|z_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|w_k\|^2\} + \mathbb{E}\{V_0\} - \mathbb{E}\{V_\infty\}. \tag{30}$$

Since $\eta_0 = 0$ and the system (9) is exponentially mean-square stable, it is straightforward to see that

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|z_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|w_k\|^2\}. \tag{31}$$

This ends the proof. ■

Note that at this stage, the condition (23) is not an LMI, hence, cannot be solved by Matlab LMI Toolbox. In the following, we first convert the condition (23) into an LMI with matrix equation constraint, and then formulate a convex problem.

Without loss of generality, we make the following assumption.

A1: The matrix B_2 is of full-column rank, i.e., $\mathrm{rank}(B_2) = m$.

The following theorem presents how to convert the condition (23) into an LMI with matrix equality constraint.

Theorem 3: Given a scalar $\gamma > 0$. System (9) is exponentially mean-square stable and the H_∞-norm constraint (11) is achieved for all nonzero w_k, if there exist positive–definite matrices $P_1 \in \mathbb{R}^{n\times n}, S_1 \in \mathbb{R}^{n\times n}, P_2 \in \mathbb{R}^{n\times n}$, and $S_2 \in \mathbb{R}^{n\times n}$, and real matrices $M \in \mathbb{R}^{m\times n}, N \in \mathbb{R}^{n\times p}, P \in \mathbb{R}^{m\times m}$ such that (32) and (33), as shown at the bottom of the page, hold. Moreover, the controller parameters are given by

$$K = P^{-1} M \quad L = S_1^{-1} N. \tag{34}$$

Proof: By Schur complement, the condition (23) is equivalent to (35), as shown at the bottom of the page, where $H_{11} = P_1 A + (1 - \bar{\beta}) P_1 B_2 K$, $H_{22} = S_1 A - (1 - \bar{\delta}) S_1 L C$.

$$\begin{bmatrix}
P_2 - P_1 & * & * & * & * & * & * & * & * & * \\
0 & S_2 - S_1 & * & * & * & * & * & * & * & * \\
0 & 0 & -P_2 & * & * & * & * & * & * & * \\
0 & 0 & 0 & -S_2 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & -\gamma^2 I & * & * & * & * & * \\
A + (1-\bar{\beta})B_2 K & -(1-\bar{\beta})B_2 K & \bar{\beta} B_2 K & -\bar{\beta} B_2 K & B_1 & -P_1^{-1} & * & * & * & * \\
0 & A - (1-\bar{\delta})LC & 0 & -\bar{\delta} LC & B_1 & 0 & -S_1^{-1} & * & * & * \\
B_2 K & -B_2 K & -B_2 K & B_2 K & 0 & 0 & 0 & -\alpha_1^{-2} P_1^{-1} & * & * \\
LC & 0 & -LC & 0 & 0 & 0 & 0 & 0 & -\alpha_2^{-2} S_1^{-1} & * \\
D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0 \tag{23}$$

$$\begin{bmatrix}
P_2 - P_1 & * & * & * & * & * & * & * & * & * \\
0 & S_2 - S_1 & * & * & * & * & * & * & * & * \\
0 & 0 & -P_2 & * & * & * & * & * & * & * \\
0 & 0 & 0 & -S_2 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & -\gamma^2 I & * & * & * & * & * \\
P_1 A + (1-\bar{\beta})B_2 M & -(1-\bar{\beta})B_2 M & \bar{\beta} B_2 M & -\bar{\beta} B_2 M & P_1 B_1 & -P_1 & * & * & * & * \\
0 & S_1 A - (1-\bar{\delta})NC & 0 & -\bar{\delta} NC & S_1 B_1 & 0 & -S_1 & * & * & * \\
B_2 M & -B_2 M & -B_2 M & B_2 M & 0 & 0 & 0 & -\alpha_1^{-2} P_1 & * & * \\
NC & 0 & -NC & 0 & 0 & 0 & 0 & 0 & -\alpha_2^{-2} S_1 & * \\
D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0 \tag{32}$$

$$P_1 B_2 = B_2 P \tag{33}$$

$$\begin{bmatrix}
P_2 - P_1 & * & * & * & * & * & * & * & * & * \\
0 & S_2 - S_1 & * & * & * & * & * & * & * & * \\
0 & 0 & -P_2 & * & * & * & * & * & * & * \\
0 & 0 & 0 & -S_2 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & -\gamma^2 I & * & * & * & * & * \\
H_{11} & -(1-\bar{\beta}) P_1 B_2 K & \bar{\beta} P_1 B_2 K & -\bar{\beta} P_1 B_2 K & P_1 B_1 & -P_1 & * & * & * & * \\
0 & H_{22} & 0 & -\bar{\delta} S_1 LC & S_1 B_1 & 0 & -S_1 & * & * & * \\
\alpha_1 P_1 B_2 K & -\alpha_1 P_1 B_2 K & -\alpha_1 P_1 B_2 K & \alpha_1 P_1 B_2 K & 0 & 0 & 0 & -P_1 & * & * \\
\alpha_2 S_1 LC & 0 & -\alpha_2 S_1 LC & 0 & 0 & 0 & 0 & 0 & -S_1 & * \\
D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0 \tag{35}$$

Letting

$$B_2 P = P_1 B_2 \quad M = PK \quad N = S_1 L \tag{36}$$

we can conclude that (35) is equivalent to (32).

On the other hand, since the matrix B_2 is full column rank, the columns of matrices B_2 and $P_1 B_2$ are all linearly independent with $P_1 > 0$. Hence, if (32) is satisfied, we have

$$\operatorname{rank}(P) \geq \operatorname{rank}(B_2 P) = \operatorname{rank}(P_1 B_2) \geq \operatorname{rank}(B_2) = m \tag{37}$$

which implies that the matrix P must be nonsingular. Therefore, (34) is obtained from (36), and the proof is complete. ■

We are now in a position to discuss how to solve the equality constraint in (33). For the matrix B_2 of full-column rank, there always exist two orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ such that

$$\tilde{B}_2 := U B_2 V = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} B_2 V = \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \tag{38}$$

where $U_1 \in \mathbb{R}^{m\times n}$ and $U_2 \in \mathbb{R}^{(n-m)\times n}$, and $\Sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, where σ_i $(i = 1, 2, \ldots, m)$ are nonzero singular values of B_2.

Lemma 2: Let the matrix $B_2 \in \mathbb{R}^{n\times m}$ be of full-column rank. If matrix P_1 has the following structure:

$$P_1 = U^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{12} \end{bmatrix} U = U_1^T P_{11} U_1 + U_2^T P_{22} U_2 \tag{39}$$

where $P_{11} \in \mathbb{R}^{m\times m} > 0$ and $P_{22} \in \mathbb{R}^{(n-m)\times(n-m)} > 0$, and U_1 and U_2 are defined in (38), then there exists a nonsingular matrix $P \in \mathbb{R}^{m\times m}$ such that $B_2 P = P_1 B_2$.

Proof: The proof is similar to that of [6, Lemma 3], and is thus omitted. ■

Theorem 4: Given a scalar $\gamma > 0$. The system (9) is exponentially mean-square stable and the H_∞-norm constraint (11) is achieved for all nonzero w_k if there exist positive–definite matrices $P_{11} \in \mathbb{R}^{m\times m}, P_{22} \in \mathbb{R}^{(n-m)\times(n-m)}, S_1 \in \mathbb{R}^{n\times n}, P_2 \in \mathbb{R}^{n\times n}$ and $S_2 \in \mathbb{R}^{n\times n}$, and real matrices $M \in \mathbb{R}^{m\times n}, N \in \mathbb{R}^{n\times p}$ such that (40), as shown at the bottom of the page, holds, where $P_1 := U_1^T P_{11} U_1 + U_2^T P_{22} U_2$, and U_1 and U_2 come from (38). Moreover, the controller parameters are given by

$$K = V\Sigma^{-1} P_{11}^{-1} \Sigma V^T M \quad L = S_1^{-1} N. \tag{41}$$

Proof: Since there exist $P_{11} > 0$ and $P_{22} > 0$ such that $P_1 = U_1^T P_{11} U_1 + U_2^T P_{22} U_2$, where U_1 and U_2 are defined in (38), we compute P as follows from $B_2 P = P_1 B_2$

$$P_1 U^T \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T = U^T \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T P \tag{42}$$

i.e.,

$$U^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{12} \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T = U^T \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T P \tag{43}$$

which implies that

$$P^{-1} = V\Sigma^{-1} P_{11}^{-1} \Sigma V^T. \tag{44}$$

Thus, (41) is obtained from (34) and (44), and the rest of the proof follows from Theorem 3. ■

So far the controller has been designed which satisfies the requirements Q1) and Q2). Due to the advantages of LMI formulations, the results in Theorem 4 also suggest the following optimization problem that would be interesting to control engineers.

P1) The optimal H_∞ control problem

$$\min_{P_{11}>0, P_{22}>0, P_2>0, S_1>0, S_2>0, M, N} \gamma \text{ subject to (40).} \tag{45}$$

V. Simulation Example

In this section, we aim to demonstrate the effectiveness and applicability of the proposed method. For this purpose, we study the networked control problem for an uninterruptible power system (UPS). Our objective is to control the PWM inverter, through networks, such that the output AC voltage is kept at the desired setting and undistorted, while maintaining robustness against the disturbances in the load. We consider the UPS with 1 KVA. The discrete-time model (1) can be obtained with sampling time 10 ms at half-load operating point as follows [15]:

$$\begin{aligned} A &= \begin{bmatrix} 0.9226 & -0.6330 & 0 \\ 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0.5 \\ 0 \\ 0.2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ D &= [0.1 \quad 0 \quad 0] \end{aligned} \tag{46}$$

$$C = [23.738 \quad 20.287 \quad 0] \quad \bar{\delta} = 0.1. \tag{47}$$

Case 1) In this case, we want to design an H_∞ controller (6) with random communication delay $\bar{\beta} = 0.1$ and disturbance attenuation level $\gamma = 1$. This case is exactly concerned with the addressed H_∞ control problem in Theorem 4. We provide one solution by solving LMI (40) and obtain the observer gain and controller gain as follows:

$$\begin{aligned} K &= [-0.5784 \quad 0.4843 \quad -0.2121] \\ L &= [0.0032 \quad 0.0078 \quad 0.0059]^T. \end{aligned}$$

When we choose the initial conditions as $x_0 = [1 \quad 0 \quad 0]^T, \hat{x}_0 = [0 \quad 0 \quad 0]^T$ and the disturbance input as nonlinear load $w_k = 1/k^2$, the simulation results of the state responses are given in Fig. 2.

$$\begin{bmatrix}
P_2 - P_1 & * & * & * & * & * & * & * & * & * \\
0 & S_2 - S_1 & * & * & * & * & * & * & * & * \\
0 & 0 & -P_2 & * & * & * & * & * & * & * \\
0 & 0 & 0 & -S_2 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & -\gamma^2 I & * & * & * & * & * \\
P_1 A + (1-\bar{\beta}) B_2 M & -(1-\bar{\beta}) B_2 M & \bar{\beta} B_2 M & -\bar{\beta} B_2 M & P_1 B_1 & -P_1 & * & * & * & * \\
0 & S_1 A - (1-\bar{\delta}) NC & 0 & -\bar{\delta} NC & S_1 B_1 & 0 & -S_1 & * & * & * \\
\alpha_1 B_2 M & -\alpha_1 B_2 M & -\alpha_1 B_2 M & \alpha_1 B_2 M & 0 & 0 & 0 & -P_1 & * & * \\
\alpha_2 NC & 0 & -\alpha_2 NC & 0 & 0 & 0 & 0 & 0 & -S_1 & * \\
D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0 \tag{40}$$

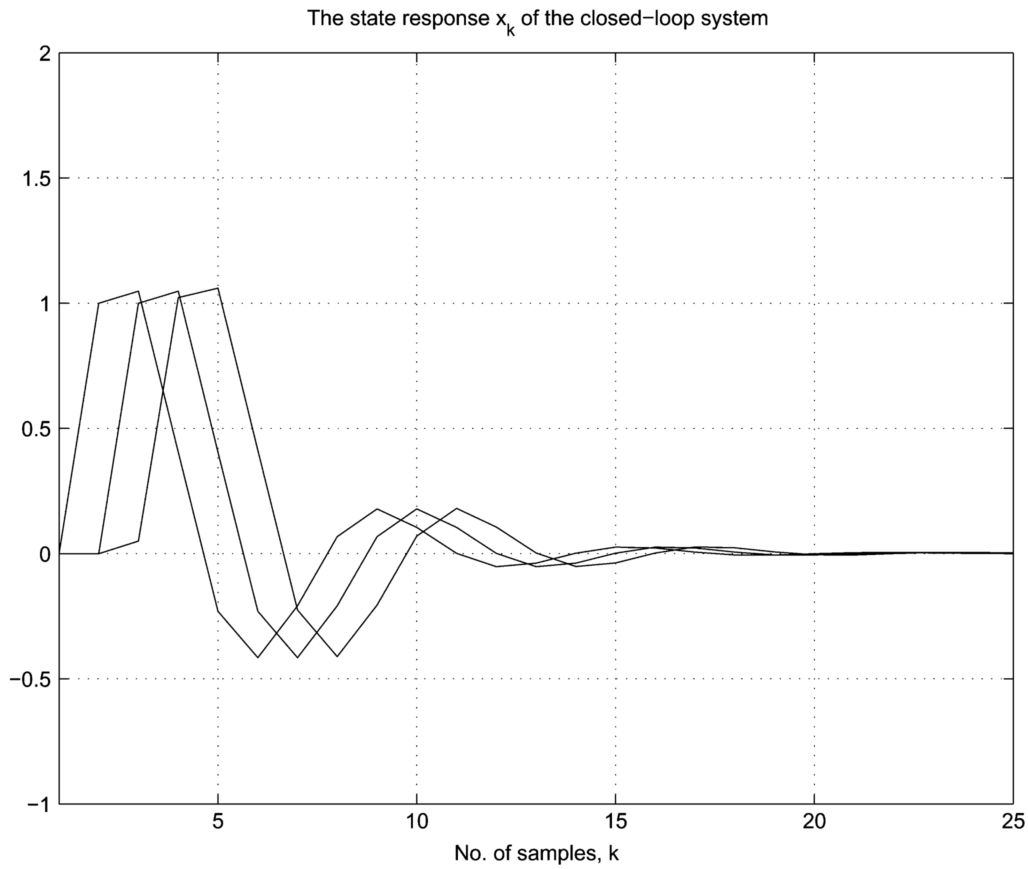

Fig. 2. H_∞ control with $\gamma = 1$.

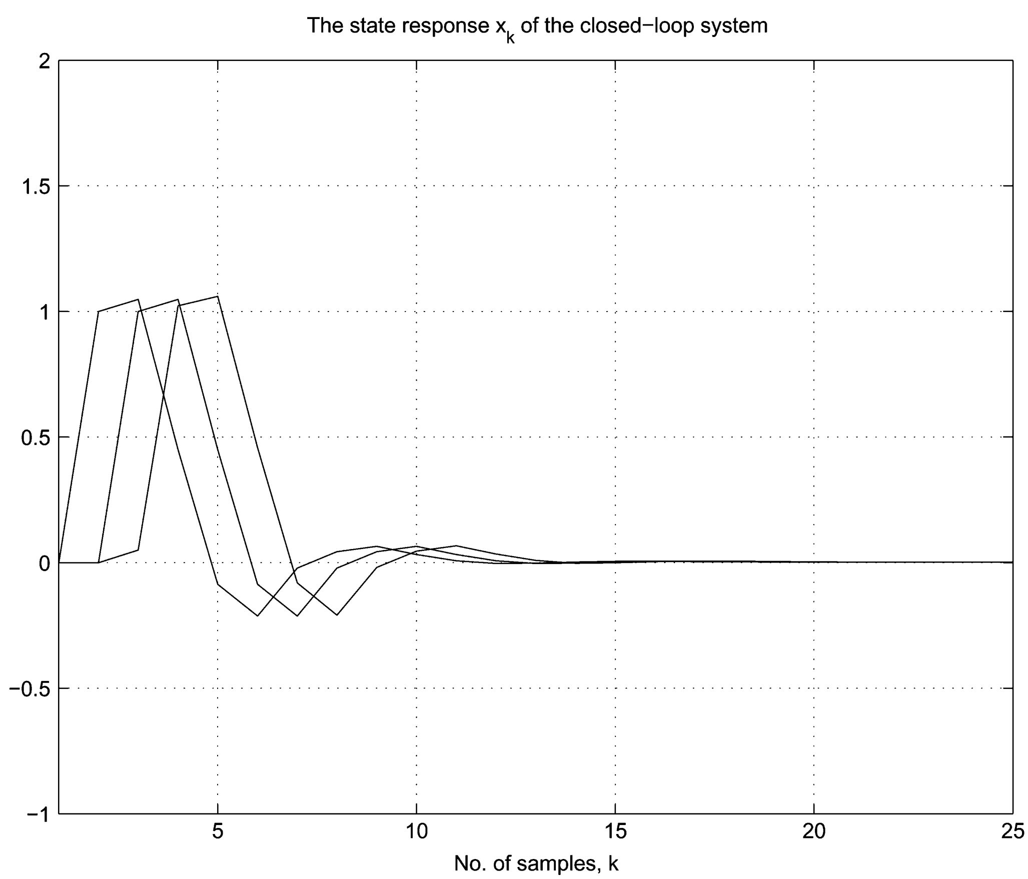

Fig. 3. H_∞ control with $\gamma_{\min} = 0.8088$.

Case 2) We now wish to design the controller (6) with random communication delay $\bar{\beta} = 0.1$ such that the H_∞ performance is minimized, i.e., we want to solve the problem P1). Solving the optimization problem (45) using LMI Toolbox yields the minimum value $\gamma_{\min} = 0.8088$ and

$$K = [-0.5960 \quad 0.5549 \quad -0.1587]$$
$$L = [0.0069 \quad 0.0147 \quad 0.0096]^T.$$

Similar to the first case, the simulation results of the state responses are given in Fig. 3.

VI. Conclusion

In this note, a novel control problem has been considered for networked systems with random communication delays. The H_∞ observer-based controller has been designed to achieve a desired H_∞ disturbance rejection level. The controller has been obtained by solving an LMI. Simulation results have demonstrated the feasibility of our control scheme. One of our future research topics would be the design of controllers for networked systems with long random delays.

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Stability of Quaternionic Linear Systems

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Abstract—**The main goal of this paper is to characterize stability and bounded-input–bounded-output (BIBO)-stability of quaternionic dynamical systems. After defining the quaternion skew-field, algebraic properties of quaternionic polynomials such as divisibility and coprimeness are investigated. Having established these results, the Smith and the Smith–McMillan forms of quaternionic matrices are introduced and studied. Finally, all the tools that were developed are used to analyze stability of quaternionic linear systems in a behavioral framework.**

Index Terms—**Behaviors, quaternions, stability.**

I. Introduction

This paper deals with stability, which is a very common issue in many areas of applied mathematics. In particular, for input/output dynamical control systems, it will focus on bounded-input– bounded-output (BIBO) stability which is especially important for control systems in the presence of disturbances: roughly speaking, it ensures that small perturbations in the control do not cause diverging errors in the output.

The systems which are here considered take values in the quaternion skew-field $\mathbb{H}$, that was discovered by Sir Rowan Hamilton in 1843. These *hypercomplex* numbers may be favorably used to describe phenomena occurring in areas such as electromagnetism and quantum physics [1] by means of a compact notation that leads to a higher efficiency in computational terms [2].

In particular, they are a powerful tool in the description of rotations. Indeed, by identifying $\mathbb{R}^3$ with a subset of $\mathbb{H}$, the expression qvq^{-1} represents the rotation of a vector $v \in \mathbb{R}^3$ by an angle and about a direction that are specified by $q \in \mathbb{H}$ (see, e.g., [3]). It is not uncommon

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