PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 4, April 2007, Pages 977–986
S 0002-9939(06)08568-6
Article electronically published on October 4, 2006

MAPPINGS PRESERVING SPECTRA OF PRODUCTS OF MATRICES

JOR-TING CHAN, CHI-KWONG LI, AND NUNG-SING SZE

(Communicated by Joseph A. Ball)

Dedicated to Professor Ahmed Sourour on the occasion of his sixtieth birthday.

ABSTRACT. Let M_n be the set of $n \times n$ complex matrices, and for every $A \in M_n$, let $\mathrm{Sp}(A)$ denote the spectrum of A. For various types of products $A_1 * \cdots * A_k$ on M_n, it is shown that a mapping $\phi : M_n \to M_n$ satisfying $\mathrm{Sp}(A_1 * \cdots * A_k) = \mathrm{Sp}(\phi(A_1) * \cdots * \phi(A_k))$ for all $A_1, \ldots, A_k \in M_n$ has the form

$$X \mapsto \xi S^{-1} X S \quad \text{or} \quad A \mapsto \xi S^{-1} X^t S$$

for some invertible $S \in M_n$ and scalar ξ. The result covers the special cases of the usual product $A_1 * \cdots * A_k = A_1 \cdots A_k$, the Jordan triple product $A_1 * A_2 = A_1 * A_2 * A_1$, and the Jordan product $A_1 * A_2 = (A_1A_2 + A_2A_1)/2$. Similar results are obtained for Hermitian matrices.

1. Introduction

Let M_n be the set of all $n \times n$ complex matrices. In [5], Marcus and Moyls proved that if a linear mapping $\phi : M_n \to M_n$ preserves the eigenvalues (counting multiplicities) of each matrix in M_n, then there exists an invertible matrix S such that ϕ has the form

$$A \mapsto S^{-1}AS \qquad \text{or} \qquad A \mapsto S^{-1}A^tS,$$

where A^t denotes the transpose of A. The assumption on multiplicity is not really necessary. Let $\mathrm{Sp}(A)$ denote the spectrum of A, i.e., the set of all eigenvalues of A without counting multiplicities. Then by a result of Jafarian and Sourour [3], the above conclusion holds if $\mathrm{Sp}(\phi(A)) = \mathrm{Sp}\,(A)$.

The result has been generalized in different directions. For example, in [8], Omladič and P. Šemrl considered spectrum-preserving mappings that are just additive. In [6] Molnár studied surjective maps ϕ on bounded linear operators such that

$$\mathrm{Sp}\,(\phi(A)\phi(B)) = \mathrm{Sp}\,(AB) \qquad \text{for all linear operators } A, B. \tag{1.1}$$

Received by the editors April 7, 2005 and, in revised form, November 10, 2005.
2000 *Mathematics Subject Classification.* Primary 15A04, 15A18.
Key words and phrases. Eigenvalue, spectrum, preserve.
This research was partially supported by Hong Kong RCG CERG grant HKU 7007/03P. The second author was also supported by a USA NSF grant.

The second author is also an honorary professor of the Heilongjiang University, and an honorary professor of the University of Hong Kong.

In particular, such a map on M_n has the form

$$A \mapsto \xi S^{-1}AS \qquad \text{or} \qquad A \mapsto \xi S^{-1}A^tS \tag{1.2}$$

for some invertible matrix S and $\xi \in \{1, -1\}$. Continuous differentiable maps on M_n preserving the spectrum were characterized in [1].

In this paper, we consider different types of products $A * B$ on M_n including the usual product $A * B = AB$, the Jordan triple product $A * B = ABA$, and the Jordan product $A * B = (AB + BA)/2$. We obtain a general result, which implies that a mapping $\phi : M_n \to M_n$ satisfying

$$\operatorname{Sp}(A * B) = \operatorname{Sp}(\phi(A) * \phi(B)) \quad \text{for all } A, B \in M_n \tag{1.3}$$

has the form (1.2) for some invertible $S \in M_n$ and scalar ξ. As we do not require the surjective assumption on ϕ, our result refines that of Molnár in the finite-dimensional case.

Note that a characterization of those $\phi : M_n \to M_n$ such that AB and $\phi(A)\phi(B)$ have the same eigenvalues counting multiplicities is given in [7]. A crucial observation is the following proposition. We include the proof for the sake of completeness.

Proposition 1.1. *Suppose $\phi : M_n \to M_n$ satisfies*

$$\operatorname{tr}(AB) = \operatorname{tr}(\phi(A)\phi(B)) \qquad \textit{for all } A, B \in \mathcal{M}.$$

Then ϕ is an invertible linear map.

Proof. For every $X = (x_{ij}) \in M_n$, let R_X be the n^2 row vector

$$R_X = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{nn}),$$

and C_X the n^2 column vector

$$C_X = (x_{11}, x_{21} \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{1n}, \ldots, x_{nn})^t.$$

Then for any $X, Y \in M_n$,

$$R_{\phi(X)}C_{\phi(Y)} = \operatorname{tr}(\phi(X)\phi(Y)) = \operatorname{tr}(XY) = R_XC_Y. \tag{1.4}$$

Let $\{Y_1, \ldots, Y_{n^2}\}$ be a basis for M_n. Let $\mathcal{Y}$ have columns $C_{Y_1}, \ldots C_{Y_{n^2}}$, and $\mathcal{Z} \in M_{n^2}$ have columns $C_{\phi(Y_1)}, \ldots, C_{\phi(Y_{n^2})}$. Then by (1.4), for any $X \in M_n$ we have

$$R_{\phi(X)}\mathcal{Z} = R_X\mathcal{Y}.$$

Next, we show that $\mathcal{Z}$ is invertible. To this end, let $\{X_1, \ldots, X_{n^2}\}$ be a basis for M_n, $\mathcal{X} \in M_{n^2}$ with rows $R_{X_1}, \ldots R_{X_{n^2}}$, and $\mathcal{W} \in M_{n^2}$ with rows $R_{\phi(X_1)}, \ldots, R_{\phi(X_{n^2})}$. Then $\mathcal{W}\mathcal{Z} = \mathcal{X}\mathcal{Y}$ for the invertible matrices $\mathcal{X}$ and $\mathcal{Y}$. So, $\mathcal{Z}$ is invertible, and for any $X \in M_n$,

$$R_{\phi(X)} = R_X\mathcal{Y}\mathcal{Z}^{-1}.$$

Hence, ϕ is an invertible linear map. □

The problem of characterizing mappings that preserve the spectra of the product of matrices is more challenging. Our results will give a characterization of mappings preserving the spectrum of various products of k matrices $X_1 * \cdots * X_k$ defined as follows.

Let $k \geq 2$, and let a sequence $(j_1, \ldots, j_m)$ be given so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$. We consider products of the form

$$X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m},$$

which cover the usual product $A * B = AB$ and the Jordan triple product $A * B = ABA$. We also consider products of the form

$$X_1 * \cdots * X_k = (X_{j_1} \cdots X_{j_m} + X_{j_m} \cdots X_{j_1})/2,$$

which cover the Jordan product $A * B = (AB + BA)/2$.

In Section 2, we obtain the results on the set M_n of $n \times n$ complex matrices. Using a transfer principle in model-theoretic algebra (see [2]), one sees that the results also hold for square matrices over an algebraically closed field. In Section 3, similar results are proved for the set H_n of $n \times n$ complex Hermitian matrices. The same results and proofs are valid for $n \times n$ real symmetric matrices as well.

2. Results on complex matrices

Theorem 2.1. *Suppose $k \geq 2$, and let a sequence $(j_1, \ldots, j_m)$ be given so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there is j_r not equal to j_s for all $s \neq r$. Consider*

$$X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m}.$$

Then a mapping $\phi : M_n \to M_n$ satisfies

$$\text{(2.1)} \quad \mathrm{Sp}\,(\phi(X_1) * \cdots * \phi(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k) \qquad \textit{for all } X_1, \ldots, X_k \in M_n$$

if and only if there exist an invertible matrix $S \in M_n$ and a scalar ξ satisfying $\xi^m = 1$ such that

(a) *ϕ has the form $A \mapsto \xi S^{-1} A S$, or*
(b) *$(j_{r+1}, \ldots, j_m, j_1, \ldots, j_{r-1}) = (j_{r-1}, \ldots, j_1, j_m, \ldots, j_{r+1})$ and ϕ has the form $A \mapsto \xi S^{-1} A^t S$.*

Note that the assumption that there is $j_r \notin \{j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_m\}$ is necessary. For example, if $A*B = ABBA$, then mappings ϕ satisfying $\mathrm{Sp}\,(\phi(A)*\phi(B)) = \mathrm{Sp}\,(A * B)$ may not have a nice structure. For instance, ϕ can send all involutions, i.e., those matrices $X \in M_n$ such that $X^2 = I_n$, to a fixed involution, and $\phi(X) = X$ for other X.

Proof of Theorem 2.1. It is clear that if (a) or (b) holds, then ϕ satisfies (2.1). We need only prove the necessity part. We divide the proof of it into several assertions. Since $\mathrm{Sp}\,(X_{j_1} \cdots X_{j_m}) = \mathrm{Sp}\,(X_{j_r} \cdots X_{j_m} X_{j_1} \cdots X_{j_{r-1}})$, we may assume that $j_1 \notin \{j_2, \ldots, j_m\}$. Define

$$\mathcal{S} = \{X \in M_n : X \text{ has } n \text{ distinct eigenvalues}\}.$$

□

Assertion 1. *For every $A \in \mathcal{S}$, there is a neighborhood of $\mathcal{N}_A$ such that the restriction of ϕ on $\mathcal{N}_A$ equals an invertible linear map L_A.*

Proof. For every $A \in \mathcal{S}$, $\mathrm{Sp}\,(AI_n^{m-1})$ has n distinct elements. By the continuity of the eigenvalues, there are neighborhoods $\mathcal{N}_{I_n}$ of I_n and $\mathcal{N}_A$ of A such that XY^{m-1} has n distinct eigenvalues for every $X \in \mathcal{N}_A$ and $Y \in \mathcal{N}_{I_n}$. By (2.1), $\phi(X)\phi(Y)^{m-1}$ has n distinct eigenvalues equal to those of XY^{m-1}. Hence

$$\text{(2.2)} \qquad \mathrm{tr}\,(\phi(X)\phi(Y)^{m-1}) = \mathrm{tr}\,(XY^{m-1}) \quad \text{for every } X \in \mathcal{N}_A \text{ and } Y \in \mathcal{N}_{I_n}.$$

As in the proof of Proposition 1.1, for every $X = (x_{ij}) \in M_n$, let R_X be the n^2 row vector

$$R_X = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{nn}),$$

and C_X the n^2 column vector

$$C_X = (x_{11}, x_{21} \dots, x_{n1}, x_{12}, \dots, x_{n2}, \dots, x_{1n}, \dots, x_{nn})^t.$$

Then

$$R_{\phi(X)} C_{\phi(Y)^{m-1}} = \mathrm{tr}\,(\phi(X)\phi(Y)^{m-1}) = \mathrm{tr}\,(XY^{m-1}) = R_X C_{Y^{m-1}} \tag{2.3}$$

for every $X \in \mathcal{N}_A$ and $Y \in \mathcal{N}_{I_n}$.

Now, suppose $\mathrm{tr}\,(XZ) = 0$ for each $Z \in \{Y^{m-1} : Y \in \mathcal{N}_{I_n}\}$. Then for any $R \in M_n$,

$$\mathrm{tr}\,(X(I + tR)^{m-1}) = \sum_{j=0}^{m} t^j \binom{m-1}{j} \mathrm{tr}\,(XR^j) = 0$$

for sufficiently small $t > 0$. We have $\mathrm{tr}\,(XR) = 0$. It follows that $X = 0$. So, $\{Y^{m-1} : Y \in \mathcal{N}_{I_n}\}$ is a spanning set of M_n, and contains a basis $\{Y_j^{m-1} : 1 \le j \le n^2\}$ for M_n with $Y_j \in \mathcal{N}_{I_n}$ for each $j = 1, \dots, n^2$. Let $\mathcal{Y}$ and $\mathcal{Z}$ be the $n^2 \times n^2$ matrices with columns $C_{Y_1^{m-1}}, \dots, C_{Y_{n^2}^{m-1}}$ and $C_{\phi(Y_1)^{m-1}}, \dots, C_{\phi(Y_{n^2})^{m-1}}$ respectively. By (2.2) and (2.3),

$$R_{\phi(X)} \mathcal{Z} = R_X \mathcal{Y} \quad \text{for every } X \in \mathcal{N}_A.$$

We claim that the matrix $\mathcal{Z}$ is invertible. To this end take a basis $\{X_1, \dots, X_{n^2}\}$ of M_n in $\mathcal{N}_A$ and let $\mathcal{X}$ and $\mathcal{W}$ be the $n^2 \times n^2$ matrices with rows $R_{X_1}, \dots, R_{X_{n^2}}$ and $R_{\phi(X_1)}, \dots, R_{\phi(X_{n^2})}$ respectively. Then $\mathcal{W}\mathcal{Z} = \mathcal{X}\mathcal{Y}$ for invertible matrices $\mathcal{X}$ and $\mathcal{Y}$. It follows that $\mathcal{Z}$ is invertible, and

$$R_{\phi(X)} = R_X \mathcal{X} \mathcal{Z}^{-1} \quad \text{for every } X \in \mathcal{N}_A.$$

Hence the restriction of ϕ to $\mathcal{N}_A$ is some invertible linear mapping L_A. The proof of Assertion 1 is complete. □

Assertion 2. *All the linear maps L_A in Assertion 1 are the same, i.e., ϕ is equal to an invertible linear mapping L on the dense subset $\mathcal{S}$.*

Proof. Note that for any $A, B \in \mathcal{S}$, there is a continuous curve $f : [0,1] \to \mathcal{S}$ such that $f(0) = A$ and $f(1) = B$. Consider the set

$$\mathcal{C} = \{t \in [0,1] : \phi = L_A \text{ on an open neighborhood of } f(t)\}.$$

Then clearly $\mathcal{C}$ is an open subset of $[0,1]$. But $\mathcal{C}$ is also closed in $[0,1]$. Let $t_0 \in \mathcal{C}^-$. There is an open neighborhood $\mathcal{N}_{f(t_0)}$ of $f(t_0)$ on which ϕ is equal to the linear mapping $L_{f(t_0)}$. Take $t \in f^{-1}(\mathcal{N}_{f(t_0)}) \cap \mathcal{C}$. Then on some open neighborhood $\mathcal{N}_{f(t)}$ of $f(t)$, $\phi = L_A$. On the nonempty open set $\mathcal{N}_{f(t_0)} \cap \mathcal{N}_{f(t)}$, $L_{f(t_0)} = \phi = L_A$. Hence $L_{f(t_0)} = L_A$, and $t_0 \in \mathcal{C}$. We conclude that $\mathcal{C} = [0,1]$, and $L_A = L_B$. The proof of Assertion 2 is complete. □

Assertion 3. *The mapping L in Assertion 2 has the form $A \mapsto \xi S^{-1} A S$ or $A \mapsto \xi S^{-1} A^t S$ for some invertible $S \in M_n$ and $\xi \in \mathbb{C}$ with $\xi^m = 1$. Moreover, if the latter case holds, then $(j_2, \dots, j_m) = (j_m, \dots, j_2)$.*

Proof. By the continuity of L and the spectrum, we have that

$$\mathrm{Sp}\,(L(X_1) * \cdots * L(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k)$$

for all $X_1, \dots, X_k \in M_n$. If A is invertible, then

$$0 \notin \mathrm{Sp}\,(A * \cdots * A) = \mathrm{Sp}\,(L(A) * \cdots * L(A)),$$

and hence $L(A)$ is also invertible. It follows that L is nonsingular and preserves invertible matrices. By [5], there are invertible matrices M, N such that L has the form

$$A \mapsto MAN \qquad \text{or} \qquad A \mapsto MA^tN. \tag{2.4}$$

We claim that NM is a scalar matrix. Otherwise, there exists an invertible $R \in M_n$ such that $RNMR^{-1}$ is a direct sum of companion matrices so that its second row has the form $(1, 0, \ldots, *)$. Let $A = R^{-1}E_{12}R$ or $A^t = R^{-1}E_{12}R$ depending on L have the first or the second form in (2.4), where E_{12} is the $n \times n$ matrix with 1 at the $(1, 2)$ position and 0 everywhere else. Then $\mathrm{Sp}\,(A^m) = \mathrm{Sp}\,(A) = \{0\}$. Now

$$\mathrm{Sp}\,(L(A)) = \mathrm{Sp}\,(M(R^{-1}E_{12}R)N) = \mathrm{Sp}\,(E_{12}RNMR^{-1}).$$

It follows that $1 \in \mathrm{Sp}\,(L(A))$ and hence $1 \in \mathrm{Sp}\,(L(A)^m)$ whereas $\mathrm{Sp}\,(A^m) = \{0\}$, which contradicts (2.1).

We have proved that L has the form $A \mapsto \xi S^{-1}AS$ or $A \mapsto \xi S^{-1}A^tS$ for some ξ. Since $\{\xi^m\} = \mathrm{Sp}\,(L(I_n)^m) = \mathrm{Sp}\,(I_n^m) = \{1\}$, $\xi^m = 1$.

Now, suppose L has the form $A \mapsto \xi S^{-1}A^tS$. Replacing L by the mapping $A \mapsto \bar{\xi}SL(A)S^{-1}$, we may assume that $L(A) = A^t$ for all $A \in M_n$. Then

$$\begin{aligned} &\mathrm{Sp}\,(X_{j_1} \cdots X_{j_m}) = \mathrm{Sp}\,(X_1 * \cdots * X_k) = \mathrm{Sp}\,(L(X_1) * \cdots * L(X_k)) \\ = \;&\mathrm{Sp}\,(X_{j_1}^t \cdots X_{j_m}^t) = \mathrm{Sp}\,(X_{j_m} \cdots X_{j_1}) = \mathrm{Sp}\,(X_{j_1}X_{j_m} \cdots X_{j_2}) \end{aligned}$$

for any $X_1, \ldots, X_k \in M_n$. We have to show that $(j_2, \ldots, j_m) = (j_m, \ldots, j_2)$. Suppose it is not true. Let $l \geq 2$ be the smallest integer such that $j_l \neq j_{m+2-l}$. Then $l \leq (m+1)/2$. Let $A_{j_l} = A = \mathrm{diag}(\lambda, 1, \ldots, 1)$, and for every $k \notin \{1, j_l\}$, let $A_k = B = B_1 \oplus I_{n-2}$, where $B_1 \in M_2$ is a symmetric invertible matrix with positive entries. Then

$$A_{j_2} \cdots A_{j_m} = RA^{r_1}B^{s_1}A^{r_2}B^{s_2} \cdots A^{r_t}B^{s_t}R^t$$

for positive integers r_i, s_i, where $R = A_{j_2} \cdots A_{j_{l-1}}$. Note that

$$A^{r_i}B^{s_i} = \begin{pmatrix} \lambda^{r_i}b_{11}^{(s_i)} & \lambda^{r_i}b_{12}^{(s_i)} \\ b_{21}^{(s_i)} & b_{22}^{(s_i)} \end{pmatrix} \oplus I_{n-2},$$

for positive numbers $b_{11}^{(s_i)}, b_{12}^{(s_i)}, b_{21}^{(s_i)}$ and $b_{22}^{(s_1)}$. An induction argument shows that the $(1, 2)$ entry of $A_{j_l} \cdots A_{j_{m-l+2}}$ is a polynomial of degree $r_1 + \cdots + r_t$ in λ. Similarly, the $(1, 2)$ entry of $A_{j_{m-l+2}} \cdots A_{j_l}$, is a polynomial of degree $r_2 + \cdots + r_t$. So, there is $\lambda > 0$ such that

$$A_{j_l} \cdots A_{j_{m-l+2}} \neq A_{j_{m-l+2}} \cdots A_{j_l}.$$

It follows that

$$A_{j_2} \cdots A_{j_m} = RA_{j_l} \cdots A_{j_{m-l+2}}R^t \neq RA_{j_{m-l+2}} \cdots A_{j_l}R^t = A_{j_m} \cdots A_{j_2}.$$

Note that if $X \in M_n$ is a rank one idempotent matrix, and $\mathrm{Sp}\,(A) = \mathrm{Sp}\,(BX)$, then $\mathrm{tr}\,(AX) = \mathrm{tr}\,(BX)$. Moreover, if $\mathrm{tr}\,(AX) = \mathrm{tr}\,(BX)$ for all rank one idempotent $X \in M_n$, then $A = B$. By these facts, we see that there exists a rank one idempotent A_1 such that

$$\mathrm{Sp}\,(A_{j_m} \cdots A_{j_2}A_{j_1}) \neq \mathrm{Sp}\,(A_{j_2} \ldots, A_{j_m}A_{j_1}) = \mathrm{Sp}\,(A_{j_1}A_{j_2} \cdots A_{j_m}),$$

which is a contradiction. Hence, $(j_2, \ldots, j_m) = (j_m, \ldots, j_2)$ as asserted. $\square$

Assertion 4. *The mapping ϕ equals the invertible linear mapping L in Assertion 3.*

Proof. From (2.1), and the continuity of L and the spectrum, we have

$$\mathrm{Sp}\,(L(A)L(B)^{m-1}) = \mathrm{Sp}\,(AB^{m-1}) = \mathrm{Sp}\,(\phi(A)L(B)^{m-1}) \quad \text{for every } A, B \in M_n.$$

Since L is surjective,

$$\mathrm{Sp}\,(\phi(A)C^{m-1}) = \mathrm{Sp}\,(L(A)C^{m-1}) \qquad \text{for every } A, C \in M_n. \tag{2.5}$$

Let $A \in M_n$. If $C \in M_n$ is a rank one idempotent, $\phi(A)C^{m-1}$ has at most one nonzero eigenvalue, which is given by $\mathrm{tr}\,(\phi(A)C)$. The same is true for $L(A)C$. By (2.5),

$$\mathrm{tr}\,(\phi(A)C) = \mathrm{tr}\,(L(A)C) \qquad \text{for every rank one idempotent matrix } C \in M_n.$$

It follows that $\phi(A) = L(A)$ for all $A \in M_n$. The proof of Assertion 4 is complete.

By Assertions 1–4, the theorem follows. □

Theorem 2.2. *Suppose $k \geq 2$, and $X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m} + X_{j_m} \cdots X_{j_1}$ for a given sequence $(j_1, \ldots, j_m)$ so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there exists j_r not equal to j_s for all $s \neq r$. Then a mapping $\phi : M_n \to M_n$ satisfies*

$$\mathrm{Sp}\,(\phi(X_1) * \cdots * \phi(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k) \qquad \text{for all } X_1, \ldots, X_k \in M_n \tag{2.6}$$

if and only if there exist an invertible matrix $S \in M_n$ and a scalar ξ satisfying $\xi^m = 1$ such that ϕ has the form

$$A \mapsto \xi S^{-1}AS \qquad \text{or} \qquad A \mapsto \xi S^{-1}A^tS.$$

Proof. The necessity of the result is clear. We consider the sufficiency part. Using similar arguments as in the proof of Theorem 2.1 (cf. Assertions 1 and 2), we can prove that ϕ is equal to a bijective linear mapping L on the dense subset

$$\mathcal{S} = \{X \in M_n : X \text{ has } n \text{ distinct eigenvalues}\}.$$

By continuity of L and the spectrum, we see that

$$\mathrm{Sp}\,(L(X_1) * \cdots * L(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k)$$

for all $X_1, \ldots, X_k \in M_n$. Thus, $\mathrm{Sp}\,(L(A)^m) = \mathrm{Sp}\,(A^m)$ for all $A \in M_n$. Using the argument in Assertion 3 in the proof of Theorem 2.1, we see that L has the form $A \mapsto \xi S^{-1}AS$ or $A \mapsto \xi S^{-1}A^tS$. Now, replace ϕ and L by the mappings $A \mapsto \bar{\xi}S\phi(A)S^{-1}$ and $A \mapsto \bar{\xi}SL(A)S^{-1}$, respectively; we may assume that $L(A) = A$ for all $A \in M_n$.

We will show that $\phi = L$ on M_n. From (2.2), we have

$$\begin{aligned}
& \mathrm{Sp}\,(X^{r-1}L(Y)X^{m-r} + X^{m-r}L(Y)X^{r-1}) \\
= \; & \mathrm{Sp}\,(L(X)^{r-1}L(Y)L(X)^{m-r} + L(X)^{m-r}L(Y)L(X)^{r-1}) \\
= \; & \mathrm{Sp}\,(X^{r-1}YX^{m-r} + X^{m-r}YX^{r-1}) \\
= \; & \mathrm{Sp}\,(\phi(X)^{r-1}\phi(Y)\phi(X)^{m-r} + \phi(X)^{m-r}\phi(Y)\phi(X)^{r-1}) \\
= \; & \mathrm{Sp}\,(L(X)^{r-1}\phi(Y)L(X)^{m-r} + L(X)^{m-r}\phi(Y)L(X)^{r-1}) \\
= \; & \mathrm{Sp}\,(X^{r-1}\phi(Y)X^{m-r} + X^{m-r}\phi(Y)X^{r-1})
\end{aligned}$$

for every $X \in \mathcal{S}$ and $Y \in M_n$. Since the set of such matrices X is dense in M_n, by continuity of the spectrum, we see that

$$\mathrm{Sp}\,(X^{r-1}\phi(Y)X^{m-r}+X^{m-r}\phi(Y)X^{r-1})=\mathrm{Sp}\,(X^{r-1}L(Y)X^{m-r}+X^{m-r}L(Y)X^{r-1}) \tag{2.7}$$

for any $X, Y \in M_n$. It remains to prove the following.

Assertion. *Let $A, B \in M_n$. Then $A = B$ if, for every $X \in M_n$,*

$$\mathrm{Sp}\,(X^{r-1}AX^{m-r} + X^{m-r}AX^{r-1}) = \mathrm{Sp}\,(X^{r-1}BX^{m-r} + X^{m-r}BX^{r-1})$$

Proof. If both $r-1$ and $m-r$ are positive, then for any rank one idempotent $X \in M_n$ we have

$$\begin{aligned}\mathrm{Sp}\,(2AX) &= \mathrm{Sp}\,(XAX + XAX) = \mathrm{Sp}\,(X^{r-1}AX^{m-r} + X^{m-r}AX^{r-1})\\ &= \mathrm{Sp}\,(X^{r-1}BX^{m-r} + X^{m-r}BX^{r-1}) = \mathrm{Sp}\,(XBX + XBX) = \mathrm{Sp}\,(2BX).\end{aligned}$$

Since AX and BX have the same spectrum and have rank at most one, we see that

$$\mathrm{tr}\,(AX) = \mathrm{tr}\,(BX).$$

It follows that $A = B$.

Suppose $r-1$ or $m-r$ is zero. Then

$$\mathrm{Sp}\,(AX + XA) = \mathrm{Sp}\,(BX + XB)$$

for all $X \in \{Z^{m-1} : Z \in M_n\}$, which is a dense set in M_n. By continuity of the spectrum, we may assume that the above equality is true for all $X \in M_n$. We shall assume without loss of generality that A is upper triangular. We claim that B is also upper triangular. Suppose $A = (a_{ij})$, $B = (b_{ij})$, and for every $t \in \mathbb{C}$, let $X_t = E_{11} + tE_{12} + \cdots + t^{n-1}E_{1n}$. Then only the first row of $AX_t + X_tA$ is nonzero and equals $(2a_{11} \ * \ \cdots \ *)$.

Hence $\mathrm{Sp}\,(AX_t + X_tA) = \{2a_{11}, 0\}$. As $\mathrm{Sp}\,(BX_t + X_tB) = \mathrm{Sp}\,(AX_t + X_tA) = \{2a_{11}, 0\}$, $BX_t + X_tB$ has eigenvalues $2a_{11}$ and 0 with certain multiplicities. So,

$$\mathrm{tr}\,(BX_t + X_tB) \in \{2a_{11}, \ldots, 2(n-1)a_{11}\}.$$

Now

$$\begin{aligned}&BX_t + X_tB\\ &\quad= \begin{pmatrix} b_{11} & b_{11}t & \cdots & b_{11}t^{n-1}\\ b_{21} & b_{21}t & \cdots & b_{21}t^{n-1}\\ \vdots & \vdots & & \vdots\\ b_{n1} & b_{n1}t & \cdots & b_{n1}t^{n-1}\end{pmatrix} + \begin{pmatrix} b_n + b_{21}t + \cdots + b_{n1}t^{n-1} & * & \cdots & *\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}\end{aligned}$$

has diagonal entries

$$2b_{11} + b_{21}t + \cdots + b_{n1}t^{n-1}, b_{21}t, \ldots, b_{n1}t^{n-1},$$

and hence

$$\mathrm{tr}\,(BX_t + X_tB) = 2b_{11} + 2b_{21}t + \cdots + 2b_{n1}t^{n-1}$$

is a polynomial in t. It cannot take on a finite number of values only unless it is a constant. The coefficients, except the constant term, are all zero. Hence $b_{21} = \cdots = b_{n1} = 0$. Similarly, by considering $X_t = E_{ii} + tE_{i,i+1} + \cdots + t^{n-i}E_{in}$, we get $b_{i+1,i} = \cdots = b_{ni} = 0$ for $i = 2, \ldots, n-1$. The matrix B is upper triangular.

To show that $A = B$, we first obtain, by putting $X = E_{ii}$, $a_{ii} = b_{ii}$ for every i. For $i < j$, we have

$$AE_{ji} + E_{ji}A = \begin{pmatrix} & & a_{1j} & & & & \\ & & \vdots & & & & \\ & & a_{ij} & & & & \\ & & \vdots & & & & \\ 0 & \cdots & a_{ii} + a_{jj} & \cdots & a_{ij} & \cdots & a_{in} \\ & & 0 & & & & \\ & & \vdots & & & & \\ & & 0 & & & & \end{pmatrix}.$$

Expanding along the jth column, say, we get

$$\det(AE_{ji} + E_{ji}A - \lambda I_n) = (-\lambda)^{n-2}(a_{ij} - \lambda)^2.$$

Hence $\mathrm{Sp}\,(AE_{ji} + E_{ji}A) = \{a_{ij}, 0\}$. Note that $\mathrm{Sp}\,(BE_{ji} + E_{ji}B) = \{b_{ij}, 0\}$. So, $a_{ij} = b_{ij}$. □

3. Results on Hermitian matrices

In this section, we study mappings on H_n that have similar preserving properties as in Section 2. First, we consider products of the form $X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m}$ such that one of the j_r appears only once in $(j_1, \ldots, j_m)$. Even though H_n may not be closed under this product, mappings that preserve the spectrum of the product are in nice form. If we insist that $X_1 * \cdots * X_m \in H_n$, then m is odd, and $r = (m+1)/2$ is the only possible value for j_r to appear once; in particular, $A * B = A^k B A^k$ is the only product we can define on two matrices.

Theorem 3.1. *Suppose $k \geq 2$, and $X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m}$ for a given sequence $(j_1, \ldots, j_m)$ so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there exists j_r not equal to j_s for all $s \neq r$. Then a mapping $\phi : H_n \to H_n$ satisfies*

(3.1) $\quad \mathrm{Sp}\,(\phi(X_1) * \cdots * \phi(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k) \qquad$ *for all $X_1, \ldots, X_k \in H_n$*

if and only if there exist a unitary matrix $S \in M_n$ and a scalar ξ satisfying $\xi^m = 1$ such that

(a) *ϕ has the form $A \mapsto \xi S^* A S$, or*
(b) *$(j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_m) = (j_{r+1}, \ldots, j_m, j_1, \ldots, j_{r-1})$ and ϕ has the form $A \mapsto \xi S^* A^t S$.*

Proof. Again, we only need to consider the sufficiency part. Using similar arguments as in the proof of Theorem 2.1 (cf. Assertions 1 and 2), we can prove that ϕ is equal to a bijective (real) linear mapping L on the dense subset

$$\mathcal{S} = \{X \in H_n : X \text{ has } n \text{ distinct eigenvalues}\},$$

and that L preserves the invertible matrices in H_n. By [4, Theorem 6], there is an invertible matrix $S \in M_n$ such that L is of the form

$$A \mapsto \pm S^* A S \qquad \text{or} \qquad A \mapsto \pm S^* A^t S.$$

From the observations

$$\{1\} = \mathrm{Sp}\,(I_n^m) = \mathrm{Sp}\,(L(I_n)^m) = \mathrm{Sp}\,((\pm S^* S)^m)$$

and that S^*S is positive definite, we conclude that $S^*S = I_n$, i.e., S is unitary. Hence L has the asserted forms.

Also, we can show that $(j_{r+1}, \ldots, j_m, j_1, \ldots, j_{r-1}) = (j_m, \ldots, j_{r+1}, j_{r-1}, \ldots, j_1)$ if L has the form $A \mapsto \xi S^* A^t S$ with the help of the following fact.

Two matrices $A, B \in M_n$ are equal if $\mathrm{Sp}\,(XA) = \mathrm{Sp}\,(XB)$ *for every rank one $X \in H_n$.*

[Note that we use real symmetric matrices in the proof of Assertion 3 in the proof of Theorem 2.1.] Using the above fact again, we can adapt the proof of Assertion 4 in the proof of Theorem 2.1. □

Theorem 3.2. *Suppose $k \geq 2$, and $X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m} + X_{j_m} \cdots X_{j_1}$ for a given sequence $(j_1, \ldots, j_m)$ so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there exists j_r not equal to j_s for all $s \neq r$. Then a mapping $\phi : H_n \to H_n$ satisfies*

$$\mathrm{Sp}\,(\phi(X_1) * \cdots * \phi(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k) \qquad \text{for all } X_1, \ldots, X_k \in H_n \tag{3.2}$$

if and only if there exist a unitary matrix $S \in M_n$ and a scalar ξ satisfying $\xi^m = 1$ such that ϕ has the form

$$A \mapsto \xi S^* A S \qquad \text{or} \qquad A \mapsto \xi S^* A^t S.$$

Proof. We use arguments similar to those in the proof of Theorem 2.2. We need only replace the Assertion in the proof by the following. □

Assertion. *Let $A, B \in H_n$. Then $A = B$ if*

$$\mathrm{Sp}\,(X^{r-1} A X^{m-r} + X^{m-r} A X^{r-1}) = \mathrm{Sp}\,(X^{r-1} B X^{m-r} + X^{m-r} B X^{r-1})$$

for every rank one idempotent $X \in H_n$.

Proof. If both $r-1$ and $m-r$ are positive, we can prove the result using a similar argument as in the proof of Theorem 2.2.

If $r-1$ or $m-r$ is zero, then we have

$$\mathrm{Sp}\,(XA + AX) = \mathrm{Sp}\,(XB + BX) \quad \text{for every rank one idempotent } X \in H_n.$$

We shall assume without loss of generality that A is the diagonal matrix $\mathrm{diag}\,(a_1, \ldots, a_n)$. Putting $X = E_{11}$, we have $AE_{11} + E_{11}A = \mathrm{diag}\,(2a_1, 0, \ldots, 0)$, and hence $\mathrm{Sp}\,(AE_{11} + E_{11}A) = \{2a_1, 0\}$. Let $B = (b_{ij})$. Then

$$BE_{11} + E_{11}B = \begin{pmatrix} 2b_{11} & b_{12} & \ldots & a_{1n} \\ \overline{b_{12}} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \overline{b_{1n}} & 0 & \ldots & 0 \end{pmatrix}.$$

The characteristic polynomial of $BE_{11} + E_{11}B$ is

$$(-\lambda)^{n-2}(\lambda^2 - 2b_{11}\lambda - (|b_{12}|^2 + \cdots + |b_{1n}|^2)).$$

The zeros of the polynomial are $2a_1$ and 0. Now it is easy to see that the polynomial cannot have a nonzero double zero. Hence if $a_1 \neq 0$, $2a_1$ is a simple zero. We have $b_{11} = a_1$ and $b_{12} = \cdots = b_{1n} = 0$. It is obvious that if $a_1 = 0$, then $b_{11} = b_{12} = \cdots = b_{1n} = 0$. Similarly, by putting $X = E_{jj}$, we conclude that $A = B$. □

References

1. L. Baribeau and T. Ransford, Non-linear spectrum preserving maps, Bull. London Math. Soc. 32 (2000), 8-14. MR1718765 (2000j:15008)
2. A. Guterman, C.K. Li, and P. Šemrl, Some general techniques on linear preserver problems, Linear Algebra Appl. 315 (2000), 61–81. MR1774960 (2002a:15003)
3. A.A. Jafarian and A.R. Sourour, Spectrum-preserving linear maps, J. Funct. Anal. 66 (1986), 255–261. MR0832991 (87m:47011)
4. C.-K. Li, L. Rodman and P. Šemrl, Linear maps on selfadjoint operators preserving invertibility, positive definiteness, numerical range, Canad. Math. Bull. 46 (2003), 216–228. MR1981676 (2004b:47063)
5. M. Marcus and B.N. Moyls, Linear transformations on algebras of matrices, Canad. J. Math. 11 (1959), 61–66. MR0099996 (20:6432)
6. L. Molnár, Some characterizations of the automorphisms of $B(H)$ and $C(X)$, Proc. Amer. Math. Soc. 130 (2001), 111–120. MR1855627 (2002m:47047)
7. L. Molnár, Local automorphisms of operator algebras on Banach spaces, Proc. Amer. Math. Soc. 131 (2003), 1867–1874. MR1955275 (2003j:47050)
8. M. Omladič and P. Šemrl, Spectrum-preserving additive maps, Linear Algebra Appl. 153 (1991), 67–72. MR1111386 (92e:47061)

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong

E-mail address: jtchan@hku.hk

Department of Mathematics, College of William and Mary, Williamsburg, Virginia 23187-8795

E-mail address: ckli@math.wm.edu

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong.

E-mail address: NungSingSze@graduate.hku.hk

Current address: Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269-3009