Induced l_2 and Generalized H_2 Filtering for Systems With Repeated Scalar Nonlinearities

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***Abstract*—This paper provides complete results on the filtering problem for a class of nonlinear systems described by a discrete-time state equation containing a repeated scalar nonlinearity as in recurrent neural networks. Both induced l_2 and generalized H_2 indexes are introduced to evaluate the filtering performance. For a given stable discrete-time systems with repeated scalar nonlinearities, our purpose is to design a stable full-order or reduced-order filter with the same repeated scalar nonlinearities such that the filtering error system is asymptotically stable and has a guaranteed induced l_2 or generalized H_2 performance. Sufficient conditions are obtained for the existence of admissible filters. Since these conditions involve matrix equalities, the cone complementarity linearization procedure is employed to cast the nonconvex feasibility problem into a sequential minimization problem subject to linear matrix inequalities, which can be readily solved by using standard numerical software. If these conditions are feasible, a desired filter can be easily constructed. These filtering results are further extended to discrete-time systems with both state delay and repeated scalar nonlinearities. The techniques used in this paper are very different from those used for previous controller synthesis problems, which enable us to circumvent the difficulty of dilating a positive diagonally dominant matrix. A numerical example is provided to show the applicability of the proposed theories.**

***Index Terms*—Diagonally dominant matrix, generalized H_2 performance, induced l_2 performance, linear matrix inequality, recurrent neural networks, repeated scalar nonlinearity.**

I. INTRODUCTION

IN a recent paper [7], Chu and Glover investigate a class of discrete-time nonlinear systems described by the following state-space equation:

$$\begin{aligned} x_i(t+1) &= \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{l} b_{ij}\omega_j(t), \quad i=1,\ldots,n \\ y_i(t) &= \sum_{j=1}^{n} c_{ij} f(x_j(t)) + \sum_{j=1}^{l} d_{ij}\omega_j(t), \quad i=1,\ldots,m \end{aligned} \tag{1}$$

where $f:\mathbb{R}\rightarrow\mathbb{R}$ is nonlinear (not necessarily linear). This system can be abbreviated to

$$\begin{aligned} x(t+1) &= Af(x(t)) + B\omega(t) \\ y(t) &= Cf(x(t)) + D\omega(t) \end{aligned} \tag{2}$$

where $x(t)$, $\omega(t)$, and $y(t)$ represent state, input, and output, respectively. As is indicated in [7], this class of nonlinear systems is analogous to the upper linear fractional transformation uncertain model [2] and could find applications in many practical situations such as recurrent artificial neural networks [1]. With the assumption of being odd and 1-Lipschitz, $f(t)$ encapsulates some typical classes of nonlinearities as its special cases.

Instead of using the popular diagonal Lyapunov function [16], in deriving stability and induced norm performance conditions for system (2), [7] introduces the diagonally dominant Lyapunov function [17] and makes full use of the fact that the nonlinearity on each state-component is the same. Upon their derived stability and performance conditions, the problems of model reduction and controller synthesis have been investigated in their series of works [5], [7], [8].

The diagonally dominant Lyapunov approach developed by Chu and Glover has been well recognized to be more effective concerning conservatism than previous one that utilizes a diagonal Lyapunov function. However, although the stability condition and induced performance conditions are formulated as linear matrix inequalities (LMIs), which can be easily checked via using standard numerical software, it is worth noting that the model reduction and controller synthesis problems based on a diagonally dominant Lyapunov function are much more complicated than that within the diagonal Lyapunov function framework. The main obstacle lies in how to dilate a positive diagonally dominant matrix X to a larger positive diagonally dominant matrix $\bar{X}$ such that the top-left corner of the inverse of $\bar{X}$ is fixed. Notably, necessary and sufficient conditions are still not available, although much effort has been made in recent years. In view of the reduced conservatism of the diagonally dominant Lyapunov approach, in the present paper, following Chu and Glover's work, we further investigate the filtering problem for this class of nonlinear systems, which has received little attention so far. The filtering problem arises in cases when we want to extract some signals, but they cannot be directly measured, which has been extensively investigated for different kinds of systems (see, for instance, [4], [25], and the references therein). Generally speaking, for a linear system, the filtering problem can be solved by applying similar techniques used for controller synthesis, but here, in this paper, to avoid the difficulty of dilating a positive diagonally dominant matrix as mentioned above, we shall seek new techniques to

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solve the filtering problem, which is significantly different from that used in previous references. In summary, the main objective of the present work is to solve the filtering problem for linear systems with repeated scalar nonlinearities based on a diagonally dominant Lyapunov function, in which novel techniques circumventing the difficulty of dilating a diagonally dominant matrix will be utilized.

In solving the filtering problem for this class of nonlinear systems, in this paper, both induced l_2 and generalized H_2 indexes are introduced to evaluate the filtering performance. For a given stable discrete-time systems with repeated scalar nonlinearities, our purpose is to design a stable full- or reduced-order filter with the same repeated scalar nonlinearities such that the filtering error system is asymptotically stable and has a guaranteed induced l_2 or generalized H_2 performance. Sufficient conditions are obtained for the existence of admissible filters. Since these conditions involve matrix equalities, the cone complementarity linearization (CCL) procedure is employed to cast the nonconvex feasibility problem into a sequential minimization problem subject to LMIs, which can be readily solved by using standard numerical software [11]. If these conditions are feasible, a desired filter can be easily constructed. These filtering results are further extended to discrete-time systems with both state delay and repeated scalar nonlinearities. A numerical example is provided to show the applicability of the proposed theories.

Notations: The notations used throughout the paper are fairly standard. The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space ($\mathbb{R}$ for $n = 1$), $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$ and the notation $P > 0$ means that P is real symmetric and positive definite; I and 0 represent identity matrix and zero matrix, respectively; the notation $|\cdot|$ refers to the Euclidean vector norm; $\text{tr}(M)$ and $\|M\|$ refer to the trace and 2-norm (spectral norm) of the matrix M, respectively. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry, and $\text{diag}\{\ldots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. For a matrix $U \in \mathbb{R}^{m\times n}$ with rank k, we denote $U^{\perp}$ as the orthogonal complement, which is defined as a (possibly nonunique) $(m-k)\times m$ matrix with rank $(m-k)$, such that $U^{\perp}U = 0$. The space of square summable infinite sequence is denoted by $l_2[0,\infty)$, and for $\omega = \{\omega(t)\} \in l_2[0,\infty)$, its norm is given by $\|\omega\|_2 = \sqrt{\Sigma_{t=0}^{\infty}|\omega(t)|^2}$.

II. Problem Description

Consider the discrete-time nonlinear system described by the following state equation

$$\begin{aligned}\mathcal{S}: x(t+1) &= Af(x(t)) + B\omega(t)\\ y(t) &= Cf(x(t)) + D\omega(t)\\ z(t) &= Hf(x(t))\end{aligned} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ represents the state vector; $y(t) \in \mathbb{R}^m$ is the measured output; $z(t) \in \mathbb{R}^p$ is the signal to be estimated; $\omega(t) \in \mathbb{R}^l$ is the disturbance input which belongs to $l_2[0,\infty)$; (A,B,C,D,H) are system matrices with compatible dimensions; $f:\mathbb{R}\to\mathbb{R}$ is nonlinear, and

$$f(x) \triangleq [f(x_1(t)) \quad f(x_2(t)) \quad \cdots \quad f(x_n(t))]^T$$

As is in [7], we make the following assumption on the nonlinear function f.

Assumption 1: The nonlinear function f in system (3) is assumed to satisfy

$$\forall x,y\in\mathbb{R}, \quad |f(x)+f(y)| \le |x+y|. \tag{4}$$

Remark 1: The assumption on f means that f is odd (by putting $y=-x$) and 1-Lipschitz (by putting $y=-y$), and therefore, f encapsulates some typical classes of nonlinearities, such as

- the hyperbolic tangent $\tanh$, which is popularly used for activation function in neural networks;
- the semilinear function, that is, the standard saturation $\text{sat}(t) \triangleq t$ if $|t| \le 1$ and $\text{sat}(t) \triangleq \text{sgn}(t)$ if $|t| > 1$;
- the sine function $\sin$ etc.

In this paper, for the nonlinear system $\mathcal{S}$ in (3), we are interested in designing a nonlinear filter $\mathcal{F}$ of the following form:

$$\begin{aligned}\mathcal{F}: x_F(t+1) &= A_Ff(x_F(t)) + B_Fy(t)\\ z_F(t) &= C_Ff(x_F(t))\end{aligned} \tag{5}$$

where $x_F(t) \in \mathbb{R}^k$ is the filter state vector, $z_F(t) \in \mathbb{R}^p$ is the output signal of the filter which is used for an estimation of $z(t)$, and (A_F,B_F,C_F) are appropriately dimensioned filter matrices to be determined. It should be pointed out that here, we are interested not only in the full-order filtering problem (when $k=n$) but in the reduced-order filtering problem as well (when $1\le k<n$). As can be seen in the subsequent sections, these two filtering problems are solved in a unified framework.

Augmenting the model of $\mathcal{S}$ to include the states of the filter $\mathcal{F}$, we obtain the filtering error system $\mathcal{E}$:

$$\begin{aligned}\mathcal{E}: \xi(t+1) &= \bar{A}f(\xi(t)) + \bar{B}\omega(t)\\ e(t) &= \bar{C}f(\xi(t))\end{aligned} \tag{6}$$

where $\xi(t) = [x^T(t)\; x_F^T(t)]^T$, $e(t) = z(t) - z_F(t)$, and

$$\bar{A} = \begin{bmatrix} A & 0\\ B_FC & A_F\end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B\\ B_FD\end{bmatrix}, \quad \bar{C} = [H \quad -C_F]. \tag{7}$$

Before presenting the main objectives of this paper, we first introduce the following definitions for the filtering error system $\mathcal{E}$ in (6).

Definition 1: The filtering error system $\mathcal{E}$ in (6) with $\omega(t)=0$ is said to be stable if, for any $\epsilon>0$, there is a $\delta(\epsilon)>0$ such that $|\xi(t)|<\epsilon$, $t>0$ when $|\xi(0)|<\delta(\epsilon)$. In addition, if $\lim_{t\to\infty}|\xi(t)|=0$ for any initial conditions, then the filtering error system $\mathcal{E}$ in (6) is said to be asymptotically stable.

Definition 2: Given a scalar $\gamma>0$, the filtering error system $\mathcal{E}$ in (6) is said to be asymptotically stable with an induced l_2 disturbance attenuation γ if it is asymptotically stable and under zero initial conditions $\|e\|_2 < \gamma\|\omega\|_2$ for all nonzero $\omega\in l_2[0,\infty)$.

Definition 3: Given a scalar $\gamma > 0$, the filtering error system $\mathcal{E}$ in (6) is said to be asymptotically stable with a generalized H_2 disturbance attenuation γ if it is asymptotically stable and under zero initial conditions $\|e\|_\infty < \gamma\|\omega\|_2$ for all nonzero $\omega \in l_2[0,\infty)$, where $\|e\|_\infty \triangleq \sup_t \sqrt{|e(t)|^2}$.

Then, the problems to be solved are expressed as follows.

Induced l_2 Filtering Problem: Given an asymptotically stable system $\mathcal{S}$ in (3), develop full-order and reduced-order filters of the form $\mathcal{F}$ in (5) such that for all admissible $\omega \in l_2[0,\infty)$, the filtering error system $\mathcal{E}$ in (6) is asymptotically stable with an induced l_2 disturbance attenuation level γ. Filters guaranteeing such a performance are called induced l_2 filters.

Generalized H_2 Filtering Problem: Given an asymptotically stable system $\mathcal{S}$ in (3), develop full- and reduced-order filters of the form $\mathcal{F}$ in (5) such that for all admissible $\omega \in l_2[0,\infty)$, the filtering error system $\mathcal{E}$ in (6) is asymptotically stable with a generalized H_2 disturbance attenuation level γ. Filters guaranteeing such a performance are called generalized H_2 filters.

The following definition and lemmas will be used extensively in the paper.

Definition 4: A square matrix $M \triangleq [m_{ij}] \in \mathbb{R}^{n\times n}$ is called diagonally dominant if for all $i = 1,\ldots,n$

$$m_{ii} \geq \sum_{j\neq i} |m_{ij}|.$$

Lemma 1 [10]: Let $W = W^T \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times m}$, and $V \in \mathbb{R}^{k\times n}$ be given matrices, and suppose $\text{rank}(U) < n$, and $\text{rank}(V) < n$. Consider the problem of finding some matrix $\mathcal{G}$ satisfying

$$W + U\mathcal{G}V + (U\mathcal{G}V)^T < 0 \tag{8}$$

Then, (8) is solvable for $\mathcal{G}$ if and only if

$$U^\perp W U^{\perp T} < 0,\ V^{T\perp} W V^{T\perp T} < 0. \tag{9}$$

Lemma 2 [7]: Supposing that $M \geq 0$ is diagonally dominant, then for all nonlinear functions f satisfying (4), it holds that

$$f^T(x) M f(x) \leq x^T M x$$

for all x.

III. Induced l_2 Filtering

In this section, we will address the induced l_2 filtering problem for systems with repeated scalar nonlinearities. We first introduce the following lemma, which is crucial to our filter synthesis development (see, for instance, [7]).

Lemma 3: Consider system $\mathcal{S}$ in (3), and suppose the filter matrices (A_F, B_F, C_F) of $\mathcal{F}$ in (5) are given. Then, the filtering error system $\mathcal{E}$ in (6) is asymptotically stable with an induced l_2 disturbance attenuation level bound γ if there exists a positive diagonally dominant matrix $\mathcal{P}$ satisfying

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix}^T \begin{bmatrix} \mathcal{P} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} - \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0. \tag{10}$$

This lemma is first presented in [7] and has been used to solve the controller design problems [8]. It is worth noting that although the form of (10) is quite similar to the bounded real lemma for linear discrete-time systems (see, for instance, [3] and [22]), the additional diagonally dominant constraint on the positive definite matrix $\mathcal{P}$ makes the controller synthesis problem as well as the model reduction problem very complicated. As indicated in [8], the main reason is that it is difficult to dilate a positive diagonally dominant matrix X to a larger positive diagonally dominant matrix $\bar{X}$ such that the top-left corner of the inverse of $\bar{X}$ is fixed. Necessary and sufficient conditions are still not available so far, and therefore, the controller synthesis results presented in [8] are only sufficient conditions for their corresponding analysis results. Since the filter design can be seen as a dual problem of control, it is usually possible to solve the filtering problem by following similar lines as used for controller synthesis. It is noted that if we still use the technique developed in [8] to solve the filtering problem, the difficulty of dilating a positive diagonally dominant matrix still exists. Since the dilation problem is basically brought by partitioning the positive diagonally dominant matrix, this difficulty would disappear if we do not partition the matrix when solving the synthesis problems. As can be seen in the following development, the positive diagonally dominant matrix $\mathcal{P}$ in (10) will remain in its original form (not to be partitioned) in solving the filtering problems, and therefore, the matrix dilation problem will not exist in our results.

In the following, we will focus on the design of full- and reduced-order induced l_2 filters of the form $\mathcal{F}$ based on Lemma 3, that is, to determine the filter matrices (A_F, B_F, C_F) that will guarantee the filtering error system $\mathcal{E}$ to be asymptotically stable with an induced l_2 performance. The following theorem provides sufficient conditions for the existence of such induced l_2 filters for system $\mathcal{S}$.

Theorem 1: Consider system $\mathcal{S}$ in (3). Then, an admissible induced l_2 filter of the form $\mathcal{F}$ in (5) exists if there exist matrices $0 < \mathcal{P} \triangleq [p_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$, $\mathcal{L} > 0$, $\mathcal{X} > 0$, $\mathcal{R} = \mathcal{R}^T \triangleq [r_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$ satisfying

$$\begin{bmatrix} -J\mathcal{L}J^T & J\bar{A}_0 & J\bar{B}_0 \\ * & -\mathcal{P}+\mathcal{X} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{11}$$

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & M^\perp \end{array}\right] \left[\begin{array}{c|cc} -\mathcal{L} & \bar{A}_0 & \bar{B}_0 \\ \hline * & -\mathcal{P}+\mathcal{X} & 0 \\ * & * & -\gamma^2 I \end{array}\right] \times \left[\begin{array}{c|c} I & 0 \\ \hline 0 & M^\perp \end{array}\right]^T < 0 \tag{12}$$

$$\begin{bmatrix} -I & \bar{C}_0 J^T \\ * & -J\mathcal{X}J^T \end{bmatrix} < 0 \tag{13}$$

$$p_{ii} - \sum_{j\neq i}(p_{ij} + 2r_{ij}) \geq 0, \quad \forall i \tag{14}$$

$$r_{ij} \geq 0, \quad \forall i \neq j \tag{15}$$

$$p_{ij} + r_{ij} \geq 0, \quad \forall i \neq j \tag{16}$$

$$\mathcal{P}\mathcal{L} = I. \tag{17}$$

Furthermore, if $(\mathcal{P},\mathcal{L},\mathcal{X},\mathcal{R})$ is a feasible solution of (11)–(17), then the system matrices of an admissible induced l_2 filter in the form of (5) are given by

$$\begin{aligned}\mathcal{G}_i =& -\Pi_i^{-1}U_i^T\Lambda_i V_i^T\left(V_i\Lambda_i V_i^T\right)^{-1}\\ &+\Pi_i^{-1}\Xi_i^{\frac{1}{2}}L_i\left(V_i\Lambda_i V_i^T\right)^{-\frac{1}{2}}\\ \Lambda_i =& \left(U_i\Pi_i^{-1}U_i^T-W_i\right)^{-1}>0\\ \Xi_i =& \Pi_i - U_i^T\left(\Lambda_i-\Lambda_i V_i^T\left(V_i\Lambda_i V_i^T\right)^{-1}V_i\Lambda_i\right)U_i \quad (18)\end{aligned}$$

where $\mathcal{G}_1 \triangleq [A_F\ B_F]$, $\mathcal{G}_2 \triangleq C_F$; Π_i, L_i, $i=1, 2$ are any appropriately dimensioned matrices satisfying $\Pi_i>0$, $\|L_i\|<1$, and

$$\begin{aligned}&\bar{A}_0=\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix},\quad \bar{B}_0=\begin{bmatrix}B\\ 0\end{bmatrix},\quad \bar{C}_0=[H\quad 0]\\ &M=\begin{bmatrix}0 & C^T\\ I & 0\\ 0 & D^T\end{bmatrix},\quad E=\begin{bmatrix}0\\ I\end{bmatrix},\quad R=\begin{bmatrix}0 & I\\ C & 0\end{bmatrix}\\ &S=\begin{bmatrix}0\\ D\end{bmatrix},\quad T=[0\quad -I],\quad J=[I\quad 0]\\ &W_1=\begin{bmatrix}-\mathcal{P}^{-1} & \bar{A}_0 & \bar{B}_0\\ * & -\mathcal{P}+\mathcal{X} & 0\\ * & * & -\gamma^2 I\end{bmatrix},\quad U_1=\begin{bmatrix}E\\ 0\\ 0\end{bmatrix}\\ &V_1=[0\quad R\quad S]\\ &W_2=\begin{bmatrix}-I & \bar{C}_0\\ * & -\mathcal{X}\end{bmatrix},\quad U_2=\begin{bmatrix}I\\ 0\end{bmatrix},\quad V_2=[0\quad T]. \quad (19)\end{aligned}$$

Proof: From Lemma 3, we know that there exists an admissible filter $\mathcal{F}$ in the form of (5) such that the filtering error system $\mathcal{E}$ in (6) is asymptotically stable with an induced l_2 disturbance attenuation level bound γ if there exists a positive diagonally dominant matrix $\mathcal{P}$ satisfying (10). By the Schur complement [3], (10) is equivalent to

$$\begin{bmatrix}-\mathcal{P}^{-1} & 0 & \bar{A} & \bar{B}\\ * & -I & \bar{C} & 0\\ * & * & -\mathcal{P} & 0\\ * & * & * & -\gamma^2 I\end{bmatrix}<0. \quad (20)$$

By introducing an additional matrix variable $\mathcal{X}$, (20) can be split into two inequalities as in

$$\begin{bmatrix}-\mathcal{P}^{-1} & \bar{A} & \bar{B}\\ * & -\mathcal{P}+\mathcal{X} & 0\\ * & * & -\gamma^2 I\end{bmatrix}<0 \quad (21)$$

$$\begin{bmatrix}-I & \bar{C}\\ * & -\mathcal{X}\end{bmatrix}<0. \quad (22)$$

Now, rewrite (7) in the following form:

$$\begin{aligned}\bar{A}&=\bar{A}_0+E[A_F\quad B_F]R\\ \bar{B}&=\bar{B}_0+E[A_F\quad B_F]S\\ \bar{C}&=\bar{C}_0+C_F T \quad (23)\end{aligned}$$

where $\bar{A}_0$, $\bar{B}_0$, $\bar{C}_0$, E, R, S, and T are defined in (19).

Noticing (23), (21) and (22) can be rewritten as

$$W_1+U_1[A_F\ B_F]V_1+(U_1[A_F\ B_F]V_1)^T<0 \quad (24)$$

$$W_2+U_2C_FV_2+(U_2C_FV_2)^T<0 \quad (25)$$

where W_i, U_i, V_i, and $i=1\ 2$, are defined in (19).

If we choose

$$U_1^{\perp}=\begin{bmatrix}J & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix},\quad V_1^{T\perp}=\left[\begin{array}{c|c}I & 0\\ \hline 0 & M^{\perp}\end{array}\right] \quad (26)$$

then, by using Lemma 1, (24) is solvable for $[A_F\ B_F]$ if and only if

$$\begin{bmatrix}-J\mathcal{P}^{-1}J^T & J\bar{A}_0 & J\bar{B}_0\\ * & -\mathcal{P}+\mathcal{X} & 0\\ * & * & -\gamma^2 I\end{bmatrix}<0 \quad (27)$$

$$\left[\begin{array}{c|c}I & 0\\ \hline 0 & M^{\perp}\end{array}\right]\left[\begin{array}{c|cc}-\mathcal{P}^{-1} & \bar{A}_0 & \bar{B}_0\\ \hline * & -\mathcal{P}+\mathcal{X} & 0\\ * & * & -\gamma^2 I\end{array}\right]\times\left[\begin{array}{c|c}I & 0\\ \hline 0 & M^{\perp}\end{array}\right]^T<0 \quad (28)$$

which are equivalent to (11) and (12) by noticing (17).

Choose

$$U_2^{\perp}=[0\quad I],\quad V_2^{T\perp}=\begin{bmatrix}I & 0\\ 0 & J\end{bmatrix}. \quad (29)$$

Then, by using Lemma 1, (25) is solvable for C_F if and only if $\mathcal{X}>0$ and (13) hold.

In addition, from (14)–(16), we have

$$p_{ii}\ge\sum_{j\neq i}(p_{ij}+2r_{ij})=\sum_{j\neq i}(|p_{ij}+r_{ij}|+|-r_{i,j}|)\ge\sum_{j\neq i}|p_{ij}|$$

which guarantees the positive definite matrix $\mathcal{P}$ to be diagonally dominant [in fact, (14)–(16) are also necessary conditions for $\mathcal{P}$ to be diagonally dominant as shown in [7, Lemma 5]). Then, the first part of the proof is completed. The second part of the theorem is immediate by [10] and [15], and the proof is completed. □

Remark 2: It is worth noting that in order to solve the filter synthesis problem, we first separate the induced l_2 performance into two matrix inequalities, which is enabled by the introduction of the additional matrix variable $\mathcal{X}$ [see (21) and (22)]. This separation is, in our opinion, crucial to solving the filter synthesis problem by using the Projection Lemma (Lemma 1).

Remark 3: Theorem 1 presents sufficient conditions for the existence of admissible induced l_2 filters. It is worth noting that the technique used for deriving these conditions is much different from that used for controller synthesis problem in [8]. Unlike in [8], here, the positive diagonally dominant matrix $\mathcal{P}$ remains in its original form; therefore, the difficulty of dilating a positive diagonally dominant matrix mentioned previously does not occur in our result. In this case, the diagonally dominant property of $\mathcal{P}$ can be easily guaranteed by the LMIs (14)–(16).

Although the difficulty of dilating matrix is circumvented in our result, one may argue that the conditions presented in Theorem 1 still cannot be directly solved by available numerical software due to the existence of the matrix equality in (17). However, such a problem can be solved by applying the cone complementarity linearization idea proposed in [9]. The basic idea in the CCL algorithm is that if the LMI $\begin{bmatrix} \mathcal{P} & I \\ I & \mathcal{L} \end{bmatrix} \geq 0$ is feasible in the $n \times n$ matrix variables $\mathcal{L} > 0$ and $\mathcal{P} > 0$, then $\text{tr}(\mathcal{PL}) \geq n$, and $\text{tr}(\mathcal{PL}) = n$ if and only if $\mathcal{PL} = I$. In view of this, it is possible to solve the equalities in (17) with the application of CCL algorithm.

Based on the above discussion, we suggest the following nonlinear minimization problem involving LMI conditions instead of the original nonconvex feasibility problem formulated in Theorem 1.

Problem Il2F (Induced l_2 Filtering):

$$\min \text{tr}(\mathcal{PL}) \text{ subject to (11)-(16) and } \begin{bmatrix} \mathcal{P} & I \\ I & \mathcal{L} \end{bmatrix} \geq 0 \tag{30}$$

According to [9], if the solution of the above minimization problem is $n+k$, that is, $\min \text{tr}(\mathcal{PL}) = n+k$, then the conditions in Theorem 1 are solvable. Although it is still not possible to always find a global optimal solution, the proposed nonlinear minimization problem is easier to solve than the original nonconvex feasibility problem. Actually, we can readily modify [9, Algo. 1] to solve the above nonlinear problem.

Algorithm Il2F

Step 1 Find a feasible set $(\mathcal{P}_{(0)}, \mathcal{L}_{(0)}, \mathcal{X}_{(0)}, \mathcal{R}_{(0)})$ satisfying (11)-(16) and (30). Set $q = 0$.

Step 2 Solve the following LMI problem:

$$\min \text{ tr}\left(\mathcal{P}\mathcal{L}_{(q)} + \mathcal{P}_{(q)}\mathcal{L}\right) \text{ subject to (11)-(16) and (30).}$$

Step 3 Substitute the obtained matrix variables $(\mathcal{P}, \mathcal{L}, \mathcal{X}, \mathcal{R})$ into (27) and (28). If conditions (27) and (28) are satisfied, then output the feasible solutions $(\mathcal{P}, \mathcal{L}, \mathcal{X}, \mathcal{R})$. EXIT.

Step 4 If $q > N$, where N is the maximum number of iterations allowed, EXIT.

Step 5 Set $q = q+1$, $(\mathcal{P}_{(q)}, \mathcal{L}_{(q)}, \mathcal{X}_{(q)}, \mathcal{R}_{(q)}) = (\mathcal{P}, \mathcal{L}, \mathcal{X}, \mathcal{R})$, and go to Step 2.

Remark 4: In Algorithm Il2F, we use (27) and (28) as stopping criterion since it can be numerically difficult in practice to obtain the optimal solution such that $\min \text{tr}(\mathcal{PL})$ is exactly equal to $n+k$. Algorithm Il2F can be used to solve the feasibility problem in Theorem 1 for a given constant γ. However, it is not difficult to further modify Algorithm Il2F to obtain the minimal value of γ in terms of the feasibility of (11)–(17).

Remark 5: It is worth pointing out that the technique used here for solving the filtering problem is also much different from previous results in the filtering area. The techniques used in [13], [18], [19], etc. fall into the variable linearization category [20], where the original nonlinear matrix inequality is transformed into linear matrix inequality by performing congruence transformations and by defining new matrix variables. However, this approach is not suitable to solve the filtering problems for this class of nonlinear systems under investigation. The main reason is that the performed congruence transformation will involve the positive diagonally dominant matrix $\mathcal{P}$, and thus, it is difficult to guarantee the diagonally dominant property of matrix $\mathcal{P}$. On the other hand, although [14] and [24] also use the well-known Projection Lemma (Lemma 1), they are quite different from the results obtained in this paper, since they also involve the partition of the positive matrix $\mathcal{P}$, and thus, the dilation difficulty will arise when used for the problem addressed here.

Remark 6: The conditions presented in Theorem 1 are not convex due to the existence of the matrix equality in (17). In the above, we have developed an iterative LMI-based algorithm to solve these conditions. One immediate question that might interest readers is that of when these conditions are reduced to convex ones. To this end, from the proof of Theorem 1, we can see that if we replace conditions (14) and (15) by

$$\begin{bmatrix} J(\mathcal{P}-2I)J^T & J\bar{A}_0 & J\bar{B}_0 \\ * & -\mathcal{P}+\mathcal{X} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{31}$$

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & M^{\perp} \end{array}\right] \left[\begin{array}{c|cc} \mathcal{P}-2I & \bar{A}_0 & \bar{B}_0 \\ \hline * & -P+\mathcal{X} & 0 \\ * & * & -\gamma^2 I \end{array}\right] \times \left[\begin{array}{c|c} I & 0 \\ \hline 0 & M^{\perp} \end{array}\right]^T < 0 \tag{32}$$

based on the fact that $(\mathcal{P}^{-1} - I)\mathcal{P}(\mathcal{P}^{-1} - I) \geq 0$ (which gives $-\mathcal{P}^{-1} \leq \mathcal{P} - 2I$). Then, the resultant filtering synthesis conditions are reduced to convex ones, leading to Corollary 1.

Corollary 1: Consider system $\mathcal{S}$ in (3). Then, an admissible induced l_2 filter of the form $\mathcal{F}$ in (5) exists if there exist matrices $0 < \mathcal{P} \triangleq [p_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$, $\mathcal{X} > 0$, $\mathcal{R} = \mathcal{R}^T \triangleq [r_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$ satisfying (13)–(16), (31), and (32).

Remark 7: It is worth mentioning that the filter designed above is of general structure. Alternatively, one may construct a full-order observer-type filter to estimate the signal $z(t)$ in (3), which takes the form

$$\begin{aligned} x_F(t+1) &= Af(x_F(t)) + B_F(y(t) - Cf(x_F(t))) \\ z_F(t) &= Hf(x_F(t)). \end{aligned} \tag{33}$$

Hence, the induced l_2 filter design finds the matrix gain B_F such that the gain from $\omega(t)$ to $z(t) - z_F(t)$ is minimized for the following incremental system:

$$\begin{aligned} x(t+1) - x_F(t+1) &= (A - B_F C)\left(f(x(t)) - f(x_F(t))\right) \\ &\quad + (B - B_F D)\omega(t) \\ z(t) - z_F(t) &= H\left(f(x(t)) - f(x_F(t))\right) \end{aligned}$$

Some results to compute the gain matrix B_F have been proposed in [6], which relax the requirement of a diagonal Lyapunov function. An interesting and meaningful future research work will be a comparison between the approaches presented in this paper and in [6].

IV. Generalized H_2 Filtering

The generalized H_2 performance was first proposed in [21]. The filtering problem based on such a performance is usually called energy-to-peak filtering and was first solved in [14], where both full- and reduced-order filters are designed for systems with exactly known matrices. Subsequent works on generalized H_2 filtering problems can be found in [12], [13], and [19]. In this section, we will investigate the problem of the generalized H_2 filtering for system $\mathcal{S}$ in (3). This problem will be solved by employing similar technique used above for the induced l_2 filtering problem. The following lemma presents a generalized H_2 performance for systems with repeated scalar nonlinearity, which has not been studied previously.

Lemma 4: Consider system $\mathcal{S}$ in (3), and suppose the filter matrices (A_F, B_F, C_F) of $\mathcal{F}$ in (5) are given. Then, the filtering error system $\mathcal{E}$ in (6) is asymptotically stable with a generalized H_2 disturbance attenuation level bound γ if there exists a positive diagonally dominant matrix $\mathcal{P}$ satisfying

$$\Pi \triangleq \begin{bmatrix} \bar{A}^T\mathcal{P}\bar{A} - \mathcal{P} & \bar{A}^T\mathcal{P}\bar{B} \\ * & \bar{B}^T\mathcal{P}\bar{B} - I \end{bmatrix} < 0 \tag{34}$$

$$\begin{bmatrix} -\gamma^2 I & \bar{C} \\ * & -\mathcal{P} \end{bmatrix} < 0. \tag{35}$$

Proof: Define the following Lyapunov function candidate

$$V(\xi(t), t) \triangleq \xi^T(t)\mathcal{P}\xi(t)$$

where $\mathcal{P}$ is the positive diagonally dominant matrix to be determined. First, (34) implies $\bar{A}^T\mathcal{P}\bar{A} - \mathcal{P} < 0$. Then, by [7, Prop. 3], the filtering error system with $\omega(t) \equiv 0$ is guaranteed to be asymptotically stable. To establish the energy-to-peak performance, we assume zero initial condition, that is, $\xi = 0$, and then, we have $V(\xi(t), t)|_{t=0} = 0$. Considering the index

$$\mathcal{J}_1 \triangleq V(\xi(t), t) - \sum_{s=0}^{t-1} \omega^T(s)\omega(s) \tag{36}$$

then for any nonzero $\omega \in l_2[0, \infty)$ and $t > 0$, it holds that

$$\begin{aligned} \mathcal{J}_1 &= V(\xi(t), t) - V(\xi(t), t)|_{t=0} - \sum_{s=0}^{t-1} \omega^T(s)\omega(s) \\ &= \sum_{s=0}^{t-1} \left[\Delta V(\xi(s), s) - \omega^T(s)\omega(s)\right] \end{aligned}$$

where

$$\begin{aligned} \Delta V(\xi(s), s) &= V(\xi(s+1), s+1) - V(\xi(s), s) \\ &\le f^T(\xi(s))\bar{A}^T\mathcal{P}\bar{A}f(\xi(s)) \\ &\quad + 2f^T(\xi(s))\bar{A}^T\mathcal{P}\bar{B}\omega(s) \\ &\quad + \omega^T(s)\bar{B}^T\mathcal{P}\bar{B}\omega(s) - f^T(\xi(s))\mathcal{P}f(\xi(s)). \end{aligned}$$

Note that the last step of the above derivation makes use of Lemma 2. Then, we have

$$\mathcal{J}_1 \le \sum_{s=0}^{t-1} \begin{bmatrix} f(\xi(s)) \\ \omega(s) \end{bmatrix}^T \Pi \begin{bmatrix} f(\xi(s)) \\ \omega(s) \end{bmatrix}$$

where Π is defined in (34). Thus, LMI (34) guarantees $\mathcal{J}_1 \le 0$, which further implies

$$f^T(\xi(t))\mathcal{P}f(\xi(t)) \le \xi^T(t)\mathcal{P}\xi(t) \le \sum_{s=0}^{t-1} \omega^T(s)\omega(s).$$

On the other hand, using the Schur complement, LMI (35) guarantees $\bar{C}^T\bar{C} < \gamma^2\mathcal{P}$. Then, it can be easily established that for all $t > 0$

$$\begin{aligned} |e(t)|^2 &= e^T(t)e(t) < \gamma^2 f^T(\xi(t))\mathcal{P}f(\xi(t)) \\ &\le \gamma^2 \sum_{s=0}^{t-1} \omega^T(s)\omega(s) \le \gamma^2 \sum_{s=0}^{\infty} \omega^T(s)\omega(s). \end{aligned} \tag{37}$$

Taking the supremum over $t > 0$ yields $\|e\|_\infty < \gamma\|\omega\|_2$ for all nonzero $\omega \in l_2[0, \infty)$. □

Then, we are in a position to solve the generalized H_2 filtering problem.

Theorem 2: Consider system $\mathcal{S}$ in (3). Then, an admissible generalized H_2 filter of the form $\mathcal{F}$ in (5) exists if there exist matrices $0 < \mathcal{P} \triangleq [p_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$, $\mathcal{L} > 0$, $\mathcal{R} = \mathcal{R}^T \triangleq [r_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$ satisfying (14)–(17), and

$$\begin{bmatrix} -J\mathcal{L}J^T & J\bar{A}_0 & J\bar{B}_0 \\ * & -\mathcal{P} & 0 \\ * & * & -I \end{bmatrix} < 0 \tag{38}$$

$$\begin{aligned} &\left[\begin{array}{c|c} I & 0 \\ \hline 0 & M^\perp \end{array}\right] \left[\begin{array}{c|cc} -\mathcal{L} & \bar{A}_0 & \bar{B}_0 \\ \hline * & -\mathcal{P} & 0 \\ * & * & -I \end{array}\right] \\ &\times \left[\begin{array}{c|c} I & 0 \\ \hline 0 & M^\perp \end{array}\right]^T < 0 \end{aligned} \tag{39}$$

$$\begin{bmatrix} -\gamma^2 I & \bar{C}_0 J^T \\ * & -J\mathcal{P}J^T \end{bmatrix} < 0. \tag{40}$$

Furthermore, if $(\mathcal{P}, \mathcal{L}, \mathcal{R})$ is a feasible solution of the above conditions, then the system matrices of an admissible generalized H_2 filter in the form of (5) are given by (18), where $\mathcal{G}_1 \triangleq [A_F \; B_F]$, $\mathcal{G}_2 \triangleq C_F$; Π_i, L_i, $i = 1, 2$ are any appropriately dimensioned matrices satisfying $\Pi_i > 0$, $\|L_i\| < 1$; $\bar{A}_0$, $\bar{B}_0$, $\bar{C}_0$, E, R, S, T, M, and J are defined in (19), and

$$\begin{aligned} W_1 &= \begin{bmatrix} -\mathcal{P}^{-1} & \bar{A}_0 & \bar{B}_0 \\ * & -\mathcal{P} & 0 \\ * & * & -I \end{bmatrix}, \quad U_1 = \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix} \\ V_1 &= [0 \quad R \quad S], \quad V_2 = [0 \quad T] \\ W_2 &= \begin{bmatrix} -\gamma^2 I & \bar{C}_0 \\ * & -\mathcal{P} \end{bmatrix}, \quad U_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}. \end{aligned} \tag{41}$$

Proof: The theorem can be proved by following similar lines as in the proof of Theorem 1. From Lemma 4, we know that there exists an admissible filter $\mathcal{F}$ in the form of (5) such that the filtering error system $\mathcal{E}$ in (6) is asymptotically stable with a generalized H_2 disturbance attenuation level bound γ if there exists a positive diagonally dominant matrix $\mathcal{P}$ satisfying (34) and (35).

First, rewrite (7) in the form (23). By the Schur complement, (34) is equivalent to

$$\begin{bmatrix} -\mathcal{P}^{-1} & \bar{A} & \bar{B} \\ * & -\mathcal{P} & 0 \\ * & * & -I \end{bmatrix} < 0. \tag{42}$$

Then, (42) and (35) can be rewritten as (24) and (25) but with different matrices W_i, U_i, V_i, $i = 1$, and 2, which are given in (41).

If we choose $U_1^{\perp}$ and $V_1^{T\perp}$ as in (26), then by using Lemma 1, (24) is solvable for $[A_F \; B_F]$ if and only if (38) and (39) hold by considering (17). Choose $U_2^{\perp}$ and $V_2^{T\perp}$ as in (29); then, by using Lemma 1, (25) is solvable for C_F if and only if $\mathcal{P} > 0$ and (40) hold. In addition, (14)–(16) guarantee the positive definite matrix $\mathcal{P}$ to be diagonally dominant.

The second part of the theorem is immediate by [10] and [15], and the proof is completed. □

Similar to the induced l_2 filtering case, the obtained conditions in Theorem 2 are not strict LMI conditions due to the matrix equality in (17). We suggest the following nonlinear minimization problem involving LMI conditions instead of the original nonconvex feasibility problem formulated in Theorem 2.

Problem GH2F (Generalized H_2 Filtering):

$$\min \; \mathrm{tr}(\mathcal{PL}) \text{ subject to (14)-(16), (30), and (38)-(40).}$$

If the solution of the above minimization problem is $n + k$, that is, $\min \mathrm{tr}\mathcal{PL} = (n + k)$, then the conditions in Theorem 2 are solvable. An iterative algorithm similar to Algorithm Il2F can be proposed to solve Problem GH2F.

V. Delay Systems

Since time delay exists commonly in dynamic systems and has been generally regarded as a main source of instability and poor performance, in recent years, much attention has been devoted to systems with state delay. In the case that time delay appears for systems with repeated scalar nonlinearities, both analysis and synthesis results have yet to be reported.

In this section, we make an attempt to investigate the filtering problems for time-delay systems with repeated scalar nonlinearity. More specifically, we consider the following system:

$$\begin{aligned} \mathcal{S}_d : x(t+1) &= Af(x(t)) + A_d g(x(t-d)) + B\omega(t) \\ y(t) &= Cf(x(t)) + C_d g(x(t-d)) + D\omega(t) \\ z(t) &= Hf(x(t)) \\ x(t) &= \phi(t), \quad t = -d, -d+1, \ldots, 0 \end{aligned} \tag{43}$$

where $x(t)$, $y(t)$, $z(t)$, and $\omega(t)$ have the same dimensions and meanings as those used for delay-free system $\mathcal{S}$ in (3). A, A_d, B, C, C_d, D, and H are system matrices with compatible dimensions; f and g are nonlinear functions satisfying Assumption 1 (with f replaced by g for the nonlinear function g); $d > 0$ represents a constant delay; and $\{\phi(t), t = -d, -d+1, \ldots, 0\}$ is a given initial condition sequence. When assuming $A_d = 0$ and $C_d = 0$, system $\mathcal{S}_d$ in (43) will reduce to the delay-free system $\mathcal{S}$ in (3).

For the nonlinear time-delay system $\mathcal{S}_d$ in (43), we are interested in designing a nonlinear filter $\mathcal{F}_d$ of the following form:

$$\begin{aligned} \mathcal{F}_d : x_F(t+1) &= A_F f(x_F(t)) + A_{dF} g(x_F(t-d)) \\ &\quad + B_F y(t) \\ z_F(t) &= C_F f(x_F(t)) \\ x_F(t) &= \psi(t), \quad t = -d, -d+1, \ldots, 0 \end{aligned} \tag{44}$$

where $x_F(t)$, $z_F(t)$ have the same dimensions and meanings as those used for (5), and (A_F, A_{dF}, B_F, C_F) are appropriately dimensioned filter matrices to be determined.

Augmenting the model of $\mathcal{S}_d$ to include the states of the filter $\mathcal{F}_d$, we obtain the filtering error system $\mathcal{E}_d$:

$$\begin{aligned} \mathcal{E}_d : \xi(t+1) &= \bar{A}f(\xi(t)) + \bar{A}_d g(\xi(t-d)) + \bar{B}\omega(t) \\ e(t) &= \bar{C}f(\xi(t)) \\ \xi(t) &= \begin{bmatrix} \phi^T(t) & \psi^T(t) \end{bmatrix}^T, \quad t = -d, -d+1, \ldots, 0 \end{aligned} \tag{45}$$

where $\xi(t) = [x^T(t) \; x_F^T(t)]^T$, $e(t) = z(t) - z_F(t)$, and

$$\bar{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} A_d & 0 \\ B_F C_d & A_{dF} \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix}, \quad \bar{C} = [H \quad -C_F]. \tag{46}$$

We introduce the following definitions for the filtering error system $\mathcal{E}_d$ in (45).

Definition 5: The filtering error system $\mathcal{E}_d$ in (45) with $\omega(t) = 0$ is said to be stable if for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that $|\xi(t)| < \epsilon$, $t > 0$ when $\sup_{-d \le t \le 0}\{|\xi(t)|\} < \delta(\epsilon)$. In addition, if $\lim_{t\to\infty} |\xi(t)| = 0$ for any initial conditions, then the filtering error system $\mathcal{E}_d$ in (45) is said to be asymptotically stable.

Definition 6: Given a scalar $\gamma > 0$, the filtering error system $\mathcal{E}_d$ in (45) is said to be asymptotically stable with an induced l_2 disturbance attenuation γ if it is asymptotically stable and under zero initial conditions, $\|e\|_2 < \gamma\|\omega\|_2$ for all nonzero $\omega \in l_2[0, \infty)$.

Definition 7: Given a scalar $\gamma > 0$, the filtering error system $\mathcal{E}_d$ in (45) is said to be asymptotically stable with generalized H_2 disturbance attenuation γ if it is asymptotically stable, and under zero initial conditions, $\|e\|_\infty < \gamma\|\omega\|_2$ for all nonzero $\omega \in l_2[0, \infty)$.

In this section, we will solve the induced l_2 and generalized H_2 filtering problems, which are formulated into the following two problems:

Induced l_2 Delay Filtering Problem: Given an asymptotically stable nonlinear time-delay system $\mathcal{S}_d$ in (43), develop full- and reduced-order filters of the form $\mathcal{F}_d$ in (44) such that for all admissible $\omega \in l_2[0, \infty)$, the filtering error system $\mathcal{E}_d$ in (45) is asymptotically stable with an induced l_2 disturbance attenuation level γ.

Generalized H_2 Delay Filtering Problem: Given an asymptotically stable nonlinear time-delay system $\mathcal{S}_d$ in (43), develop full- and reduced-order filters of the form $\mathcal{F}_d$ in (44) such that for all admissible $\omega \in l_2[0,\infty)$, the filtering error system $\mathcal{E}_d$ in (45) is asymptotically stable with a generalized H_2 disturbance attenuation level γ.

A. Stability Condition

This subsection is devoted to deriving a stability criterion for time-delay systems with repeated scalar nonlinearity, which has not been investigated in previous references.

Lemma 5: Consider system $\mathcal{S}_d$ in (43), and suppose the filter matrices (A_F, A_{dF}, B_F, C_F) of $\mathcal{F}_d$ in (44) are given. Then, the filtering error system $\mathcal{E}_d$ in (45) is asymptotically stable if there exist diagonally dominant matrices $\mathcal{Q} > 0$ and $\mathcal{M} \geq 0$ satisfying

$$\begin{bmatrix} \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}-\mathcal{M} & \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}_d \\ * & \bar{A}_d^T(\mathcal{Q}+\mathcal{M})\bar{A}_d-\mathcal{Q} \end{bmatrix} < 0. \quad (47)$$

Proof: Define the following Lyapunov functional candidate:

$$V(\xi(t),t) \triangleq \xi^T(t)(\mathcal{Q}+\mathcal{M})\xi(t) + \sum_{i=t-d}^{t-1} \xi^T(i)\mathcal{Q}\xi(i) \quad (48)$$

where $\mathcal{Q} > 0$ and $\mathcal{M} \geq 0$ are diagonally dominant. Then, along the trajectory of the filtering error system $\mathcal{E}_d$ in (45) with $\omega(t) = 0$, we have

$$\begin{aligned} \Delta V(\xi(t),t) &= \left[\bar{A}f(\xi(t))+\bar{A}_d g(\xi(t-d))\right]^T(\mathcal{Q}+\mathcal{M}) \\ &\quad \times \left[\bar{A}f(\xi(t))+\bar{A}_d g(\xi(t-d))\right] - \xi^T(t-d)\mathcal{Q}\xi(t-d) \\ &\quad - \xi^T(t)(\mathcal{Q}+\mathcal{M})\xi(t)+\xi^T(t)\mathcal{Q}\xi(t) \\ &= \begin{bmatrix} f(\xi(t)) \\ g(\xi(t-d)) \end{bmatrix}^T \\ &\quad \times \begin{bmatrix} \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}-M & \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}_d \\ * & \bar{A}_d^T(\mathcal{Q}+\mathcal{M})\bar{A}_d-Q \end{bmatrix} \\ &\quad \times \begin{bmatrix} f(\xi(t)) \\ g(\xi(t-d)) \end{bmatrix} - \left[\xi^T(t)\mathcal{M}\xi(t) - f^T(\xi(t))\mathcal{M}f(\xi(t))\right] \\ &\quad - \left[\xi^T(t-d)\mathcal{Q}\xi(t-d) - g^T(\xi(t-d))\mathcal{Q}g(\xi(t-d))\right]. \quad (49) \end{aligned}$$

Note that (47) guarantees the first part of $\Delta V(\xi(t),t)$ to be nonpositive, and by noticing that $\mathcal{Q} > 0$ and $\mathcal{M} \geq 0$ are all diagonally dominant, the second and third parts of $\Delta V(\xi(t),t)$ are all non-negative, according to Lemma 2. Therefore, it can be easily established that $\Delta V(\xi(t),t) < 0$ for all $[f^T(\xi(t))\ g^T(\xi(t-d))] \neq 0$. Then, by the standard Lyapunov stability theory, the filtering error system $\mathcal{E}_d$ with $\omega(t) = 0$ is guaranteed to be asymptotically stable. □

Remark 8: It is worth noting that for delay systems with repeated scalar nonlinearities, two matrix variables (namely $\mathcal{Q}$ and $\mathcal{M}$) need to be determined. These two matrices are all required to be diagonally dominant. A minor point needs to be pointed out that $\mathcal{M}$ is in fact only required to be positive semidefinite. In particular, $\mathcal{M}$ could be chosen as 0, which is sometimes preferred. In what follows, however, to keep the filtering synthesis problems tractable, we will restrict the matrix $\mathcal{M}$ to be strictly positive definite (see Theorems 3 and 4).

Remark 9: For delay-free system with repeated scalar nonlinearity, the stability obtained in [7] has exactly the same form as that for general linear systems (without repeated scalar nonlinearity). The only difference between them is the diagonally dominant constraint on the positive definite matrix.

It is noted that for a general delay system without repeated scalar nonlinearity, the stability condition takes the following form:

$$\begin{bmatrix} \bar{A}^T\mathcal{P}\bar{A}-\mathcal{P}+\mathcal{Q} & \bar{A}^T\mathcal{P}\bar{A}_d \\ * & \bar{A}_d^T\mathcal{P}\bar{A}_d-\mathcal{Q} \end{bmatrix}. \quad (50)$$

It seems that the stability condition obtained in Lemma 5 is quite different from (50). However, by defining $\mathcal{P} \triangleq \mathcal{M}+\mathcal{Q}$, (47) becomes (50), and the additional requirement is that $\mathcal{Q}$ and $\mathcal{P}-\mathcal{Q}$ are positive diagonally dominant. With this connection, we can conclude that Lemma 5 generalizes the previous stability condition for delay systems to more general repeated scalar nonlinear delay systems.

Remark 10: When assuming $\bar{A}_d = 0$, that is, system $\mathcal{E}_d$ in (45) reduces to its corresponding delay-free system $\mathcal{E}$ in (6), then (47) becomes

$$\begin{bmatrix} \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}-\mathcal{M} & 0 \\ * & -\mathcal{Q} \end{bmatrix} < 0$$

which can be reduced to (10) (since $\mathcal{Q}$ is a free parameter). With this connection, Lemma 6 extends the stability condition obtained in [7] to the time-delay case.

Remark 11: It is worth pointing out that the repeated scalar nonlinearities on $x(t)$ and $x(t-d)$ in (43) are not necessarily going to be the same, which enhances the flexibility of this model (for example, there could be nonlinearities on $x(t)$ but no nonlinearities on $x(t-d)$). In the case when we assume that the nonlinear functions $f = g$, that is, the same nonlinearities appear on $x(t)$ and $x(t-d)$, some tighter stability conditions may be obtained, which deserves some effort in future research.

B. Induced l_2 Filtering

The following lemma presents an induced l_2 performance condition, which will be used in the following development.

Lemma 6: Consider system $\mathcal{S}_d$ in (43), and suppose the filter matrices (A_F, A_{dF}, B_F, C_F) of $\mathcal{F}_d$ in (44) are given. Then, the filtering error system $\mathcal{E}_d$ in (45) is asymptotically stable with an induced l_2 disturbance attenuation level bound γ if there exist diagonally dominant matrices $\mathcal{Q} > 0$ and $\mathcal{M} \geq 0$ satisfying (51), shown at the bottom of the page.

$$\Psi \triangleq \begin{bmatrix} \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}-\mathcal{M}+\bar{C}^T\bar{C} & \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}_d & \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{B} \\ * & \bar{A}_d^T(\mathcal{Q}+\mathcal{M})\bar{A}_d-\mathcal{Q} & \bar{A}_d^T(\mathcal{Q}+\mathcal{M})\bar{B} \\ * & * & \bar{B}^T(\mathcal{Q}+\mathcal{M})\bar{B}-\gamma^2 I \end{bmatrix} < 0 \quad (51)$$

Proof: First, (51) implies (47), and then, from Lemma 5, we know that the filtering error system $\mathcal{E}_d$ in (45) with $\omega(t) = 0$ is asymptotically stable.

Define the Lyapunov functional candidate as in (48). Then, along the trajectory of the filtering error system $\mathcal{E}_d$ in (45), we have

$$\begin{aligned}\Delta V(\xi(t), t) &= \left[\bar{A} f(\xi(t)) + \bar{A}_d g(\xi(t-d)) + \bar{B}\omega(t)\right]^T (\mathcal{Q}+\mathcal{M}) \\ &\quad \times \left[\bar{A} f(\xi(t)) + \bar{A}_d g(\xi(t-d)) + \bar{B}\omega(t)\right] \\ &\quad - \xi^T(t)(\mathcal{Q}+\mathcal{M})\xi(t) + \xi^T(t)\mathcal{Q}\xi(t) - \xi^T(t-d)\mathcal{Q}\xi(t-d) \\ &= \lambda^T(t)\bar{\Psi}\lambda(t) - \left[\xi^T(t)\mathcal{M}\xi(t) - f^T(\xi(t))\mathcal{M} f(\xi(t))\right] \\ &\quad - \left[\xi^T(t-d)\mathcal{Q}\xi(t-d) - g^T(\xi(t-d))\mathcal{Q} g(\xi(t-d))\right]\end{aligned} \tag{52}$$

where we have (53), shown at the bottom of the page. By noticing $\mathcal{Q} > 0$ and $\mathcal{M} \geq 0$ are all diagonally dominant, we have

$$\Delta V(\xi(t), t) \leq \lambda^T(t)\bar{\Psi}\lambda(t). \tag{54}$$

Consider the following index:

$$\mathcal{J}_2 \triangleq \sum_{t=0}^{\infty} \left[e^T(t)e(t) - \gamma^2 \omega^T(t)\omega(t)\right].$$

Under zero initial condition, $V(\xi(t), t)|_{t=0} = 0$. Then, considering (54), we have

$$\begin{aligned}\mathcal{J}_2 &\leq \sum_{t=0}^{\infty} \left[e^T(t)e(t) - \gamma^2\omega^T(t)\omega(t) + \Delta V(\xi(t), t)\right] \\ &\leq \sum_{t=0}^{\infty} \lambda^T(t)\Psi\lambda(t)\end{aligned}$$

where $\lambda(t)$ is defined in (53), and Ψ is defined in (51). Therefore, for all nonzero $\omega = \{\omega(t)\} \in l_2[0, \infty)$, we have $\mathcal{J}_2 < 0$ and $e = \{e(t)\} \in l_2[0, \infty)$, which means $\|e\|_2 < \gamma\|\omega\|_2$, and the proof is completed. □

Then, the following theorem presents the corresponding induced l_2 filtering result for time-delay systems with repeated scalar nonlinearity (the theorem can be proved by following similar lines, as in the proof of Theorem 1).

Theorem 3: Consider system $\mathcal{S}_d$ in (43). Then, an admissible induced l_2 filter of the form $\mathcal{F}_d$ in (44) exists if there exist matrices $0 < \mathcal{M} \triangleq [m_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$, $\mathcal{N} > 0$, $0 < \mathcal{Q} \triangleq [q_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$, $\mathcal{L} > 0$, $\mathcal{X} > 0$, $\mathcal{R} = \mathcal{R}^T \triangleq [r_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$, and $\mathcal{T} = \mathcal{T}^T \triangleq [t_{ij}] \in \mathbb{R}^{(n+k)\times(n+k)}$ satisfying (13), and

$$\begin{bmatrix} -J\mathcal{N}J^T & 0 & J\bar{A}_0 & J\bar{A}_{d0} & J\bar{B}_0 \\ * & -J\mathcal{L}J^T & J\bar{A}_0 & J\bar{A}_{d0} & J\bar{B}_0 \\ * & * & -\mathcal{M}+\mathcal{X} & 0 & 0 \\ * & * & * & -\mathcal{Q} & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{55}$$

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & M^{\perp} \end{array}\right] \left[\begin{array}{cc|ccc} \mathcal{N} & 0 & \bar{A}_0 & \bar{A}_{d0} & \bar{B}_0 \\ * & \mathcal{L} & \bar{A}_0 & \bar{A}_{d0} & \bar{B}_0 \\ \hline * & * & -\mathcal{M}+\mathcal{X} & 0 & 0 \\ * & * & * & -\mathcal{Q} & 0 \\ * & * & * & * & -\gamma^2 I \end{array}\right] \times \left[\begin{array}{c|c} I & 0 \\ \hline 0 & M^{\perp} \end{array}\right]^T < 0 \tag{56}$$

$$m_{ii} - \sum_{j\neq i}(m_{ij} + 2r_{ij}) \geq 0$$

$$q_{ii} - \sum_{j\neq i}(q_{ij} + 2t_{ij}) \geq 0, \quad \forall i \tag{57}$$

$$r_{ij} \geq 0, \quad t_{ij} \geq 0, \quad \forall i \neq j \tag{58}$$

$$m_{ij} + r_{ij} \geq 0, \quad q_{ij} + t_{ij} \geq 0, \quad \forall i \neq j \tag{59}$$

$$\mathcal{M}\mathcal{N} = I, \quad \mathcal{Q}\mathcal{L} = I \tag{60}$$

Furthermore, if $(\mathcal{M}, \mathcal{N}, \mathcal{Q}, \mathcal{L}, \mathcal{X}, \mathcal{R}, \mathcal{T})$ is a feasible solution of the above conditions, then the system matrices of an admissible induced l_2 filter in the form of (44) are given by (18), where $\mathcal{G}_1 \triangleq [A_F\ A_{dF}\ B_F], \mathcal{G}_2 \triangleq C_F; \Pi_i, L_i, i = 1$, and 2 are any appropriately dimensioned matrices satisfying $\Pi_i > 0$, $\|L_i\| < 1$, and

$$\begin{aligned}
&\bar{A}_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{d0} = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_0 = \begin{bmatrix} B \\ 0 \end{bmatrix} \\
&\bar{C}_0 = [H \quad 0], \quad J = [I \quad 0] \\
&E = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad R = \begin{bmatrix} 0 & I \\ 0 & 0 \\ C & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 0 & I \\ C_d & 0 \end{bmatrix} \\
&T = \begin{bmatrix} 0 \\ 0 \\ D \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & C^T \\ I & 0 & 0 \\ 0 & 0 & C_d^T \\ 0 & I & 0 \\ 0 & 0 & D^T \end{bmatrix}, \quad U_1 = \begin{bmatrix} E \\ E \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&W_1 = \begin{bmatrix} -\mathcal{M}^{-1} & 0 & \bar{A}_0 & \bar{A}_{d0} & \bar{B}_0 \\ * & -\mathcal{Q}^{-1} & \bar{A}_0 & \bar{A}_{d0} & \bar{B}_0 \\ * & * & -\mathcal{M}+\mathcal{X} & 0 & 0 \\ * & * & * & -\mathcal{Q} & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} \\
&W_2 = \begin{bmatrix} -I & \bar{C}_0 \\ * & -\mathcal{X} \end{bmatrix}, \quad U_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad V_2 = [0 \quad L] \\
&V_1 = [0 \quad 0 \quad R \quad S \quad T], \quad L = [0 \quad -I].
\end{aligned} \tag{61}$$

Similar to the delay-free case, the obtained conditions in Theorem 3 are not strict LMI conditions due to the matrix equality in (17). We suggest the following nonlinear minimization problem involving LMI conditions instead of the original nonconvex feasibility problem formulated in Theorem 3.

$$\lambda(t) = \begin{bmatrix} f(\xi(t)) \\ g(\xi(t-d)) \\ \omega(t) \end{bmatrix}, \quad \bar{\Psi} \triangleq \begin{bmatrix} \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A} - M & \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}_d & \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{B} \\ * & \bar{A}_d^T(\mathcal{Q}+\mathcal{M})\bar{A}_d - Q & \bar{A}_d^T(\mathcal{Q}+\mathcal{M})\bar{B} \\ * & * & \bar{B}^T(\mathcal{Q}+\mathcal{M})\bar{B} \end{bmatrix} \tag{53}$$

Problem Il2DF (Induced l_2 Delay Filtering):

$\min$ tr$(\mathcal{M}\mathcal{N}+\mathcal{Q}\mathcal{L})$ subject to (55)-(59) and

$$\begin{bmatrix}\mathcal{M} & I\\ I & \mathcal{N}\end{bmatrix}\geq 0,\quad \begin{bmatrix}\mathcal{Q} & I\\ I & \mathcal{L}\end{bmatrix}\geq 0. \tag{62}$$

If the solution of the above minimization problem is $2(n+k)$, then the conditions in Theorem 3 are solvable. Moreover, we can readily modify the algorithms presented in the above sections to solve Problem Il2DF.

C. Generalized H_2 Filtering

The following lemma provides a generalized H_2 performance condition for time-delay systems with repeated scalar nonlinearity upon which the generalized H_2 filtering problem will be solved in this subsection (this lemma can be proved by following similar lines as in the proofs of Lemmas 4 and 6).

Lemma 7: Consider system $\mathcal{S}_d$ in (43), and suppose the filter matrices (A_F, A_{dF}, B_F, C_F) of $\mathcal{F}_d$ in (44) are given. Then, the filtering error system $\mathcal{E}_d$ in (45) is asymptotically stable with a generalized H_2 disturbance attenuation level bound γ if there exist diagonally dominant matrices $\mathcal{Q}>0$ and $\mathcal{M}\geq 0$ satisfying (63), shown at the bottom of the page, and

$$\begin{bmatrix}-\gamma^2 I & \bar{C}\\ * & -\mathcal{M}-\mathcal{Q}\end{bmatrix}<0. \tag{64}$$

Then, the generalized H_2 filter synthesis result follows immediately (the proof is omitted).

Theorem 4: Consider system $\mathcal{S}_d$ in (43). Then, an admissible generalized H_2 filter of the form $\mathcal{F}_d$ in (44) exists if there exist matrices $0<\mathcal{M}\triangleq[m_{ij}]\in\mathbb{R}^{(n+k)\times(n+k)}$, $\mathcal{N}>0$, $0<\mathcal{Q}\triangleq[q_{ij}]\in\mathbb{R}^{(n+k)\times(n+k)}$, $\mathcal{L}>0$, $\mathcal{R}=\mathcal{R}^T\triangleq[r_{ij}]\in\mathbb{R}^{(n+k)\times(n+k)}$, and $\mathcal{T}=\mathcal{T}^T\triangleq[t_{ij}]\in\mathbb{R}^{(n+k)\times(n+k)}$ satisfying (57)–(60), and

$$\begin{bmatrix}-J\mathcal{N}J^T & 0 & J\bar{A}_0 & J\bar{A}_{d0} & J\bar{B}_0\\ * & -J\mathcal{L}J^T & J\bar{A}_0 & J\bar{A}_{d0} & J\bar{B}_0\\ * & * & -\mathcal{M} & 0 & 0\\ * & * & * & -\mathcal{Q} & 0\\ * & * & * & * & -I\end{bmatrix}<0 \tag{65}$$

$$\left[\begin{array}{c|c}I & 0\\ \hline 0 & M^{\perp}\end{array}\right]\left[\begin{array}{cc|ccc}\mathcal{N} & 0 & \bar{A}_0 & \bar{A}_{d0} & \bar{B}_0\\ * & \mathcal{L} & \bar{A}_0 & \bar{A}_{d0} & \bar{B}_0\\ \hline * & * & -\mathcal{M} & 0 & 0\\ * & * & * & -\mathcal{Q} & 0\\ * & * & * & * & -I\end{array}\right]\times\left[\begin{array}{c|c}I & 0\\ \hline 0 & M^{\perp}\end{array}\right]^T<0 \tag{66}$$

$$\begin{bmatrix}-\gamma^2 I & \bar{C}_0 J^T\\ * & -J(\mathcal{Q}+\mathcal{M})J^T\end{bmatrix}<0. \tag{67}$$

Furthermore, if $(\mathcal{M},\mathcal{N},\mathcal{Q},\mathcal{L},\mathcal{R},\mathcal{T})$ is a feasible solution of the above conditions, then the system matrices of an admissible generalized H_2 filter in the form of (44) are given by (18), where $\mathcal{G}_1\triangleq[A_F\ A_{dF}\ B_F]$, $\mathcal{G}_2\triangleq C_F$; Π_i, L_i, $i=1$, and 2 are any appropriately dimensioned matrices satisfying $\Pi_i>0$, $\|L_i\|<1$; $\bar{A}_0$, $\bar{A}_{d0}$, $\bar{B}_0$, $\bar{C}_0$, E, R, S, T, L, M, and J are defined in (61), and

$$\begin{aligned}W_1&=\begin{bmatrix}-\mathcal{M}^{-1} & 0 & \bar{A}_0 & \bar{A}_{d0} & \bar{B}_0\\ * & -\mathcal{Q}^{-1} & \bar{A}_0 & \bar{A}_{d0} & \bar{B}_0\\ * & * & -\mathcal{M} & 0 & 0\\ * & * & * & -\mathcal{Q} & 0\\ * & * & * & * & -I\end{bmatrix}\\ U_1^T&=[E^T\quad E^T\quad 0\quad 0\quad 0]\\ W_2&=\begin{bmatrix}-\gamma^2 I & \bar{C}_0\\ * & -\mathcal{M}-\mathcal{Q}\end{bmatrix},\quad U_2=\begin{bmatrix}I\\ 0\end{bmatrix}\\ V_2&=[0\quad L],\quad V_1=[0\quad 0\quad R\quad S\quad T].\end{aligned} \tag{68}$$

The following nonlinear minimization problem involving LMI conditions can be used to solve the nonconvex feasibility problem formulated in Theorem 4.

Problem GH2DF (Generalized H_2 Delay Filtering):

$\min$ tr$(\mathcal{M}\mathcal{N}+\mathcal{Q}\mathcal{L})$ subject to (57)-(59), (65)-(67), and (62).

If the solution of the above minimization problem is $2(n+k)$, then the conditions in Theorem 2 are solvable. Moreover, we can readily modify the algorithms presented in the above sections to solve Problem GH2DF.

VI. Illustrative Example

In this section, we provide a numerical example to show the effectiveness of the developed theories. Consider the following system:

$$\begin{aligned}x(t+1)&=\begin{bmatrix}0 & -0.5\\ 1 & 1\end{bmatrix}f(x(t))+\begin{bmatrix}-6 & 0\\ 1 & 0\end{bmatrix}\omega(t)\\ y(t)&=[-100\quad 10]f(x(t))+\begin{bmatrix}0\\ 1\end{bmatrix}\omega(t)\\ z(t)&=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}f(x(t)).\end{aligned} \tag{69}$$

It is assumed that the nonlinear function f satisfies Assumption 1. Note that this example is the same as that in [23], except that (69) contains repeated scalar nonlinearity f. Our purpose here is to design full-order ($k=2$) and reduced-order ($k=1$) filters in the form of $\mathcal{F}$ in (5) to estimate the signal $z(t)$ such that the filtering error system $\mathcal{E}$ in (6) is asymptotically stable with an induced l_2 performance.

$$\Omega\triangleq\begin{bmatrix}\bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}-\mathcal{M} & \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{A}_d & \bar{A}^T(\mathcal{Q}+\mathcal{M})\bar{B}\\ * & \bar{A}_d^T(\mathcal{Q}+\mathcal{M})\bar{A}_d-\mathcal{Q} & \bar{A}_d^T(\mathcal{Q}+\mathcal{M})\bar{B}\\ * & * & \bar{B}^T(\mathcal{Q}+\mathcal{M})\bar{B}-I\end{bmatrix}<0 \tag{63}$$

By applying Theorem 1 with the application of Algorithm Il2F, we obtain admissible induced l_2 filters and their associated matrix parameters as follows:

Full-order Induced l_2 Filter ($\gamma = 15.5$):

$$\mathcal{P} = \begin{bmatrix} 5.2157 & 2.3653 & 0.0047 & -2.0247 \\ 2.3653 & 3.3761 & -0.2743 & -0.4614 \\ 0.0047 & -0.2743 & 0.4400 & -0.0119 \\ -2.0247 & -0.4614 & -0.0119 & 2.9197 \end{bmatrix}$$
$$A_F = \begin{bmatrix} 0.0981 & 0.9416 \\ -0.0261 & -0.2504 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0.0010 \\ -0.0063 \end{bmatrix}$$
$$C_F = \begin{bmatrix} -0.0585 & 1.4120 \\ 0.2218 & -0.0346 \end{bmatrix}.$$

First-order Induced l_2 Filter ($\gamma = 16.5$):

$$\mathcal{P} = \begin{bmatrix} 5.4126 & 2.5039 & 1.6394 \\ 2.5039 & 3.9154 & 0.7559 \\ 1.6394 & 0.7559 & 2.8433 \end{bmatrix}$$
$$A_F = -0.1372, \quad B_F = 0.0054, \quad C_F = \begin{bmatrix} -1.3891 \\ -0.6117 \end{bmatrix}.$$

It can be seen that the matrix variable $\mathcal{P}$ obtained in each of the above cases is positive diagonally dominant. When we assume $f(x(t)) = x(t)$, system (69) becomes a linear system. The filtering results in [18] can be readily applied to this linear system. By [18], it is found that for the full-order induced l_2 filtering, the obtained optimal noise attenuation level $\gamma = 10.06$. Compared with the results obtained above, it can be seen that system (69) with $f(x(t)) = x(t)$ can achieve relatively smaller noise attenuation level, which is quite reasonable because the general nonlinear function $f(x(t))$ with Assumption 1 includes $f(x(t)) = x(t)$ as a special case.

VII. Conclusion

This paper contributes further to the study of discrete-time systems with repeated scalar nonlinearities. The contribution of this paper can be summarized as follows:

1) Complete results are developed for the filtering problems of discrete-time systems with repeated scalar nonlinearities. Induced l_2 filtering and generalized H_2 filtering problems have been solved for such systems either with or without state delay.
2) The technique used in solving these problems is quite different from previous ones used for the controller synthesis problems. The most obvious advantage of this technique is its capability to circumvent the difficulty of dilating a positive diagonally dominant matrix X to a larger positive diagonally dominant matrix $\bar{X}$ such that the top-left corner of the inverse of $\bar{X}$ is fixed, which has been recognized to be a main obstacle for synthesis problems of systems with repeated scalar nonlinearities.

Finally, it is worth pointing out that the stability and filtering results developed for delay systems with repeated scalar nonlinearities are all delay independent, which might be conservative when the delays are small. Therefore, further research can be directed at developing delay-dependent approaches for solving the related problems to reduce the underlying conservativeness.

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