$k\Omega$ and LM741 for the operational amplifier with dc bias of ± 15 V. The upper trace shows the input triangular waveform with a frequency of 500 mHz. The bottom trace shows the rectangular waveform at the output of the operational amplifier. Thus, a good differentiation action is obtained by using the proposed circuit. It was not necessary to add a resistor between the negative input terminal of the isolation operational amplifier and ground since a dc input to a differentiator produces a zero output voltage.

D. Comparison Among the Theoretical, Simulation, and Experimental Results

The simulation and experimental results verify the predicted frequency ranges of (7) for the integrator and (14) for the differentiator. Note also that the clean waveforms of Figs. 10 and 11 indicate high signal to noise ratios for both circuits.

It should be pointed out that the case of infinite k, with the feedback resistor kr removed, produced good experimental and simulation results which were omitted for brevity.

VI. Conclusion

An active-network synthesis of inverse system design is presented. The synthesis is general and can be applied with different impedances. Its application to invert a passive differentiator resulted in a versatile low-frequency differential integrator. Its application to invert a passive *RC* integrator yielded a versatile low-frequency differential differentiator. Each employs a single time constant, has a resistive input, and a reasonably high Q value. Simulation and experimental results verify the theoretical expectations. The active-network synthesis can be applied to obtain other varied realizations. The differential integrators and differentiators could easily be modified to obtain inverting and non-inverting integrators and differentiators by simply grounding one of the two inputs in each of the differential configurations. Additionally, the limited bandwidths of the circuits mitigates the contribution of the noise and yield output waveforms with large signal to noise ratios.

Acknowledgment

The author wishes to thank S. K. Mitra for providing the atmosphere conducive to research by inviting him to spend the summer of 1997 at the Signal and Image Processing Laboratory of University of California at Santa Barbara, where this research was initiated, and R. Ferzli, F. El-Zoghet, F, Elias, and B. Alawieh for their help in the production of the figures, simulation, and experimental results.

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Robust Stabilization of Singular-Impulsive-Delayed Systems With Nonlinear Perturbations

Zhi-Hong Guan, C. W. Chan, Andrew Y. T. Leung, and Guanrong Chen

***Abstract*—Many dynamic systems in physics, chemistry, biology, engineering, and information science have impulsive dynamical behaviors due to abrupt jumps at certain instants during the dynamical process, and these complex dynamic behaviors can be modeled by singular impulsive differential systems. This paper formulates and studies a model for singular impulsive delayed systems with uncertainty from nonlinear perturbations. Several fundamental issues such as global exponential robust stabilization of such systems are established. A simple approach to the design of a robust impulsive controller is then presented. A numerical example is given for illustration of the theoretical results. Meanwhile, some new results and refined properties associated with the M-matrices and time-delay dynamic systems are derived and discussed.**

***Index Terms*—Impulsive systems, nonlinear perturbation, robust stabilization, singular systems, time-delay, uncertainty.**

I. Introduction

In recent years, considerable efforts have been devoted to the analysis and synthesis of singular systems (known also as descriptor systems, semistate systems, differential algebraic systems, generalized state-space systems, etc.). These systems arise naturally in various fields including electrical networks [25], robotics [22], [23], social, biological, and multisector economic systems [21], [29], dynamics of thermal nuclear reactors [26], automatic control systems [27], among many others such as singular perturbation systems. Progress in the investigation of singular systems can be found in books [1], [4], [6], [8] and survey papers [5], [15], [16].

Although most singular systems are analyzed either in the continuous- or discrete-time setting, many singular systems exhibit both continuous-time and discrete-time behaviors. Examples include many evolutionary processes, especially those in biological systems such as biological neural networks and bursting rhythm models in pathology. Other examples exist in optimal control of economic systems, frequency-modulated signal processing systems, and some flying object motions. These systems are characterized by abrupt changes in the states at certain instants [3], [9], [10], [11], [14]. This type of impulsive phenomena can also be found in the fields of information science, electronics, automatic control systems, computer networks, artificial intelligence, robotics, and telecommunications [10]. Many sudden and sharp changes occur instantaneously in singular systems, in the form of impulses which cannot be well described by a pure continuous-time or discrete-time model. For instance, if the initial conditions is inconsistent, then a singular system will have a finite

Manuscript received September 11, 2000. This work was supported in part by the National Natural Science Foundation of China under Grant 69774038 and Grant 60074009, in part by the Doctorate Foundation of the Education Ministry of China under Grant 199048718, and in part by the Foundation for University Key Teacher, Education Ministry of China. This paper was recommended by Associate Editor M. Gilli.

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Publisher Item Identifier S 1057-7122(01)06082-2.

instantaneous jump at the initial time [20]. For singular systems with time delays, infinite impulses as well as finite jumps can occur in the solutions of the systems [6], [17]. Therefore, it is very important, and indeed necessary, to study singular impulsive systems, perhaps also with time delays.

On top of the singular, impulsive, and time-delayed features of such dynamic systems, there might also be uncertainties such as perturbations, usually arising from modeling errors, data-measurement errors, changes in environmental conditions, and component variations, etc. These altogether lead the system to an unexpectedly complicated situation, thereby leading to very complex dynamical behaviors. In the design of a controller for such a complex system, it is important to ensure that the system be stable with respect to these uncertainties. Robust stabilization is a concept addressing this issue of stability for uncertain systems. In particular, robust stabilization for a singular and delayed system has recently attracted increasing interest (see, e.g., [7], [18], [6], [17], [28] and the references therein). These existing studies, however, are not for impulsive type of control systems.

Given the above background, this paper attempts to study the robust stabilization problem for a complex dynamic system of the singular, impulsive, and time-delayed type with uncertainties from nonlinear perturbations. This work is based on our previous investigations on singular and impulsive systems [11], [19]. Basically, in this paper, we first introduce a model for singular impulsive delayed system with nonlinear perturbations, and then study its exponential robust stabilization problem.

The paper is organized as follows. In Section II, the singular impulsive delayed system is described and modeled, and in Section III, some preliminary results are derived and discussed. The main result of global exponential robust-stabilization criteria for the established model is given in Section IV, where a systematic design procedure for obtaining a robust impulsive controller is presented. For illustration, a numerical example is described in Section V. Finally, some conclusions are drawn in Section VI.

II. Problem Formulation

Let $R_+ = [0, +\infty)$, $J = [t_0, +\infty)$, $(t_0 \geq 0)$ and R^n denote the n-dimensional Euclidean space. The norm of z is $\|z\| := \sum_{i=1}^n |z_i|$ and $|z| := (|z_1|, \ldots, |z_n|)^T$, where $z = (z_1, \ldots, z_n)^T \in R^n$. Similarly, for $A = (a_{ij})_{n\times n} \in R^{n\times n}$

$$\|A\| = \max_j \sum_{i=1}^n |a_{ij}| \quad |A| = (|a_{ij}|)_{n\times n}$$
$$\sigma_{\min}(A) = \lambda_{\min}^{1/2}(A^T A)$$
$$\mu(A) = \max_j \left\{ a_{jj} + \sum_{i=1,\, i\neq j}^n |a_{ij}| \right\}$$

where $\mu(A)$ and $\lambda_{\min}(A)$ are the matrix measure and the minimum eigenvalue of A, respectively. We use the notation $A(a_{ij}) \geq B(b_{ij})$ and $(z_1, \ldots, z_n)^T \geq (y_1, \ldots, y_n)^T$ to mean that $a_{ij} \geq b_{ij}$ and $z_i \geq y_i$ for all $i, j = 1, \ldots, n$, respectively. The identity matrix of order n is denoted as I_n, or simply I, if no confusion arises.

Consider the following nonlinear uncertain time-delay impulsive and singular dynamic system:

$$\begin{cases} Dx_1 = [A_1 x_1(t) + B_1 x_1(t-\tau)]Dv_1 + C_1 u_1 \\ \qquad + f_1(t, x(t), x(t-\tau))Dw_1 \\ 0 = [A_2 x_2(t) + B_2 x_2(t-\tau)]Dv_2 + C_2 u_2 \\ \qquad + f_2(t, x(t), x(t-\tau))Dw_2 \end{cases} \tag{2.1}$$

with the initial conditions

$$x_i(t) = \phi_i(t), \qquad t_0 - \tau \leq t \leq t_0 \qquad i = 1, 2 \tag{2.2}$$

where

$x_i \in R^{n_i}$, $x = (x_1, x_2)^T$	state vector with $n_1 + n_2 = n$;
$u_i \in R^{m_i}$, $u = (u_1, u_2)^T$	control vector with $m_1 + m_2 = m$;
A_i, B_i, and C_i	known real constant matrices of appropriate dimensions;
$f_i(t, x(t), x(t-\tau))$	nonlinear uncertain vertor function with $f_i(t, 0, 0) \equiv 0$ for all $t \in J$;
$\tau > 0$	is a constant.

Here, Dx_i, Dv_i and Dw_i denote the distributional derivatives of the functions $x_i \in R^{n_i}$, v_i, and w_i, respectively. v_i, w_i: $R_+ \to R$ are functions of bounded variation and right-continuous on every compact subinterval of J. This implies that Dv_i and Dw_i can be identified with the Lebesgue–Stieltjes measure, which have the effect of suddenly changing the state of the system at the points of discontinuity of v_i and u_i. Moreover, ϕ_i: $[t_0 - \tau, t_0] \to R^{n_i}$ are functions of bounded variation and right-continuous. Finally, the initial condition is denoted by the vector $\Phi(t) = (\phi_1(t), \phi_2(t))^T$.

In general, a function of bounded variation and right-continuous consists of two parts: one is an absolutely continuous function and the other is a singular function. When discontinuous points of the function are isolated and at most countable, the singular part has the form $\sum_{k=1}^\infty a_k H_k(t)$. Without loss of generality, we therefore assume that

$$v_i(t) = t + \sum_{k=1}^\infty \alpha_{ik} H_k(t)$$
$$w_i(t) = t + \sum_{k=1}^\infty \beta_{ik} H_k(t), \qquad i = 1, 2 \tag{2.3}$$

where α_{ik} and β_{ik} are constants, with discontinuity points

$$t_1 < t_2 < \cdots < t_k < \cdots, \qquad \lim_{k\to\infty} t_k = \infty$$

where $t_1 > t_0$, and $H_k(t)$ are the Heaviside functions defined by

$$H_k(t) = \begin{cases} 0, & t < t_k \\ 1, & t \geq t_k. \end{cases}$$

It is easy to see that

$$Dv_i = 1 + \sum_{k=1}^\infty \alpha_{ik} \delta(t - t_k)$$
$$Dw_i = 1 + \sum_{k=1}^\infty \beta_{ik} \delta(t - t_k)$$

where $\delta(t)$ is the Dirac impulse function.

Provided that all the states are available, the robust-state feedback controller $u_i(t)$ are given by

$$u_i(t) = K_i x_i(t) Dv_i \tag{2.4}$$

where K_i is a constant matrix (called the *gain matrix* hereafter) of appropriate dimension. Obviously, $u_i(t)$ is an impulsive controller.

Substituting (2.4) into (2.1) yields a nonlinear uncertain closed-loop time-delay singular and impulsive dynamical systems, in the form of

$$\begin{cases} Dx_1 = \left[\overline{A}_1 x_1(t) + B_1 x_1(t-\tau)\right] Dv_1 \\ \qquad +f_1(t, x(t), x(t-\tau))Dw_1 \\ \quad 0 = \left[\overline{A}_2 x_2(t) + B_2 x_2(t-\tau)\right] Dv_2 \\ \qquad +f_2(t, x(t), x(t-\tau))Dw_2 \end{cases} \tag{2.5}$$

where

$$\overline{A}_1 = A_1 + C_1 K_1 \qquad \overline{A}_2 = A_2 + C_2 K_2. \tag{2.6}$$

Now, the problem is to find some conditions for robust control $u_i(t)$, given by (2.4), such that the overall nonlinear uncertain time-delay singular impulsive system (2.5) is asymptotically stable in the presence of nonlinear uncertainties.

III. Time-Delay and Impulsive Systems

In this section, some necessary concepts and refined properties associated with time-delay systems and impulsive systems are derived and discussed.

Consider the following time-delay system:

$$x'(t) = Ax(t) + Bx(t-\tau) \tag{3.1}$$

$$x'(t) = Ax(t) + Bx(t-\tau) + f(t). \tag{3.2}$$

It is well known that the asymptotic stability of (3.1) implies that the solution λ of equation

$$\det Q(\lambda) = 0, \quad Q(\lambda) := \lambda I - A - B\exp(-\lambda\tau) \tag{3.3}$$

satisfies $\text{Re}(\lambda) < 0$, and vice versa.

Definition 3.1 [24]: System (3.1) is said to be *stable with decay rate* α, if the solution of (3.1) satisfies $\text{Re}(\lambda) + \alpha < 0$ for some $\alpha > 0$.

Remark 3.1: There are many results associated with the estimate of the decay rate for the time-delay system (3.1). For instance, if $-\mu(A) - \alpha > \|B\|\exp(\tau\alpha)$, then system (3.1) is stable with the decay rate α [24].

Lemma 3.1 [13]: The following time-delay matrix system

$$X'(t) = AX(t) + BX(t-\tau) \quad X(t) \in R^{n\times n}$$

with initial condition

$$X(t) = \begin{cases} 0, & t < 0 \\ I_n, & t = 0 \end{cases}$$

has a unique solution on $t > 0$ given by

$$X(t) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} Q^{-1}(s)\exp(ts)\,ds, \qquad t > 0 \tag{3.4}$$

where $Q(s)$ is given by (3.3), $c > \beta$, $\beta = \|A\| + \|B\|$, and $X(t)$ is also a matrix-valued function of bounded variation on every compact subinterval of J.

In addition, for any $-\alpha > \max\{\text{Re}(\lambda) | \det Q(\lambda) = 0\}$, there exists a constant $M = M(\alpha)$ such that

$$\|X(t)\| \le M\exp(-\alpha t), \qquad t \ge 0. \tag{3.5}$$

In Lemma 3.1, $X(t)$ is called the *fundamental solution* or *fundamental solution matrix* of (3.1).

Remark 3.2: Lemma 3.1 and Definition 3.1 imply that if system (3.1) is stable with a decay rate α, then inequality (3.5) holds. For the estimate of (3.5), we have the refined result given in Lemma 3.2 below.

Lemma 3.2: If $\alpha_0 = \max\{\text{Re}(s) | \det Q(s) = 0\}$, then for any $-\alpha > \alpha_0$, the fundamental solution $X(t)$ of (3.1) satisfies the inequality

$$\|X(t)\| \le \frac{\sqrt{n}}{\pi}\exp(-\alpha t), \qquad t \ge 0. \tag{3.6}$$

Proof: It follows from Lemma 3.1 that $X(t)$ has the expression of (3.4). For simplicity, the following notation is used

$$\int_{(c)} := \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} = \lim_{T\to\infty}\frac{1}{2i\pi}\int_{c-iT}^{c+iT}$$

where c is some sufficiently large real number. First, we want to show that

$$\int_{(c)} Q^{-1}(s)\exp(st)\,ds = \int_{(-\alpha)} Q^{-1}(s)\exp(st)\,ds \tag{3.7}$$

where $c > -\alpha$. Consider the integration of the function $Q^{-1}(s)e^{st}$ along the boundary of the box in the complex plane, with boundary $L_1M_1L_2M_2$, in the counterclockwise direction, where

$$L_1\colon \{c + i\xi\colon -T \le \xi \le T\} \quad L_2\colon \{-\alpha + i\xi\colon -T \le \xi \le T\}$$
$$M_1\colon \{\sigma + iT\colon -\alpha \le \sigma \le c\} \quad M_2\colon \{\sigma - iT\colon -\alpha \le \sigma \le c\}.$$

Since $Q^{-1}(s)$ has no zeros in the box and on its boundary, it follows that $Q^{-1}(s)e^{st}$ is an analytic function in the box, leading to

$$\left\{\int_{L_1} + \int_{M_1} + \int_{L_2} + \int_{M_2}\right\} Q^{-1}(s)\exp(st)\,ds = 0.$$

Thus, (3.7) is verified if we can show that

$$\int_{M_1} Q^{-1}(s)\exp(st)\,ds \to 0 \quad \int_{M_2} Q^{-1}(s)\exp(st)\,ds \to 0 \tag{3.8}$$

as $T \to \infty$.

For an $n \times n$ matrix P, let $\|P\|_2 := \lambda_{\max}^{1/2}(P\overline{P}^T)$, where $\overline{P}^T$ is the conjugate transpose matrix of P and $\lambda_{\max}(P\overline{P}^T)$ is the maximum eigenvalue of $P\overline{P}^T$. Then $\|P\| \le \sqrt{n}\|P\|_2$. It is now readily shown that

$$\begin{aligned} &\|Q^{-1}(s)\|_2^2 \\ &\quad = \max_i \lambda_i\left\{Q^{-1}(s)(\overline{Q^{-1}(s)})^T\right\} \\ &\quad = \frac{1}{\min_i \lambda_i\left\{Q^{-1}(s)(\overline{Q^{-1}(s)})^T\right\}} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\lambda_i\left\{Q^{-1}(s)\left(\overline{Q^{-1}(s)}\right)^T\right\} \\ &\quad = \lambda_i\left\{(sI - A - B - e^{-s\tau})\left(\overline{sI - A - Be^{-s\tau}}\right)^T\right\} \\ &\quad \ge |s|^2 + \lambda_{\min}(AA^T) + \lambda_{\min}(BB^T)e^{-2\tau\text{Re}(s)} \\ &\qquad - 2\|AB^T\|e^{-\tau\text{Re}(s)} - 2|s|\|A\| - 2|s|\|B\|e^{-\tau\text{Re}(s)}. \end{aligned} \tag{3.10}$$

Choose T_0 such that

$$\left(1+\frac{\alpha^2}{T^2}\right)-\frac{2}{T}\left(1+\frac{\alpha^2}{T^2}\right)^{1/2}(\|A\|+\|B\|e^{\alpha\tau}) + \frac{1}{T^2}[\lambda_{\min}(AA^T)+\lambda_{\min}(BB^T)e^{2\alpha\tau}-2\|AB^T\|e^{\alpha\tau}] \geq \frac{1}{4}$$

for all $T \geq T_0$. If $T \geq T_0$ and $s \in M_1$, then (3.9) and (3.10) implies that

$$\|Q^{-1}(s)\|_2^2 \leq [(\sigma^2+T^2)+\lambda_{\min}(AA^T)+\lambda_{\min}(BB^T)e^{2\alpha\tau} -2\|AB^T\|e^{\alpha\tau}-2\sqrt{\sigma^2+T^2}(\|A\|+\|B\|e^{\alpha\tau})]^{-1} \leq \frac{4}{T^2}$$

namely, $\|Q^{-1}(s)\|_2 \leq 2/T$. Therefore

$$\begin{aligned}\left\|\int_{M_1} Q^{-1}(s)e^{st}\,ds\right\| &\leq \sqrt{n}\left\|\int_{M_1} Q^{-1}(s)e^{st}\,ds\right\|_2 \\ &\leq \sqrt{n}\max_{T\geq T_0,\, s\in M_1}\|Q^{-1}(s)\|_2|e^{st}|\left|\int_{M_1} ds\right| \\ &\leq \frac{2\sqrt{n}}{T}e^{ct}(c+\alpha)\to 0\end{aligned}$$

as $T \to \infty$. Similarly, $\|\int_{M_2} Q^{-1}(s)e^{st}\,ds\| \to 0$ as $T \to \infty$. Thus, (3.8) holds.

Next, let T_0 be as above. If $G(s) = Q^{-1}(s) - (I/(s-\alpha_0))$, then

$$G(s) = Q^{-1}(s)\left[I - \frac{Q(s)}{s-\alpha_0}\right] = Q^{-1}(s)\left[\frac{-\alpha_0 I + A + Be^{-s\tau}}{s-\alpha_0}\right]$$

$$\|G(s)\|_2 \leq \frac{2}{T^2}[|\alpha_0|+\|A\|_2+\|B\|_2e^{\alpha\tau}]$$

for $s = -\alpha + iT$, $|T| \geq T_0$. Accordingly

$$\begin{aligned}\left\|\int_{(-\alpha)} G(s)e^{st}\,ds\right\| &\leq \sqrt{n}\left\|\int_{(-\alpha)} G(s)e^{st}\,ds\right\|_2 \\ &\leq \sqrt{n}\max_{s=-\alpha+iT,\,|T|\geq T_0}\|G(s)\|_2|e^{st}|\left|\int_{(-\alpha)} ds\right| \\ &\leq \frac{\sqrt{n}}{\pi T}(|\alpha_0|+\|A\|_2+\|B\|_2e^{\alpha\tau})\exp(-\alpha t)\to 0 \qquad (3.11)\end{aligned}$$

as $T \to \infty$. Furthermore

$$\begin{aligned}\left\|\int_{(-\alpha)} (s-\alpha_0)^{-1}Ie^{st}\,ds\right\| &\leq \sqrt{n}\left\|\int_{(-\alpha)} (s-\alpha_0)^{-1}Ie^{st}\,ds\right\|_2 \\ &\leq \sqrt{n}\max_{s=-\alpha+iT,\,|T|\geq T_0}\|(s-\alpha_0)^{-1}I\|_2|e^{st}|\left|\int_{(-\alpha)} ds\right| \\ &\leq \frac{\sqrt{n}}{\pi}\exp(-\alpha t), \qquad t\geq 0. \qquad (3.12)\end{aligned}$$

It follows from (3.4), (3.7), (3.11) and (3.12) that

$$\begin{aligned}\|X(t)\| &= \left\|\int_{(c)} Q^{-1}(s)\exp(st)\,ds\right\| \\ &= \left\|\int_{(-\alpha)} Q^{-1}(s)\exp(st)\,ds\right\| \\ &\leq \left\|\int_{(-\alpha)} G(s)\exp(st)\,ds\right\| + \left\|\int_{(-\alpha)} (s-\alpha_0)^{-1}I\,\exp(st)\,ds\right\| \\ &\leq \frac{\sqrt{n}}{\pi}\exp(-\alpha t), \qquad t\geq 0.\end{aligned}$$

This completes the proof. □

Lemma 3.3 [12]: The general solution of (3.1) with initial condition

$$x(t) = \phi_{k-1}(t) \quad t\in[t_{k-1}-\tau,\, t_{k-1}] \qquad (3.13)$$

is given by

$$x(t) = X(t-t_{k-1})\phi_{k-1}(t_{k-1}) + B\int_{t_{k-1}-\tau}^{t_{k-1}} X(t-s-\tau)\phi_{k-1}(s)\,ds, \qquad t\geq t_{k-1}$$

where $\phi_{k-1}(t)$ is an arbitrarily given initial function on $[t_{k-1}-\tau,\, t_{k-1}]$ and $X(t)$ is the fundamental solution matrix of system (3.1).

By Lemmas 3.1 and 3.3, the following result is immediate.

Lemma 3.4: The general solution of system (3.2) with initial condition (3.13) is given by

$$x(t) = X(t-t_{k-1})\phi_{k-1}(t_{k-1}) + B\int_{t_{k-1}-\tau}^{t_{k-1}} X(t-s-\tau)\phi_{k-1}(s)\,ds + \int_{t_{k-1}}^{t} X(t-s)f(s)\,ds, \qquad t\geq t_{k-1}$$

where $X(t)$ is the fundamental solution matrix of (3.1).

IV. Robust Stabilization

In this section, we discuss the robust stabilization problem of system (2.1), or effectively, the robust-stability problem of the closed-loop system (2.5).

Assume that $t_k - t_{k-1} \geq \delta\tau$, $\delta > 1$

$$\|f_i(t,\, x(t),\, x(t-\tau))\| \leq c_{i1}\|x(t)\| + c_{i2}\|x(t-\tau)\|, \quad i=1,\,2 \qquad (4.1)$$

and system

$$x_1'(t) = \overline{A}_1 x_1(t) + B_1 x_1(t-\tau) \qquad (4.2)$$

is stable with decay rate $\alpha_1 > 0$, where c_{ij} are constants and $\overline{A}_1,\ B_1$ are given by (2.5) and (2.6). It follows from Remark 3.2 and Lemma 3.2 that there exists a constant M_0 such that

$$\|X_1(t-s)\| \leq M_0\exp[-\alpha_1(t-s)], \qquad t\geq s \qquad (4.3)$$

where $X_1(t)$ is the fundamental solution matrix of system (4.2) and M_0 can be taken as $M_0 = \sqrt{n_1}/\pi$. For convenience, define the following notation:

$$\theta_k = \min\left\{\frac{1}{\sqrt{n_1}}\sigma_{\min}(I-\overline{A}_1\alpha_{1k}),\ \frac{1}{\sqrt{n_2}}\sigma_{\min}(\overline{A}_2\alpha_{2k})\right\} - (|\beta_{1k}|c_{11} + |\beta_{2k}|c_{21}) \tag{4.4}$$

$$\beta_k = \frac{1}{\theta_k}$$

$$\gamma_k = \frac{1}{\theta_k}[\max\{\|B_1\alpha_{1k}\|,\ \|B_2\alpha_{2k}\|\} + (|\beta_{1k}|c_{12} + |\beta_{2k}|c_{22})] \tag{4.5}$$

$$P(\alpha) = (p_{ij}(\alpha))_{2\times 2} \tag{4.6}$$

where α_{ik} and β_{ik} are defined by (2.3), and

$$p_{11}(\alpha) = \frac{M_0}{\alpha_1 - \alpha}(c_{11} + c_{12}e^{\alpha\tau})$$
$$p_{12}(\alpha) = p_{11}(\alpha)$$
$$p_{21}(\alpha) = \frac{\sqrt{n_2}}{\sigma_{\min}(\overline{A}_2)}(c_{21} + c_{22}e^{\alpha\tau})$$
$$p_{22}(\alpha) = \frac{\sqrt{n_2}}{\sigma_{\min}(\overline{A}_2)}[c_{21} + (c_{22} + \|B_2\|)e^{\alpha\tau}]$$

with M_0 given by (4.3).

Theorem 4.1: For the closed-loop system (2.5), and for $k = 1, 2, \ldots$, assume that:

1) $\sigma_{\min}(\overline{A}_2) > 0, \theta_k > 0$;
2) there exists a constant α satisfying $0 < \alpha < \alpha_1$ such that $I - P(\alpha)$ is an M-matrix;
3) $\max\{e^{\alpha\tau}, (\beta_k + \gamma_k e^{\alpha\tau})\} \le M_k \le c$ where c is a constant, β_k and γ_k are given by (4.4) and (4.5).

Then, $(\ln(cM)/\delta\tau) - \alpha < 0$ implies that (2.5) is robustly exponentially stable in the large, and the solution of (2.5) has the following estimate:

$$\|x(t)\| \le \|\Phi\|M\ \exp\left[\left(\frac{\ln(cM)}{\delta\tau} - \alpha\right)(t-t_0)\right], \qquad t \ge t_0$$

where $M = M_0(1 + (\|B_1\|e^{\alpha_1\tau}/\alpha_1))\|(I-P(\alpha))^{-1}\|$, and M_0 is given by (4.3).

Proof: It follows readily from (2.3) that v_i' and w_i' exist on $[t_{k-1}, t_k)$. Thus, for $t \in [t_{k-1}, t_k)$, (2.5) becomes

$$\begin{cases} x_1'(t) = \left[\overline{A}_1 x_1(t) + B_1 x_1(t-\tau)\right] + f_1(t, x(t), x(t-\tau)) \\ 0 = \left[\overline{A}_2 x_2(t) + B_2 x_2(t-\tau)\right] + f_2(t, x(t), x(t-\tau)), \end{cases}$$
$$t \in [t_{k-1}, t_k). \tag{4.7}$$

Let the initial condition of system (4.7) be

$$x_i(t) = \phi_{ik-1}(t) \quad t \in [t_{k-1}-\tau, t_{k-1}]$$

and $\Phi_{k-1}(t) = (\phi_{1k-1}(t), \phi_{2k-1}(t))^T$, where $\phi_{ik-1}(t)$ is a function of bounded variation and right-continuous on $[t_{k-1}-\tau, t_{k-1}]$.

By Lemma 3.3, it follows from (4.7) with the associated initial condition that for $t \in [t_{k-1}, t_k)$

$$x_1(t) = X_1(t-t_{k-1})\phi_{1k-1}(t_{k-1}) + B_1\int_{t_{k-1}-\tau}^{t_{k-1}} X_1(t-s-\tau)\phi_{1k-1}(s)\,ds + \int_{t_{k-1}}^{t} X_1(t-s)f_1(s, x(s), x(s-\tau))\,ds$$

where $X_1(t)$ is the fundamental solution matrix of (4.2), together with (4.3) and (4.1). This leads to

$$\|x_1(t)\| \le M_0\left(1+\frac{\|B_1\|e^{\alpha_1\tau}}{\alpha_1}\right)\exp[-\alpha_1(t-t_{k-1})]\|\phi_{1k-1}\| + \int_{t_{k-1}}^{t} M_0\ \exp[-\alpha_1(t-s)] \cdot [c_{11}(\|x_1(s)\|+\|x_2(s)\|)+c_{12}(\|x_1(s-\tau)\| + \|x_2(s-\tau)\|)]\,ds \tag{4.8}$$

where $t \in [t_{k-1}, t_k)$, $\alpha_1 > 0$ and c_{ij} are given by (4.1) and (4.3), and $\|\phi_{ik-1}\| = \sup_{t_{k-1}-\tau\le t\le t_{k-1}}\|\phi_{ik-1}(t)\|$.

Multiplying both sides of (4.8) by $\exp[\alpha(t-t_{k-1})]$, where α is given by assumption 2), yields

$$\|x_1(t)\|\exp[\alpha(t-t_{k-1})] \le M_0\left(1+\frac{\|B_1\|e^{\alpha_1\tau}}{\alpha_1}\right)\|\phi_{1k-1}\| + \int_{t_{k-1}}^{t} M_0\ \exp[-(\alpha_1-\alpha)(t-s)] \cdot \{c_{11}(\|x_1(s)\|+\|x_2(s)\|)\exp[\alpha(s-t_{k-1})]+c_{12}e^{\alpha\tau} \cdot (\|x_1(s-\tau)\|+\|x_2(s-\tau)\|)\exp[\alpha(s-\tau-t_{k-1})]\}\,ds. \tag{4.9}$$

Let

$$y_i(t) = \sup_{t_{k-1}-\tau\le s\le t}\{\|x_i(s)\|\exp[\alpha(s-t_{k-1})]\}, \qquad i = 1, 2. \tag{4.10}$$

Then, it follows from (4.8)–(4.10) that

$$y_1(t) \le M_0\left(1+\frac{\|B_1\|e^{\alpha_1\tau}}{\alpha_1}\right)\|\phi_{1k-1}\| + \frac{M_0}{\alpha_1-\alpha}[(c_{11}+c_{12}e^{\alpha\tau})(y_1(t)+y_2(t))],$$
$$t \in [t_{k-1}, t_k). \tag{4.11}$$

For $t \in [t_{k-1}, t_k)$, the second equation of (4.7) becomes

$$\overline{A}_2 x_2(t) = -B_2 x_2(t-\tau) - f_2(t, x(t), x(t-\tau))$$

which leads to

$$\left\|\overline{A}_2 x_2(t)\right\| \le \|B_2\|\|x_2(t-\tau)\| + c_{21}\|x(t)\| + c_{22}\|x(t-\tau)\|. \tag{4.12}$$

Since

$$\|\overline{A}_2 x_2(t)\| \ge \frac{\sigma_{\min}(\overline{A}_2)}{\sqrt{n_2}}\|x_2(t)\|$$

and $\sigma_{\min}(\overline{A}_2) > 0$, we have, from (4.12), the following:

$$\|x_2(t)\| \le \frac{\sqrt{n_2}}{\sigma_{\min}(\overline{A}_2)}[\|B_2\|\|x_2(t-\tau)\| + c_{21}\|x(t)\| + c_{22}\|x(t-\tau)\|].$$

Moreover,

$$y_2(t) \le \frac{\sqrt{n_2}}{\sigma_{\min}(\overline{A}_2)}\{(c_{21}+c_{22}e^{\alpha\tau})y_1(t) + [c_{21} + (c_{22}+\|B_2\|)e^{\alpha\tau}]y_2(t)\} \tag{4.13}$$

where $y_i(t)$ are defined by (4.10).

Combining (4.11) and (4.13), we obtain

$$y(t) \le M_0 \left(1 + \frac{\|B_1\| e^{\alpha_1 \tau}}{\alpha_1}\right) \Psi_{k-1} + P(\alpha) y(t), \quad t \in [t_{k-1}, t_k), \tag{4.14}$$

where $y(t) = (y_1(t), y_2(t))^T$, $\Psi_{k-1} = (\|\phi_{1k-1}\|, 0)^T$, $P(\alpha)$ is given by (4.6), and $I - P(\alpha)$ is an M-matrix. Thus, from Lemma A3 in the Appendix, (4.14) implies that

$$y(t) \le M_0 \left(1 + \frac{\|B_1\| e^{\alpha_1 \tau}}{\alpha_1}\right) (I - P(\alpha))^{-1} \Psi_{k-1}, \quad t \in [t_{k-1}, t_k)$$

which reduces to

$$\|x(t)\| \le M \|\Phi_{k-1}\| \exp[-\alpha(t - t_{k-1})], \qquad t \in [t_{k-1}, t_k) \tag{4.15}$$

where $x(t) = (x_1(t), x_2(t))^T$, $M = M_0(1 + (\|B_1\| e^{\alpha_1 \tau}/\alpha_1)) \|(I - P(\alpha))^{-1}\|$.

On the other hand, system (2.5) implies that

$$\begin{aligned} x_1(t_k) - x_1(t_k - h) &= \int_{t_k - h}^{t_k} \big[(\overline{A}_1 x_1(s) + B_1 x_1(s-\tau))\, dv_1(s) \\ &\quad + f_1(s, x(s), x(s-\tau))\, dw_1(s)\big] \end{aligned}$$

and

$$\begin{aligned} 0 = \int_{t_k - h}^{t_k} &\big[(\overline{A}_2 x_2(s) + B_2 x_2(s-\tau))\, dv_2(s) \\ &+ f_2(s, x(s), x(s-\tau))\, dw_2(s)\big] \end{aligned}$$

where $h > 0$ is sufficiently small, as $h \to 0^+$, which reduces to

$$\begin{cases} x_1(t_k) - x_1(t_k^-) = \overline{A}_1 \alpha_{1k} x_1(t_k) + B_1 \alpha_{1k} x_1(t_k - \tau) \\ \quad + \beta_{1k} f_1(t_k, x(t_k), x(t_k - \tau)) \\ 0 = \overline{A}_2 \alpha_{2k} x_2(t_k) + B_2 \alpha_{2k} x_2(t_k - \tau) \\ \quad + \beta_{2k} f_2(t_k, x(t_k), x(t_k - \tau)) \end{cases}$$

or

$$\begin{cases} (I - \overline{A}_1 \alpha_{1k}) x_1(t_k) = x_1(t_k^-) + B_1 \alpha_{1k} x_1(t_k - \tau) \\ \quad + \beta_{1k} f_1(t_k, x(t_k), x(t_k - \tau)) \\ \overline{A}_2 \alpha_{2k} x_2(t_k) = -B_2 \alpha_{2k} x_2(t_k - \tau) \\ \quad - \beta_{2k} f_2(t_k, x(t_k), x(t_k - \tau)). \end{cases} \tag{4.16}$$

It follows from (4.16) and (4.1) that

$$\begin{aligned} &\frac{\sigma_{\min}\left(I - \overline{A}_1 \alpha_{1k}\right)}{\sqrt{n_1}} \|x_1(t_k)\| \\ &\quad \le \|x_1(t_k^-)\| + \|B_1 \alpha_{1k}\| \|x_1(t_k - \tau)\| \\ &\qquad + |\beta_{1k}| (c_{11} \|x(t_k)\| + c_{12} \|x(t_k - \tau)\|) \end{aligned} \tag{4.17}$$

$$\begin{aligned} &\frac{\sigma_{\min}\left(\overline{A}_2 \alpha_{2k}\right)}{\sqrt{n_2}} \|x_2(t_k)\| \\ &\quad \le \|B_2 \alpha_{2k}\| \|x_2(t_k - \tau)\| \\ &\qquad + |\beta_{2k}| (c_{21} \|x(t_k)\| + c_{22} \|x(t_k - \tau)\|). \end{aligned} \tag{4.18}$$

Observe that $\|x\| = \|x_1\| + \|x_2\|$. Therefore, (4.17) and (4.18) reduce to

$$\begin{aligned} \theta_k \|x(t_k)\| \le \|x_1(t_k^-)\| + [&\max\{\|B_1 \alpha_{1k}\|, \|B_2 \alpha_{2k}\|\} + |\beta_{1k}| c_{12} \\ &+ |\beta_{2k}| c_{22})] \|x(t_k - \tau)\| \end{aligned}$$

where θ_k is given by (4.4). Based on assumption 1), we immediately arrive at

$$\|x(t_k)\| \le \beta_k \|x(t_k^-)\| + \gamma_k \|x(t_k - \tau)\| \tag{4.19}$$

where β_k and γ_k are defined by (4.5).

From (4.15) and (4.19), we can obtain the following results: When $k = 1$, take $\Phi_0(t) = \Phi(t) = (\phi_1(t), \phi_2(t))^T$, $t \in [t_0 - \tau, t_0]$, so that

$$\|x(t)\| \le \|\Phi\| M \exp[-\alpha(t - t_0)], \qquad t \in [t_0, t_1) \tag{4.20}$$

which reduces to

$$\|x(t_1^-)\| \le \|\Phi\| M \exp[-\alpha(t_1 - t_0)] \tag{4.21}$$

$$\|x(t_1)\| \le \beta_1 \|x(t_1^-)\| + \gamma_1 \|x(t_1 - \tau)\|. \tag{4.22}$$

When $k = 2$, naturally, take $\Phi_1(t) = x(t)$, $t \in [t_1 - \tau, t_1]$, so that in view of (4.20)–(4.22)

$$\begin{aligned} \|\Phi_1\| &= \sup_{t_1 - \tau \le t \le t_1} \|x(t)\| \\ &\le \|\Phi\| M \exp[-\alpha(t_1 - t_0)] \max\{e^{\alpha\tau}, \beta_1 + \gamma_1 e^{\alpha\tau}\} \\ &\le \|\Phi\| M M_1 \exp[-\alpha(t_1 - t_0)] \end{aligned}$$

where M_1 is given in assumption 3). Thus, for $t \in [t_1, t_2)$

$$\begin{aligned} \|x(t)\| &\le \|\Phi_1\| M \exp[-\alpha(t - t_1)] \\ &\le \|\Phi\| M^2 M_1 \exp[-\alpha(t - t_0)]. \end{aligned}$$

Generally, as $t \in [t_{k-1}, t_k)$,

$$\|x(t)\| \le \|\Phi\| M^k M_1 \cdots M_{k-1} \exp[-\alpha(t - t_0)]. \tag{4.23}$$

Since $M_k \le c$, $t_k - t_{k-1} \ge \delta\tau$, $(\delta > 1)$ and $cM \ge 1$

$$\begin{aligned} M^{k-1} M_1 \cdots M_{k-1} &\le (cM)^{k-1} \\ &\le \exp\left[\frac{\ln(cM)}{\delta\tau}(t_{k-1} - t_0)\right] \\ &\le \exp\left[\frac{\ln(cM)}{\delta\tau}(t - t_0)\right] \end{aligned} \tag{4.24}$$

where $t \in [t_{k-1}, t_k)$. From (4.23) and (4.24), we have

$$\|x(t)\| \le \|\Phi\| M \exp\left[\left(\frac{\ln(cM)}{\delta\tau} - \alpha\right)(t - t_0)\right], \qquad t \ge t_0. \tag{4.25}$$

This completes the proof. □

Remark 4.1: For assumption 2) in Theorem 4.1, if $I - P(0)$ is an M-matrix, then by Lemma A2 in the Appendix there exists a constant α, $0 < \alpha < \alpha_1$ such that $I - P(\alpha)$ is an M-matrix.

Note that the matrices $I - \overline{A}_1 \alpha_{1k}$ and $\overline{A}_2 \alpha_{2k}$ may be invertible. To study this case, we introduce the following notation:

$$\tilde{\theta}_k = 1 - c_{11} \|(I - \overline{A}_1 \alpha_{1k})^{-1} \beta_{1k}\| - c_{21} \|(\overline{A}_2 \alpha_{2k})^{-1} \beta_{2k}\|$$

$$\tilde{\beta}_k = \frac{1}{\tilde{\theta}_k} \|(I - \overline{A}_1 \alpha_{1k})^{-1}\| \tag{4.26}$$

$$\begin{aligned} \tilde{\gamma}_k = \frac{1}{\tilde{\theta}_k} [&c_{12} \|(I - \overline{A}_1 \alpha_{1k})^{-1} \beta_{1k}\| + c_{22} \|(\overline{A}_2 \alpha_{2k})^{-1} \beta_{2k}\| \\ &+ \max\{\|(I - \overline{A}_1 \alpha_{1k})^{-1} B_1 \alpha_{1k}\|, \|\overline{A}_2^{-1} B_2\|\}] \end{aligned} \tag{4.27}$$

$$\tilde{P}(\alpha) = (\tilde{p}_{ij}(\alpha))_{2\times 2} \tag{4.28}$$

where $\tilde{p}_{11}(\alpha) = p_{11}(\alpha)$, $\tilde{p}_{12}(\alpha) = p_{12}(\alpha)$, given by (4.6), and

$$\tilde{p}_{21}(\alpha) = \|\overline{A}_2^{\,-1}\| (c_{21} + c_{22} e^{\alpha\tau})$$

$$\tilde{p}_{22}(\alpha) = \tilde{p}_{21}(\alpha) + \|\overline{A}_2^{\,-1} B_2\| e^{\alpha\tau}.$$

If $\overline{A}_2$ is invertible, then it follows from (4.7) that

$$\begin{aligned}\|x_2(t)\| \le& \|\overline{A}_2{}^{-1}B_2\|\|x_2(t-\tau)\| \\ &+ \|\overline{A}_2{}^{-1}\|(c_{21}\|x(t)\| + c_{22}\|x(t-\tau)\|)\end{aligned}$$

which leads to

$$y_2(t) \le \tilde{p}_{21}(\alpha)y_1(t) + \tilde{p}_{22}(\alpha)y_2(t).$$

Moreover, together with (4.11), it yields

$$y(t) \le M_0\left(1 + \frac{\|B_1\|e^{\alpha_1\tau}}{\alpha_1}\right)\Psi_{k-1} + \tilde{P}(\alpha)y(t), \quad t \in [t_{k-1},\, t_k).$$

If the matrices $I - \overline{A}_1\alpha_{1k}$ and $\overline{A}_2\alpha_{2k}$ are invertible, then it follows from (4.16) that

$$\begin{aligned}x_1(t_k) =& (I - \overline{A}_1\alpha_{1k})^{-1}[x_1(t_k^-) + B_1\alpha_{1k}x_1(t_k-\tau) \\ &+ \beta_{1k}f_1(t_k,\, x(t_k),\, x(t_k-\tau))] \\ x_2(t_k) =& -(\overline{A}_2\alpha_{2k})^{-1}[B_2\alpha_{2k}x_2(t_k-\tau) \\ &+ \beta_{2k}f_2(t_k,\, x(t_k),\, x(t_k-\tau))].\end{aligned}$$

These imply that

$$\|x(t_k)\| \le \tilde{\beta}_k\|x(t_k^-)\| + \tilde{\gamma}_k\|x(t_k-\tau)\|$$

provided that $\tilde{\theta}_k > 0$. Similar to the inference of Theorem 4.1, we obtain the following result.

Corollary 4.1: For the closed-loop system (2.5), assume that

1) $\tilde{\theta}_k > 0$, $I - \overline{A}_1\alpha_{1k}$ and $\overline{A}_2\alpha_{2k}$ are invertible, $k = 1, 2, \ldots$;
2) there exists a constant α satisfying $0 < \alpha < \alpha_1$ such that $I - \tilde{P}(\alpha)$ is an M-matrix;
3) $\max\{e^{\alpha\tau},\, (\tilde{\beta}_k + \tilde{\gamma}_k e^{\alpha\tau})\} \le M_k \le c$, where c is a constant.

Then, the conclusion of Theorem 4.1 holds with $M = M_0(1 + (\|B_1\|e^{\alpha_1\tau}/\alpha_1))\|(I - \tilde{P}(\alpha))^{-1}\|$, where $\tilde{\theta}_k$, $\tilde{\beta}_k$, $\tilde{\gamma}_k$, and $\tilde{P}(\alpha)$ are given by (4.26)–(4.28).

Theorem 4.2: If assumption 2) of Theorem 4.1 is replaced by the following condition:

2)′ there exists a constant α satisfying $0 < \alpha < \alpha_1$ such that

$$\eta(\alpha) = \max\{p_{11}(\alpha) + p_{12}(\alpha),\, p_{21}(\alpha) + p_{22}(\alpha)\} < 1 \qquad (4.29)$$

where $p_{ij}(\alpha)$ are given by (4.6), then the conclusion of Theorem 4.1 holds with $M = M_0/(1 - \eta(\alpha))(1 + (\|B_1\|e^{\alpha_1\tau}/\alpha_1))$.

Proof: Similar to the argument used in Theorem 4.1, we obtain the inequalities (4.11) and (4.13). If we define

$$y(t) = \max_i\{y_i(\xi)|\xi \in [t_{k-1}-\tau,\, t],\, i = 1,\, 2\}$$

then, by assumption 2)′, (4.11) and (4.13), we have

$$y(t) \le M_0\left(1 + \frac{\|B_1\|e^{\alpha_1\tau}}{\alpha_1}\right)\|\phi_{1k-1}\| + \eta(\alpha)y(t)$$

namely

$$y(t) \le \frac{M_0}{1-\eta(\alpha)}\left(1 + \frac{\|B_1\|e^{\alpha_1\tau}}{\alpha_1}\right)\|\phi_{1k-1}\|, \quad t \in [t_{k-1},\, t_k)$$

which reduces to (4.15) with $M = [M_0/(1 - \eta(\alpha))](1 + (\|B_1\|e^{\alpha_1\tau}/\alpha_1))$. The rest of the proof is similar to that of Theorem 4.1, therefore, details are omitted. □

Corollary 4.2: If assumption 2)′ of Corollary 4.1 is replaced by the following condition:

2)′′ there exists a constant α satisfying $0 < \alpha < \alpha_1$ such that

$$\tilde{\eta}(\alpha) = \max\{\tilde{p}_{11}(\alpha) + \tilde{p}_{12}(\alpha),\, \tilde{p}_{21}(\alpha) + \tilde{p}_{22}(\alpha)\} < 1, \qquad (4.30)$$

where $\tilde{p}_{ij}(\alpha)$ are given by (4.28), then the conclusion of Corollary 4.1 holds with $M = [M_0/(1 - \tilde{\eta}(\alpha))](1 + (\|B_1\|e^{\alpha_1\tau}/\alpha_1))$.

V. Controller Design with an Example

In this section, we describe a systematic-design procedure for the robust impulsive control law developed above, for the singular impulsive delayed system (2.1). The procedure is established on the basis of the analysis given in Section IV above. An example is given to illustrate this design procedure, which also serves for interpretation of the theoretical results obtained in the paper.

The suggested design procedure is based on Theorems 4.1 and 4.2, and is summarized as follows.

Step 1) For the system (2.1), select the gain matrix K_1 such that the system (4.2) is stable with decay rate $\alpha_1 > 0$. For convenience, one may pick a constant $\alpha_1 > 0$ such that $-\mu(\overline{A}_1) - \alpha_1 > \|B_1\|\exp(\tau\alpha_1)$.

Step 2) Select the gain matrix K_2 and compute $\sigma_{\min}(\overline{A}_2)$ and θ_k, which are defined by (4.4). If $\sigma_{\min}(\overline{A}_2) > 0$ and $\theta_k > 0$, then go to Step 3; otherwise, go back to Step 1.

Step 3) Pick a constant α, $0 < \alpha < \alpha_1$, and then calculate $P(\alpha)$ or $\eta(\alpha)$, which are given by (4.6) and (4.29), respectively. If $I - P(\alpha)$ is an M-matrix or if $\eta(\alpha) < 1$, then go to Step 4; otherwise, go back to Step 2.

Step 4) Compute $\max\{e^{\alpha\tau},\, (\beta_k + \gamma_k e^{\alpha\tau})\} \le M_k$, where β_k and γ_k are given by (4.5). If $M_k \le c$, then go to Step 5; otherwise, go back to Step 3.

Step 5) Calculate the constant M with the expressions $M = (\sqrt{n}_1/\pi)(1 + (\|B_1\|e^{\alpha_1\tau}/\alpha_1))\|(I - P(\alpha))^{-1}\|$ or $M = (\sqrt{n}_1/(1 - \eta(\alpha))\pi)(1 + (\|B_1\|e^{\alpha_1\tau}/\alpha_1))$. If $(\ln(cM)/\delta\tau) - \alpha < 0$, then go to Step 6; otherwise, go back to Step 1.

Step 6) Substitute the decentralized gain matrices K_1 and K_2, determined in Step 2, into (2.4) to obtain the controller $u_i(t)$ for the system (2.1).

Remark 5.1: From Corollaries 4.1 and 4.2, we can derive a similar control design procedure for system (2.1); the details are omitted for brevity.

Example: Consider the uncertain, delayed, singular and impulsive system (2.1) and (2.2) with nonlinear perturbations, where $n_1 = 2$, $n_2 = 2$ and $n = 4$

$$A_1 = \begin{pmatrix}1 & 2\\ 0 & -3\end{pmatrix} \quad B_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad C_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$A_2 = \begin{pmatrix}-1 & 0\\ 1 & 2\end{pmatrix} \quad B_2 = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \quad C_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

and with nonlinear perturbation $f_i(t,\, x(t),\, x(t-\tau))$ satisfying (4.1), with $c_{11} = 2$, $c_{12} = 1$, $c_{21} = 1$, $c_{22} = 2$, $v_i(t)$, and $w_i(t)$ given by (2.3), where $t_k - t_{k-1} \ge \delta\tau$, $\tau = 1/10$, and

$$\alpha_{1k} = \frac{1}{9} \quad \alpha_{2k} = \frac{(-1)^{k-1}}{10} \quad \beta_{1k} = \frac{(-1)^k}{4}$$

$$\beta_{2k} = \tfrac{1}{12}[(-1)^k - 1].$$

Now, one can easily design a robust impulsive controller by following the above procedure, such that system (2.1) specified above is exponentially stable:

Step 1) Select $K_1 = \left(\begin{smallmatrix}-46 & -2\\ 0 & -43\end{smallmatrix}\right)$, and $\alpha_1 = 20$. Then

$$\overline{A}_1 = A_1 + C_1K_1 = \begin{pmatrix}-45 & 0\\ 0 & -46\end{pmatrix}$$

$$25 = -\mu(\overline{A}_1) - \alpha_1 > \|B_1\|\exp(\tau\alpha_1) = 7.39.$$

Step 2) Select $K_2 = \begin{pmatrix} -1 & -45 \\ -45 & 0 \end{pmatrix}$. Then

$$\overline{A}_2 = A_2 + C_2 K_2 = \begin{pmatrix} -46 & 0 \\ 0 & -47 \end{pmatrix}.$$

From (4.4), we have

$$\sigma_{\min}(\overline{A}_2) = 46 > 0 \quad \theta_k = \begin{cases} 2.27, & k = 2n-1 \\ 2.58, & k = 2n \end{cases} \qquad n \in \mathbf{N}.$$

Step 3) Select a constant, $\alpha = 10$, $0 < \alpha < \alpha_1$, and calculate

$$I - P(\alpha) = \begin{pmatrix} 1 - p_{11}(\alpha) & -p_{12}(\alpha) \\ -p_{21}(\alpha) & 1 - p_{22}(\alpha) \end{pmatrix} = \begin{pmatrix} 0.86 & -0.14 \\ -0.09 & 0.85 \end{pmatrix}$$

which is an $M-$matrix. Then $\|(I - P(\alpha))^{-1}\| = 1.39$.

Step 4) Compute $\max\{e^{\alpha\tau}, (\beta_k + \gamma_k e^{\alpha\tau})\} \le \max\{1, (0.38 + 0.39)\} \le 1 := c$.

Step 5) Calculate constant $M = (\sqrt{n}_1/\pi)(1 + (\|B_1\| e^{\alpha_1 \tau}/\alpha_1)) \|(I - P(\alpha))^{-1}\| = 0.86$. Thus, for any $\delta > 0$

$$\frac{\ln(cM)}{\delta\tau} - \alpha < 0.$$

Step 6) Based on the above results, the corresponding controller $u_i(t)$ is obtained as

$$u_1(t) = \begin{pmatrix} -46 & -2 \\ 0 & -43 \end{pmatrix} x_1(t) D v_1$$

$$u_2(t) = \begin{pmatrix} -1 & -45 \\ -45 & 0 \end{pmatrix} x_2(t) D v_2$$

which exponentially stabilizes the given nonlinearly perturbed, time-delayed, singular, and impulsive system.

VI. Conclusions

In this paper, we have formulated and studied the stabilization problem for a general singular-impulsive delayed system with nonlinear perturbations. Such a complex system cannot be handled by traditional techniques that can only deal with pure continuous-time or discrete-time models, perhaps with either time delays or uncertainties. Some specific properties prescribing this hybrid model have been analyzed. Global exponential robust stabilization of the system equilibrium via stabilizing controller design has been investigated. A systematic procedure for designing the robust impulsive controller has been suggested, along with an explicit example for illustration.

Future research along the same line will be devoted to possible engineering applications of the proposed model and its stabilization methodology.

Appendix

Some basic properties associated with the M-matrix are given here for the reader's convenience. A matrix A satisfying any one of the six conditions listed in Lemma A1 below is called an *M-matrix*.

Lemma A1 [2]: Let A be a real square matrix with nonpositive off-diagonal elements. Then the following six conditions are equivalent.

1) The principal minors of A are all positive.
2) The leading principal minors of A are all positive.
3) There is a vector x (or y) whose elements are all positive such that the elements of Ax (or $A^T y$) are all positive.
4) A is nonsingular and the elements of A^{-1} are all nonnegative.
5) The real parts of the eigenvalues of A are all positive.
6) There is a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$, with $d_i > 0$, such that $DA + A^T D$ is a positive definite matrix.

In what follows, we will give some results concerning M-matrices.

Lemma A2: Let $A(\xi) = (a_{ij}(\xi))_{n\times n}$ be a continuous matrix-valued function, with nonpositive off-diagonal elements in $(-\epsilon, \epsilon)$, $\epsilon > 0$. If $A(0) = (a_{ij}(0))_{n\times n}$ is an M-matrix, then there exists a constant η, $0 < \eta < \epsilon$, such that $A(\xi)$ is an M-matrix-valued function on the interval $(-\eta, \eta)$.

Proof: Since $A(0) = (a_{ij}(0))_{n\times n}$ is an M-matrix, from Lemma A1, there exists a vector $x = (x_1, \ldots, x_n)^T > 0$, $x \in R^n$, such that $\sum_{j=1}^n a_{ij}(0) x_j > 0$, $i = 1, \ldots, n$.

Let $f_i(\xi) = \sum_{j=1}^n a_{ij}(\xi) x_j$, $i = 1, \ldots, n$. Then $f_i(0) = \sum_{j=1}^n a_{ij}(0) x_j > 0$. Clearly, $f_i(\xi)$ is a continuous function on $(-\epsilon, \epsilon)$, and there exists a constant, $\eta_i \in (0, \epsilon)$, such that

$$f_i(\xi) = \sum_{j=1}^n a_{ij}(\xi) x_j > 0, \qquad \xi \in (-\eta_i, \eta_i),\ i = 1, \ldots, n.$$

Let $\eta = \min_i\{\eta_i\}$. Then

$$f_i(\xi) = \sum_{j=1}^n a_{ij}(\xi) x_j > 0, \qquad \xi \in (-\eta, \eta),\ 0 < \eta < \epsilon,$$
$$i = 1, \ldots, n$$

implying that $A(\xi)$ is an M-matrix-valued function on the interval $(-\eta, \eta)$. This completes the proof. □

Lemma A3: For a vector inequality $Ax \le b$, where $A = (a_{ij})_{n\times n}$, $b = (b_1, \ldots, b_n)^T$, $x = (x_1, \ldots, x_n)^T \in R^n$, if A is an M-matrix, then $x \le A^{-1} b$.

Proof: It is readily seen from Lemma A1 that A is nonsingular and elements of A^{-1} are nonnegative. Thus, the vector inequality $Ax \le b$ implies that $x \le A^{-1} b$. This completes the proof. □

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A Deterministic Nonlinear-Capacitor Model for Single-Electron Tunneling Junctions

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***Abstract*—Single-electron tunneling junctions (SETJs) have intriguing properties which make them a primary nanoelectronic device for highly compact, fast, and low-power circuits. However, standard models for SETJs are based on a quantum mechanical approach which is very impractical for the analysis and design of SETJ-based circuitry, where a simple, preferably deterministic model is a prerequisite. We verify by physics-based Monte Carlo simulations that the tunneling junction can in fact be modeled by a piecewise linear voltage-charge relation, which, from the circuit-theoretic perspective, is a nonlinear capacitor.**

***Index Terms*—Nonlinear capacitor, single-electron tunneling junction.**

I. Single-Electron Tunneling Junctions

Single-electron tunneling junctions (SETJs) are perhaps the most compact of all electronic devices. It is theoretically possible to create double-junction switches or logic gates within areas smaller than 100 nm^2 corresponding to a density of 10^{12} devices per cm^2, and the small capacitance implies extremely high switching speeds. This combination of density and speed make it difficult to imagine any other alternative technology that could match the long-term possibilities of single-electronics.

Manuscript received April 7, 2000; revised January 10, 2001. This work is supported by the Marine Underwater Research Institute, Office of Naval Research under Grant N00014-98-1-0594 and Grant N00014-99-1-0339. This paper was recommended by Associate Editor Y. Park.

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Publisher Item Identifier S 1057-7122(01)06085-8.

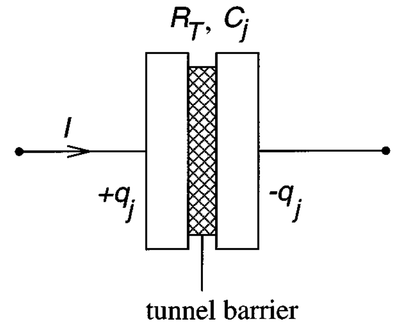

Fig. 1. Physical schematics of a single-electron tunneling junction.

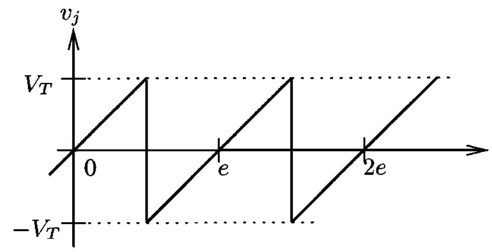

Fig. 2. Linear-periodic voltage-charge relation for a SETJ.

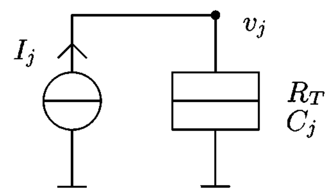

Fig. 3. Current-biased SET junction.

To explain single-electron effects, an "orthodox theory" based on a phenomenological Hamiltonian approach with a tunneling term and the electrostatic energy has proved successful [1].

To analyze circuits with single-electron junctions (SETJs), however, simplified models of the junction characteristics are required. One example is the Monte Carlo model in which classical electrons tunnel through the junctions stochastically with a probability that is a function of the temperature and the change in electrostatic energy. In the limiting case of zero temperature and small average current, it further reduces to a deterministic model where electron tunneling occurs as soon as it decreases the overall electrostatic energy of the system.

Based on these considerations, a deterministic model for the junction characteristics has been proposed which avoids any unnecessary complexities due to the stochastic nature of quantum mechanics and thermal fluctuation [2].

In this model, it is assumed that an electron tunnels when the junction voltage v_j reaches the tunneling voltage

$$V_T = \frac{e}{2C_j}. \tag{1}$$

The behavior of the junction (shown in Fig. 1) can therefore be modeled by a single-valued piecewise linear voltage–charge relation (Fig. 2). This model has been applied for the investigation of