PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 4, Pages 1035–1042
S 0002-9939(01)06383-3
Article electronically published on October 1, 2001

RIGIDITY OF PROPER HOLOMORPHIC MAPPINGS BETWEEN EQUIDIMENSIONAL BOUNDED SYMMETRIC DOMAINS

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(Communicated by Steven R. Bell)

Abstract. We prove that any proper holomorphic mapping between two equidimensional irreducible bounded symmetric domains with rank ≥ 2 is a biholomorphism. The proof of the main result in this paper will be achieved by a differential-geometric study of a special class of complex geodesic curves on the bounded symmetric domains with respect to their Bergman metrics.

1. Introduction

In 1977, Alexander [1] proved the following fundamental result.

Theorem 1.A (Alexander [1]). *Any proper holomorphic mapping of the unit ball in $\mathbb{C}^n$ $(n \geq 2)$ into itself is an automorphism.*

Alexander's theorem has been generalized to several classes of domains. Especially, there are many important results concerning proper holomorphic mappings between bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary, and these researches are often heavily based on analytic techniques (e.g., see survey articles of Bell and Narasimhan [5] and Forstnerič [7]). In this topic, respectively, Bedford and Bell [2] and Diederich and Fornæss [6] proved the following results.

Theorem 1.B (Bedford and Bell [2]). *Let D be a bounded weakly pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 2)$ with smooth real-analytic boundary. Then any proper holomorphic self-mapping of D is an automorphism.*

Theorem 1.C (Diederich and Fornæss [6]). *If $\Omega, D \subset \mathbb{C}^n$ $(n \geq 2)$ are smoothly bounded pseudoconvex domains and Ω is strongly pseudoconvex, then any proper holomorphic mapping of Ω onto D is a local biholomorphism. Thus, if D is simply connected, then the mapping is biholomorphic.*

Even though the bounded homogeneous domains in $\mathbb{C}^n$ are always pseudoconvex, there are, of course, many such domains (e.g., all bounded symmetric domains of rank ≥ 2) without smooth boundary. The lack of boundary regularity usually presents a serious analytical difficulty. In 1984, by using results of Bell [3] and

Received by the editors September 29, 2000.
2000 *Mathematics Subject Classification.* Primary 32H02, 32M15.
Key words and phrases. Bounded symmetric domains, Hermitian symmetric manifolds, proper holomorphic mappings, rigidity, totally geodesic submanifolds.

Tumanov-Henkin [20], Henkin and Novikov [9] proved the following result (see Theorem 3.3 in Forstnerič [7] for references).

Theorem 1.D (Henkin and Novikov [9])**.** *Any proper holomorphic self-mapping on an irreducible bounded symmetric domain of rank* ≥ 2 *is an automorphism.*

The subject of Hermitian symmetric manifolds plays a central role in the domain of complex differential geometry. Of particular importance is the phenomenon of rigidity of complex structures of such manifolds (e.g., see Siu [15, 16]). In 1987, Mok [10, 11] obtained metric rigidity theorems on Hermitian locally symmetric manifolds, and their proofs are applied to the study of holomorphic mappings between Hermitian locally symmetric manifolds of the same type, yielding various rigidity theorems on holomorphic mappings (see Mok [12]). Inspired by the idea in Mok's work [10, 11], Tsai [17, 18] obtained some interesting results on this topic. Following this line, we shall prove the following.

Theorem 1.1. *Let* Ω_1 *and* Ω_2 *be two equidimensional bounded symmetric domains. Assume that* Ω_1 *is irreducible and rank*$(\Omega_1) \geq 2$*. Then, any proper holomorphic mapping* f *from* Ω_1 *to* Ω_2 *is a biholomorphism.*

In Theorem 1.1, the assumption "Ω_1 is irreducible" cannot be removed. For example, $f(z_1, ..., z_n) = (z_1^2, ..., z_n^2)$ is a proper holomorphic mapping from the polydisc $D^n \subset \mathbb{C}^n$ into itself but f is not a biholomorphism. If the assumption "rank$(\Omega_1) \geq 2$" in Theorem 1.1 is removed, then, by Bell [4] and Theorem 1.A, Theorem 1.1 holds when $\dim(\Omega_1) \geq 2$.

We shall now present an outline of the argument in our proof of Theorem 1.1. Let D be a bounded symmetric domain with Bergman metric. The minimal discs on D are a special class of complex geodesic curves on D such that their unit holomorphic tangent vectors realize the minimum of the holomorphic sectional curvature of D. The proof of Theorem 1.1 will be achieved by a differential-geometric study of the minimal discs on Ω_1 and Ω_2. In order to prove that $f\colon\ \Omega_1 \to \Omega_2$ is a biholomorphism, the key step in our proof is to verify that f maps all minimal discs on Ω_1 to minimal discs on Ω_2. We finish this step by using Bell [4] (to conclude that f is algebraic) and exploiting the characteristic symmetric submanifolds and holomorphic boundary components of two equidimensional bounded symmetric domains. Then, an elementary proof of Theorem 1.1 can be given.

2. Background and preliminaries

In this section we collect some basic facts about Hermitian symmetric manifolds, including some results of Mok and Tsai [13], and refer to Mok [12] and Wolf [21] for further details.

Let X_o be an irreducible Hermitian symmetric space of non-compact type. Fix a point $o \in X_o$. Write $X_o = G_o/K$ as usual, where G_o is the identity component of the isometry group of X_o and K is the isotropy subgroup at o. The Lie algebra $\mathfrak{g}_o$ of G_o has a Cartan decomposition $\mathfrak{g}_o = \mathfrak{l} + \mathfrak{m}_o$ with respect to the symmetry at o. Complexifying $\mathfrak{m}_o$ gives $\mathfrak{m} = \mathrm{m}_o^{\mathbb{C}} \cong \mathfrak{m}^+ \oplus \mathfrak{m}^-$ so that $\mathfrak{m}^+$ is identified with $T_o(X_o)$, the holomorphic tangent space at o. Denote by G and $\mathfrak{g}$ the complexified Lie group and Lie algebra of G_o and $\mathfrak{g}_o$ respectively. Write $X_c = G/P$ for the compact dual of X_o as usual, where P is the parabolic subgroup of G corresponding to the parabolic subalgebra $\mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{m}^-$. The map $gK \to gP$ from G_o/K to G/P embeds X_o holomorphically as an open subset of X_c. It is called the Borel embedding. Let

exp denote the exponential map from $\mathfrak{m}^+$ to G. The Harish-Chandra embedding theorem tells us that the map $\tau : \mathfrak{m}^+ \to X_c$ given by $\tau(m) = \exp(m)P$ is a complex analytic diffeomorphism of $\mathfrak{m}^+$ onto a dense open subset of X_c that contains X_o and $\tau^{-1}(X_o)$ is a bounded symmetric domain in $\mathfrak{m}^+$. Here and henceforth we identify $\mathfrak{m}^+$ with $\mathbb{C}^N$ and the Euclidean coordinates on $\mathbb{C}^N$ are called Harish-Chandra coordinates.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{l}$. Then $\mathfrak{h}^{\mathbb{C}}$ is also a Cartan subalgebra of $\mathfrak{g}$. Denote by Δ the $\mathfrak{h}^{\mathbb{C}}$-root system of $\mathfrak{g}$. Two roots are called strongly orthogonal if and only if $\phi \pm \varphi \notin \Delta$. We construct a maximal strongly orthogonal set $\{\phi_1, ..., \phi_r\}$ ($r =$ rank of X_o), denoted by Π, of noncompact positive roots according to Harish-Chandra. Let Λ be a proper subset of Π. Denote by $\mathfrak{g}_\Lambda$ the derived algebra of $\mathfrak{h}^{\mathbb{C}} + \sum_{\phi \perp \Pi \setminus \Lambda} \mathfrak{g}^\phi$. Let G_Λ be the Lie subgroup of G for $\mathfrak{g}_\Lambda$ and define $G_{\Lambda,o} = G_\Lambda \cap G_o$. Let $X_\Lambda = G_\Lambda(o) \subset X_c$ and $X_{\Lambda,o} = G_{\Lambda,o}(o) \subset X_o$. With these notations we have:

Proposition 2.A (Wolf [21]). *X_Λ and $X_{\Lambda,o}$ are totally geodesic Hermitian symmetric subspaces of X_c and X_o respectively with rank $|\Lambda|$. If $|\Lambda_1| = |\Lambda_2|$, where $|\Lambda|$ denotes the number of elements in it, then $X_{\Lambda_1,o}$ is isomorphic to $X_{\Lambda_2,o}$ via a holomorphic isometry of X_o.*

$X_{\Lambda,o}$ (resp. X_Λ) are called the characteristic symmetric subspaces of X_o (resp. X_c). In particular, $X_{\Lambda,o}$ (resp. X_Λ) with $|\Lambda| = |\Pi| - 1$ are called the maximal characteristic symmetric subspaces of X_o (resp. X_c) and all intermediate characteristic symmetric subspaces of X_o (resp. X_c) can be obtained from the maximal characteristic symmetric subspaces of X_o (resp. X_c) inductively.

We now define the boundary components of $X_o \subset X_c$. Two points of ∂X_o are said to be in the same equivalence class if one can be connected to the other by a chain of holomorphic arcs. The equivalence classes are called the boundary components of X_o. Let's define the Cayley transformation $c_{\Pi \setminus \Lambda} \in G$ to be

$$c_{\Pi \setminus \Lambda} := \exp(\frac{\pi}{4} \sum_{\phi \in \Pi \setminus \Lambda} e_{-\phi} - e_\phi).$$

Then we have

Proposition 2.B (Wolf [21]). *The boundary components of $X_o \subset X_c$ are just the sets $k \cdot c_{\Pi \setminus \Lambda} \cdot X_{\Lambda,o}$ with $k \in K$ and $\Lambda \subset \Pi$. They are Hermitian symmetric manifolds of the noncompact type with rank $|\Lambda|$. The action of $G_o \subset G$ is transitive on the union of those boundary components of the same rank. The topological boundary of X_o is therefore the disjoint union of boundary orbits under the action of G_o.*

Thus the maximal boundary components and the maximal characteristic symmetric subspaces of a bounded symmetric domain must have the same dimension. Now we describe the boundary components explicitly.

Proposition 2.C (Wolf [21]). *Let $X_o \subset X_c$ be the Borel embedding of an irreducible Hermitian symmetric manifold of the noncompact type with rank r. For each integer m, $0 \le m < r$, there is just one G_o-equivalence class $\{X_{m,o}\}$ of boundary component with rank m for $X_o \subset X_c$. $X_{0,o}$ is a single point and $X_{m,o}$ are given as follows with $X_{r,o} = X_o$:*

I. *For $X_o = D^I_{p,q}$ ($r = p \le q$), $X_{m,o} = D^I_{m,q-p+m}$.*
II. *For $X_o = D^{II}_{2r}$, $X_{m,o} = D^{II}_{2m}$ and for $X_o = D^{II}_{2r+1}$, $X_{m,o} = D^{II}_{2m+1}$.*
III. *For $X_o = D^{III}_r$, $X_{m,o} = D^{III}_m$.*

IV. *For* $X_o = D_n^{IV}$ *(here* $r = 2$*),* $X_{1,o}$ *is the unit disc in* $\mathbb{C}^1$.
V. *For* $X_o = D^V$ *(here* $r = 2$*),* $X_{1,o}$ *is the unit ball in* $\mathbb{C}^5$.
VI. *For* $X_o = D^{VI}$*, (here* $r = 3$*),* $X_{2,o} = D_{10}^{IV}$ *and* $X_{1,o}$ *is the unit disc in* $\mathbb{C}^1$.

For a proof of Proposition 2.C, see pg. 292 of Wolf [21]. A very elementary proof by function theory technique for (I) of Proposition 2.C is given in pp. 93-95 of Pyatetskii-Shapiro [14]. Since characteristic symmetric submanifolds are isomorphic to boundary components via $k \cdot c_{\Pi \setminus \Lambda}$ with $k \in K$ and $\Lambda \subset \Pi$, this explicitly describes characteristic symmetric submanifolds as well.

The minimal rational curves are precisely those totally geodesic cuves on X_c with respect to any possible choice of canonical (Kähler-Einstein) metrics on X_c. Via the Harish-Chandra embedding $X_o \cong \Omega \subset\subset \mathbb{C}^N \subset X_c$ we call the intersections of C with $\mathbb{C}^N$ (resp. Ω) minimal affine lines (resp. minimal discs). The characteristic vectors of X_o (resp. X_c) are defined as those nonzero vectors tangent to minimal discs $D = C \cap \Omega$ (resp. minimal rational curves C). Then the unit characteristic vectors in $T_p(X_o)$ (resp. $T_p(X_c)$) are exactly those unit vectors which realize the minimum (resp. maximum) of the holomorphic sectional curvature of X_o (resp. X_c) with the canonical metric at p. Let X denote X_o or X_c. Denote by $\tilde{\mathcal{S}}_p(X) \subset T_p(X)$ the cone of all characteristic vectors of X at p, and the characteristic variety $\mathcal{S}_p(X) \subset \mathbb{P}T_p(X)$ is defined as the projectivization of $\tilde{\mathcal{S}}_p(X) - \{0\}$, which is a connected complex submanifold of $\mathbb{P}T_p(X)$. For an irreducible bounded symmetric domain D (in their standard realizations) in $\mathbb{C}^n$, define $\mathcal{S}(D) := \bigcup_{p \in D} \mathcal{S}_p(D) \subset \mathbb{P}T(D)$. If we identify $\mathbb{P}T(D)$ with $D \times \mathbb{P}^{n-1}$ by using the Euclidean coordinates, then $\mathcal{S}(D) \cong D \times \mathcal{S}_o(D)$, i.e., $\mathcal{S}(D)$ is parallel on D in the Euclidean sense (see Proposition 2 on pg. 103 of Mok [12]). The characteristic varieties $\mathcal{S}_p(X) \subset \mathbb{P}T_p(X)$ are given explicitly corresponding to 6 types of irreducible bounded symmetric domains X on pg. 249 of Mok [12]. Fix a characteristic vector $\alpha \in T_o(X)(X = X_c$ or $X_o)$; by Mok [12] we have an orthogonal decomposition $T_o(X) = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$ into eigenspaces of the Hermitian form $Q_\alpha(\xi, \eta) = R_{\alpha\bar{\alpha}\xi\bar{\eta}}$ corresponding to eigenvalues $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$, $\frac{1}{2}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ and 0. Then we have $\mathcal{N}_\alpha = T_o(X_{\Lambda,o})$ for $\Lambda = \Pi \backslash \alpha$ by Mok and Tsai [13]. Let $n(X) = \dim_{\mathbb{C}} X_{\Lambda,o} = \dim_{\mathbb{C}} \mathcal{N}_\alpha$. Then $\dim_{\mathbb{C}} \mathcal{N}_\alpha$ is maximum among non-zero vectors $\mu \in T_o(X)$ and we call $n(X)$ the null dimension of X.

Like Tsai [18], our idea in this paper is based on the following two propositions.

Proposition 2.1 (Tsai [18])**.** *Let* Ω_1 *and* Ω_2 *be two bounded symmetric domains and let* $f : \Omega_1 \to \Omega_2$ *be a proper holomorphic mapping. Suppose that* Ω_1 *is irreducible and rank*$(\Omega_1) \geq 2$ *. Then we have the following result:*

(i) *For any characteristic symmetric subspace* Ω_Λ *of* Ω_1*,* $f(\Omega_\Lambda)$ *is contained in some characteristic symmetric subspace of* Ω_2*.*

(ii) *rank*$(\Omega_1) \leq$ *rank*(Ω_2)*.*

(iii) *If rank*$(\Omega_1) =$ *rank*(Ω_2)*, then* $df(\tilde{\mathcal{S}}_p(\Omega_1)) \subset \tilde{\mathcal{S}}_{f(p)}(\Omega_2)$ *for any* $p \in \Omega_1$*.*

(iv) *If rank*$(\Omega_1) =$ *rank*(Ω_2)*,* Ω_2 *is irreducible.*

Proposition 2.2. *Let* Ω_1 *and* Ω_2 *be two irreducible bounded symmetric domains with rank*$(\Omega_i) \geq 2$ $(i = 1, 2)$ *and let* $f : \Omega_1 \to \Omega_2$ *be a proper holomorphic mapping. If* f *maps any minimal disc of* Ω_1 *into a minimal disc of* Ω_2*, then* f *is necessarily a totally geodesic isometric embedding with respect to their Bergman metrics* (*up to normalizing constants*)*.*

Remark. The idea in the proof of Proposition 2.2 comes from Mok [10] (see Chapter 6 of Mok [12] for references). The proof of Proposition 2.2 in a special case is given in Tsai [18] and the proof in Tsai [18] can be easily checked to be valid in our case.

3. Proof of Theorem 1.1

In this section we shall give the proof of Theorem 1.1.

A domain Ω in $\mathbb{C}^n$ is said to be circular if and only if it is invariant under the action of the circle group S^1 given by $(z_1, ..., z_n) \to (e^{i\theta}z_1, ..., e^{i\theta}z_n)$. A circular domain is said to be complete if it contains the origin.

Proposition 3.1 (Bell [4])**.** *Suppose that Ω_1 and Ω_2 are bounded complete circular domains in $\mathbb{C}^n$. If $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping with $f(0) = 0$, then f must be algebraic.*

Proposition 3.2. *Let Ω_1 and Ω_2 be two equidimensional bounded symmetric domains in $\mathbb{C}^n$ and let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping. Suppose Ω_1 is irreducible and rank$(\Omega_1) \geq 2$. Then every minimal disc of Ω_1 is mapped by f onto a minimal disc of Ω_2.*

Proof. Let ${\Omega_1}'$ be a maximal characteristic symmetric subspace of Ω_1 (note that $\operatorname{rank}({\Omega_1}') = \operatorname{rank}(\Omega_1) - 1$ here). Then, by (i) of Proposition 2.1, there exists a maximal characteristic symmetric subspace ${\Omega_2}'$ of Ω_2 such that $f({\Omega_1}') \subset {\Omega_2}'$. Since $f : \Omega_1 \to \Omega_2$ is algebraic by Proposition 3.1, $f : \Omega_1 \to \Omega_2$ extends a holomorphic mapping $\tilde{f} : \mathbb{C}^n - R \to \mathbb{C}^n$ for a nowhere dense complex analytic set $R \subset \mathbb{C}^n$. Since $f : \Omega_1 \to \Omega_2$ is proper, we have $\tilde{f}(\partial\Omega_1 \cap (\mathbb{C}^n - R)) \subset \partial\Omega_2$. Since the ramification locus of $\tilde{f}$ is a nowhere dense complex analytic set, there exist $p \in \partial\Omega_1$, $q = \tilde{f}(p) \in \partial\Omega_2$ such that $\tilde{f} : U(p) \to U(q)$ is a biholomorphism with $\tilde{f}(U(p) \cap \partial\Omega_1) = U(q) \cap \partial\Omega_2$, where $U(p), U(q)$ are two small domains in $\mathbb{C}^n$ with $p \in U(p)$ and $q \in U(q)$. Since the disjoint union of maximal boundary components of a bounded symmetric domain $\Omega \subset \mathbb{C}^n$ is dense in the topological boundary of $\Omega \subset \mathbb{C}^n$ by Propositions 2.B and 2.C, the maximal boundary components of $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^n$ have the same dimension. Since the maximal boundary components and the maximal characteristic symmetric subspaces of a bounded symmetric domain have the same dimension by Proposition 2.B, the maximal characteristic symmetric subspaces of Ω_1 and Ω_2 have the same dimension also. Thus $f|_{{\Omega_1}'} : {\Omega_1}' \to {\Omega_2}'$ is a proper holomorphic mapping with $\dim {\Omega_1}' = \dim {\Omega_2}'$ and $\operatorname{rank}({\Omega_1}') = \operatorname{rank}(\Omega_1) - 1$. By induction on rank, for any characteristic symmetric subspace B_1 of Ω_1 with $\operatorname{rank}(B_1) = 1$, there exists a characteristic symmetric subspace B_2 of Ω_2 such that $f(B_1) \subset B_2$ with $\dim B_1 = \dim B_2$. Since $f|_{B_1} : B_1 \to B_2$ is an algebraic proper holomorphic mapping by Proposition 3.1, there is no complex analytic subvariety in ∂B_2, and then B_2 must be of rank 1. This means that B_1 and B_2 are characteristic symmetric subspaces of Ω_1 and Ω_2, respectively, such that $\dim B_1 = \dim B_2$, $f(B_1) = B_2$ and $\operatorname{rank}(B_1) = \operatorname{rank}(B_2) = 1$.

(i) If $\dim B_1 = 1$, then minimal discs on Ω_1 and Ω_2 are characteristic symmetric subspaces of rank 1, and then every minimal disc of Ω_1 is mapped by f onto a minimal disc of Ω_2.

(ii) If $\dim B_1 \geq 2$, then $f|_{B_1} : B_1 \to B_2$ is a biholomorphism by Theorem 1.A. Since all minimal discs of the unit ball B^n in $\mathbb{C}^n$ ($n \geq 2$) are $\operatorname{Aut}(B^n)$-orbits of the unit disc $D \hookrightarrow B^n$ and the isometric group of Ω_1 acts transitively on the set of

minimal discs of Ω_1, every minimal disc of Ω_1 is mapped by f onto a minimal disc of Ω_2.

The proof of Proposition 3.2 is completed.

Proposition 3.3. *Let Ω_1 and Ω_2 be two equidimensional bounded symmetric domains and let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping. Suppose Ω_1 is irreducible and rank$(\Omega_1) \geq 2$. Then we have $n(\Omega_1) \leq n(\Omega_2)$ and Ω_2 is irreducible.*

Proof. $n(\Omega_1) \leq n(\Omega_2)$ immediately comes from (i) of Proposition 2.1.

Now we shall prove that Ω_2 is irreducible. Without loss of generality we assume that $\Omega_2 = \Omega_2{}' \times \Omega_2{}''$, where $\Omega_2{}'$ and $\Omega_2{}''$ are irreducible. Let $R \subset \Omega_1$ be the ramification locus of f. Fix $x \in \Omega_1 - R$. Then by Proposition 3.2 we have

$$df(\mathcal{S}_x(\Omega_1)) \subset \mathcal{S}_{f(x)}(\Omega_2) = \mathcal{S}_{f_1(x)}(\Omega_2{}') \cup \mathcal{S}_{f_2(x)}(\Omega_2{}'') \text{ (a disjoint union)}$$

where $f(x) = (f_1(x), f_2(x)) \in \Omega_2{}' \times \Omega_2{}''$.

Since $\mathcal{S}_x(\Omega_1)$ is connected, by the continuity argument we have

$$df(T(\Omega_1)) \subset T(\Omega_2{}') \quad \text{or} \quad df(T(\Omega_1)) \subset T(\Omega_2{}'').$$

This is impossible as a proper holomorphic mapping cannot decrease dimension. We get Proposition 3.3.

We can get Theorem 1.1 by Proposition 2.2 and 3.2 now. But we shall not use Proposition 2.2 to prove Theorem 1.1 and shall give a more elementary proof here.

Proof of Theorem 1.1. The proof consists of two steps:

Step (1). $\dim_{\mathbb{C}}(\mathcal{S}_o(\Omega_1)) = \dim_{\mathbb{C}}(\mathcal{S}_o(\Omega_2))$.

Proof of Step (1). Let $R \subset \Omega_1$ be the ramification locus of f. For $x \in \Omega_1 - R$ and $y = f(x)$, f induces an isomorphism $\sigma = [df(x)] : \mathbb{P}T_x(\Omega_1) \to \mathbb{P}T_y(\Omega_2)$. By Proposition 3.2 we have $\sigma(\mathcal{S}_x(\Omega_1)) \subset \mathcal{S}_y(\Omega_2)$. On the other hand, Since $n(\Omega_1) \leq n(\Omega_2)$ by Proposition 3.3 and $\dim_{\mathbb{C}}(\mathcal{S}_p(X)) + n(X) = \dim_{\mathbb{C}}(X) - 1$ (where $p \in X$ and $X = X_c$ or X_o) by Prop. 4 on pg. 105 of Mok [12], we have $\dim_{\mathbb{C}}(\mathcal{S}_x(\Omega_1)) \geq \dim_{\mathbb{C}}(\mathcal{S}_y(\Omega_2))$. Since $\mathcal{S}_y(\Omega_2)$ is connected by Proposition 3.3, we have $\sigma(\mathcal{S}_x(\Omega_1)) = \mathcal{S}_y(\Omega_2)$. As $\dim_{\mathbb{C}}(\mathcal{S}_p(X))$ is independent of $p \in X$, we get Step (1).

Step (2). f: $\Omega_1 \to \Omega_2$ is a biholomorphism.

Proof of Step (2). Let $J_f(z)$ be the Jacobian determinant of f at $z \in \Omega_1$ and let $R_f = \{z \in \Omega_1 : J_f(z) = 0\}$ be the ramification locus. Evidently, f is a local biholomorphism at any point of $\Omega_1 - R_f$. If $R_f \neq \emptyset$, then R_f must be an analytic set of complex codimension 1.

Now fix a point $x \in R_f$ and assume that $df(x) : T_x(\Omega_1) \to T_{f(x)}(\Omega_2)$ satisfies $df(x)(T_x(\Omega_1)) \neq 0$. The projective linear map $[df(x)] : \mathbb{P}T_x(\Omega_1) \to \mathbb{P}T_{f(x)}(\Omega_2)$ induces a rational map $[df(x)]|_{\mathcal{S}_x(\Omega_1)} : \mathcal{S}_x(\Omega_1) \to \mathcal{S}_{f(x)}(\Omega_2)$ (by Proposition 3.2) with image lying in a hyperplane section H of $\mathcal{S}_{f(x)}(\Omega_2)$, $H \subset$ but $\neq \mathcal{S}_{f(x)}(\Omega_2)$, as $\mathcal{S}_{f(x)}(\Omega_2)$ is not contained in any hyperplane section of $\mathbb{P}T_{f(x)}(\Omega_2)$. It follows that $[df(x)]|_{\mathcal{S}_x(\Omega_1)} : \mathcal{S}_x(\Omega_1) \to \mathcal{S}_{f(x)}(\Omega_2)$ must have positive dimensional generic fibers by Step (1). In particular, there exists a holomorphic one-parameter family of minimal discs $\{C_z : z \in D\}$ (where D is a disc of the complex plane) passing through x such that each C_z is mapped by f onto the same minimal disc in Ω_2. This is impossible, as $f^{-1}(w)$ is a finite set for any point $w \in \Omega_2$.

So $df(x) : T_x(\Omega_1) \to T_{f(x)}(\Omega_2)$ satisfies $df(x)(T_x(\Omega_1)) = 0$ for all $x \in R_f$. In this case, all smooth points of R_f are ramification points of $f|_{R_f} : R_f \to f(R_f)$. This is impossible, as $f|_{R_f} : R_f \to f(R_f)$ is a proper holomorphic mapping.

Hence $R_f = \emptyset$, so that $df(z) : T_z(\Omega_1) \to T_{f(z)}(\Omega_2)$ for any $z \in \Omega_1$ is an isomorphism. Consequently, f is a holomorphic covering map. As any Hermitian symmetric manifold of non-compact type is simply connected, $f : \Omega_1 \to \Omega_2$ is in fact a biholomorphism. So we have Step (2).

The proof of Theorem 1.1 is completed.

Acknowledgments

This work is part of the author's Ph.D. thesis [19]. The author would like to express his hearty thanks to his advisor, Professor Ngaiming Mok, who kindly introduced this topic to him. Without Professor Mok's patient guidance, encouragement and many useful suggestions, the author would not have been able to complete this work. In addition, the author wishes to thank Professors Wing-Keung To, I-Hsun Tsai and Sai-Kee Yeung for helpful disscussions.

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