A RATIONAL SUBDIVISION SCHEME USING COSINE-MODULATED WAVELETS

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ABSTRACT

This paper proposes a rational subdivision scheme using cosine-modulated wavelets. Subdivision schemes constructed from iterated filter banks can be used to generate wavelets and limit functions for multiresolution analysis. The proposed subdivision scheme is based on a kind of nonuniform filter banks called recombination nonuniform filterbanks (RN FB). It is shown that if the component FBs in a RNFB are wavelet FBs, then the necessary condition for convergence to limit functions in the subdivision scheme is also satisfied. Therefore, the design of different rational subdivision schemes is considerably simplified. An efficient RNFB, called RN cosine modulated FBs (CMFB), constructed from uniform CMFBs and cosine-modulated transmultiplexers (TMUX) are further investigated. Using a design technique for designing RN CMFB and cosine modulated wavelets (CMW) previously reported by the authors, very smooth limit functions can be generated from the rational subdivision scheme. A design example is given to illustrate the proposed method.

I. INTRODUCTION

Rational subdivision schemes based on iterated filtering with nonuniform filter banks (FBs) can be used to generate wavelets and limit functions for multiresolution analysis. The theory and properties of this subdivision scheme with integer subdivision was studied in [14]. It is closely related to the generation of M-adic (or M-band wavelets). The rational case was first studied by Blu [2] and then in more detailed by Rioul and Blu [10]. Rational subdivision schemes present many interesting properties which are quite different from the M-adic case. For example, unlike the M-adic case, the shift property of the limit functions no longer hold, and it involves an infinite set dilated functions. The design and convergence analysis of the rational case are more involved than the integer case. The design of two-band orthonormal rational filter banks and wavelets were further studied in a recent article [3], where nonlinear optimization is employed to achieve perfect reconstruction and imposing the regularity condition.

In this paper, we consider a rational subdivision scheme constructed from a kind of nonuniform filter banks called recombination nonuniform filterbanks (RN FB) [4]. In RNFBs, consecutive subchannels of a uniform FB are merged together to yield different rational sampling rates. The advantages of RNFBs are their simplicity in imposing the perfect reconstruction (PR) condition, and their ability to perform dynamic recombination [5,11,13]. It is shown in this paper that if the component FBs in a RNFB are wavelet FBs, then the necessary condition for convergence to limit functions in the subdivision scheme is also satisfied. Therefore, the design of different rational subdivision scheme is considerably simplified. An efficient RNFB, called RN cosine modulated FBs (CMFB), constructed from uniform CMFBs and cosine-modulated transmultiplexers (TMUX) are further investigated. The advantages of using CMFBs in RNFB are that the design and implementation complexities can be drastically reduced, because the analysis and synthesis filters are generated from a single prototype filter. Further, in RN CMFBs, the uniform CMFB and recombination TMUXs can be designed separately by imposing a simple matching condition. Using a technique for designing RN CMFB [5,13] and cosine modulated wavelets (CMW) [1] previously reported by the authors, very smooth limit functions can be generated from the rational subdivision scheme. A design example is given to illustrate the proposed method. Not only can multiresolution analysis with different sampling factors be generated by combination of these basic filterbanks, it can also be made dynamic reconfigurable.

The paper is organized as follows: Section II reviews the theory of rational subdivision schemes. The new cosine-modulated wavelets-based subdivision scheme is given in section III. Further information on the pseudo wavelet series generated from this subdivision scheme is examined in section IV, followed by a simple example in section V. Finally, conclusions are drawn in section IV.

II. THEORY OF RATIONAL SUBDIVISION SCHEMES

A. Rational Subdivision Schemes

Consider a "*p*/*q*-adic" subdivision scheme, which is an infinite collection of sequences $g^{(l)}(n)$, $(n \in Z)$, labeled by $(l \in \aleph)$ and is computed using a recursion of form

$$g^{(l+1)}(n) = \Lambda\{g^{(l)}(n)\}, \quad (1)$$

where $\Lambda\{.\}$ is an linear operator, which interpolates its input by a factor of p following by the filtering with $g(n)$ and a decimation by a factor q. That is

$$u^{(1)}(n) = \Lambda\{u^0(n)\} = \sum_{k\in Z} u^{(0)}(k) g(qn - pk), \quad (2)$$

where $g(n)$ is called the subdivision mask. This is illustrated diagrammatically in Figure 1. This subdivision, also known as iterated filterbanks, can be used to generate wavelets and limit functions (as $l \to \infty$) for performing multi-resolution analysis. For dyadic subdivision scheme where $p = 2$ and $q = 1$, (2) reduces to

$$u^{(1)}(n) = \Lambda\{u^0(n)\} = \sum_{k\in Z} u^{(0)}(k) g(n - 2k), \quad (3)$$

In the dyadic case, an integer shift of s in the input gives rise to an integer shift of $2^{-l}s$ at the *l*-th stage output. That is

$$u^{(l)}(n - 2^l s) = \Lambda^{(l)}\{u^{(0)}(n - s)\}. \quad (5)$$

Because of this shift property, it is only necessary to study the limit functions for $g^{(0)}(n) = \delta(n)$. For the rational case, a similar shift property does not hold and it is necessary to consider the convergence of the sequence

$$g^{(l,s)}(n) = \Lambda^{(l)}\{\delta(n - s)\}. \quad (5)$$

i.e. the impulse response $g^{(l,s)}(n)$ of the LPTV system $\Lambda^{(l)}\{x(n)\}$ to an impulse at $\delta(n - s)$. Under certain conditions, $g^{(l,s)}(n)$, if plotted against $x = n(p/q)^{-l}$, will converge to a function $\phi^{(s)}(x)$. In the dyadic case, $\phi^{(s)}(x) = \phi^{(0)}(x - s)$ is the integer shifts of the scaling function. For the rational case, it consists of an infinite set of distinct compactly supported limit function $\phi^{(s)}(x)$. Fortunately, the shift error can be made to a small value by increasing the regularity imposed.

B. Necessary and Sufficient Condition for Convergence

For a given shift parameter $s \in Z$, the *p*/*q*-adic subdivision scheme $g^{(l,s)}(n)$ converges uniformly to a limit function $\phi^{(s)}(n)$ if, for any sequence of integers n_l satisfying

$$| n_l - (p/q)^l x | \le c , \tag{6}$$

where *c* is constant, we have

$$\sup | \phi^{(s)}(x) - g^{(l,s)}(n_l) | \to 0 \text{ , as } l \to \infty . \tag{7}$$

This definition was introduced in [10]. The meaning of this definition is that, by choosing n_l as in (6) with *c* a given but small enough constant, then n_l will be arbitrarily close to $(p/q)^l x$ as $l \to \infty$. In this case, condition (7) implies that the discrete sequence $g^{(l,s)}(n_l)$ will approach the limit function $\phi^{(s)}(x)$. The reason for including the factor $(p/q)^l$ for *x* in (6) is that because of the upsampling and downsampling in the LPTV system, each stage will interpolator the input by a factor of *p*/*q*. To compensate for this expansion, the factor $(p/q)^l$ is multiplied to *x* in computing the sup function in (7). A necessary and sufficient for convergence is [10]:

A *p*/*q*-adic subdivision scheme $g^{(l,s)}(n)$ converges uniformly, for all $s \in Z$, to (continuous) limit functions $\phi^{(s)}(x)$ if and only if $G(z)$ satisfies the basic conditions

$$G(1) = p \text{ , and } \frac{1 - z^{-p}}{1 - z^{-1}} \text{ divides } G(z) , \tag{8}$$

$$\max_n | g^{(l,s)}(n+1) - g^{(l,s)}(n) | \to 0 \text{ as } j \to \infty . \tag{9}$$

Moreover, there exists $\alpha > 0$ such that

$$\max_n | g^{(l,s)}(n+1) - g^{(l,s)}(n) | \le c(p/q)^{-l\alpha} . \tag{10}$$

and $e^{j\frac{2\pi k}{p}}$, $k \ne 0$, are roots of $G(z)$. Thus, $\frac{1 - z^{-p}}{1 - z^{-1}}$ is a factor of $G(z)$. In [2,3], many designs based on the necessary condition (8) alone do converge. This is also true in our cosine modulated subdivision scheme. Therefore, we shall only focus on this necessary condition later in this paper. If *g*(*n*) is obtained from (the 1st branch $g(n) \overset{Z}{\longleftrightarrow} G(z) = \widetilde{H}_0(z)$) a two-band nonuniform PR filterbank (Fig. 2, with *L*=2, and $q_0 = q$, $q_1 = p - q$, and $p_0 = p_1 = p$) then the limits functions $\phi^{(s)}(x)$ are biorthonormal and they generate a wavelet-like expansion or multiresolution analysis, although the condition for expanding every square integrable function is still an open question.

In the next section, we shall show that the necessary condition (8) are automatically satisfied for a class of nonuniform FBs called recombination FB (RNFB), when the component FBs are all wavelet FBs.

III. SUBDIVISION SCHEME BASED ON RNFB

A. Principle of Recombination Nonuniform Filter Bank

Fig. 3 shows the structure of a recombination nonuniform filter bank, which was first proposed in [4]. It is obtained by merging certain consecutive subbands of an *M*-channel uniform FB by sets of m_l -channel TMUXs. Each merged output represents one output of the nonuniform FB with sampling rate m_l / M, where m_l is the number of TMUXs used to produce this subband (*l*-th subband in this case).

It was shown in [5] that if the *M*-channel FB and the m_l -channel TMUXs are all PR, the whole system will be PR, provided that the delays introduced by the inserted TMUXs are compensated in other branches. If m_l and *M* are coprime, then the decimator (*M*) and interpolator (m_l) can be interchanged. Moreover, by using the noble identity [9], $H_{r_l+i}(z)$, and $G_{l,i}(z)$ can be moved across the interpolator and decimator and it gives rise to an equivalent LTI representation of the analysis filter as shown in Fig. 2, where $\widetilde{H}_l(z) = \sum_{i=0}^{m_l-1} H_{r_l+i}(z^{m_l}) G_{l,i}(z^M)$, with $q_l = m_l$ and $p_l = M$. In other word, the *l*-th equivalent analysis filter of the nonuniform filter bank is the sum of the product filters between *H* 's and *G* 's with the powers in the *z* variables raised appropriately.

Likewise, the 1st branch of the RNFB can be used to generate limits function in a rational sub-division scheme if the necessary and sufficient condition in (8) and (9) are satisfied. For reason mentioned earlier, we shall focus on (8), which is equivalent to saying that $e^{j\frac{2\pi k}{p}}$, $k \ne 0$, are roots of $G(z)$. Suppose that both the uniform FB and the TMUX are derived from wavelet FBs satisfying the *K*-regularity condition. That is:

$e^{j\frac{2\pi k}{M}}$, $k \ne 0$, are *K*-th order roots of $H_0(z)$;

and $e^{j\frac{2\pi k}{m_l}}$, $k \ne 0$, are *K*-th order roots of $G_{l,0}(z)$. Substituting $z = e^{j\frac{2\pi k}{M}}$ into $\widetilde{H}_0(z)$ gives

$$\widetilde{H}_0(e^{j2\pi k/M}) = \sum_{i=0}^{m_l-1} H_i(e^{2\pi k m_l/M}) G_{l,i}(1) . \tag{11}$$

From the admissibility condition of wavelet filter banks, we have $G_{l,i}(1) = 0$, $i \ne 0$, and (11) reduces to

$$\widetilde{H}_0(e^{j2\pi k/M}) = H_0(e^{2\pi k m_l/M}) .$$

Since $e^{j\frac{2\pi k}{M}}$, $k \ne 0$, are *K*-th order roots of $H_0(z)$, so $e^{j\frac{2\pi k}{M}}$, $k \ne 0$ is also a root of $\widetilde{H}_0(z)$, since m_l and *M* are coprime. Hence the necessary condition (8) is satisfied.

B. Sub-division Scheme based on RN CMFB

Here, we consider an efficient RNFB called the RN cosine modulated FB (RN CMFB), where the uniform FB and TMUXs are derived from CMFBs. The analysis filters $h_k(n)$ and synthesis filters $f_k(n)$ are given by

$$h_k(n) = 2h(n)\cos\left[\frac{(2k+1)\pi}{2M}\left(n - \frac{N-1}{2}\right) + (-1)^k \frac{\pi}{4}\right], \tag{12}$$

$$f_k(k) = h_k(N-k-1) \text{ , } k = 0,1,\cdots,M-1 ,$$

$n = 0,1,\cdots,N-1$, where *h*(*n*) is the impulse response of the prototype filter and *N* is the filter length. For simplicity, we shall consider the case of $N = 2mM$. Let $H(z) = \sum_{q=0}^{2M-1} z^{-q} P_q(z^{2M})$ be the type-I polyphase decomposition of the prototype filter, it can be shown that the PR conditions of the CMFB are given by

$$P_k(z)P_{2M-k-1}(z) + P_{M+k}(z)P_{M-k-1}(z) = \beta \cdot z^{-\sigma} , \tag{13}$$

$$k = 0,1,\cdots,M-1 ,$$

where β is a nonzero constant and σ is a positive integer. Since the analysis and synthesis filters are frequency shifted version of the prototype filter, it is only necessary to minimize the passband and stopband ripples of the prototype filter. More precisely, the design problem can be formulated as the following constrained optimization problem

$$\min_h \Phi = \alpha \int_0^{\omega_p} \left|1 - H(e^{j\omega})\right|^2 d\omega + (1-\alpha) \int_{\omega_s}^{\pi} \left|H(e^{j\omega})\right|^2 d\omega \quad (14)$$

subject to the PR constraint in (13),

The theory and design of RN CMFB have been discussed in [5,11]. The design procedure is summarized as follows: Given the sampling factors $\{m_l / M\}$, $l = 0,1,\cdots, L-1$ with $\Sigma_{l=0}^{L-1} m_l = M$.

1. Design an *M*-channel uniform CMFB with prototype filter $H(e^{j\omega})$, filter length $N_M = 2mM$, and cutoff frequency $\omega_s = (1+2\rho)\pi/(2M)$ using (14).
2. Set $N_{m_l} = m_l N_M / M = 2mm_l$
3. Design the m_l-channel uniform recombination CMFB with length N_{m_l} , cutoff frequency $\omega_s = (1+2\rho)\pi/(2m_l)$, and prototype filter $G_l(e^{j\omega})$ using (4). The weighting in the passband and stopband should be identical to the *M*-channel CMFB.

Interested readers are referred to [5,11] for more details. RN CMFBs are attractive because of their low design and implementation complexities and good frequency characteristics. Moreover, the uniform FB and TMUXs can be designed separately, so that dynamic recombination is feasible [13]. The construction of cosine modulated wavelets is also very simple, which amounts to imposing certain zeros in the prototype filters [1]. By imposing this additional condition, the RN CMFB can be used to generate limit functions for rational multiresolution analysis.

IV. PSEUDO-WAVELET SERIES

It has been shown in [2] that the limit functions satisfy

$$\phi^{(s)}(x) = \sum_{k\in Z} g(qk - ps)\phi^{(k)}((p/q)x) \quad (15)$$

This is different from the *M*-adic case, where $\phi(x) = \sum_{k\in Z} g(k)\phi(Mx-k)$, or in the Fourier transform domain:

$$\Phi(\xi) = G(\frac{\omega}{M})\Phi(\frac{\xi}{M}) = \prod_{l=1}^{\infty} G(M^{-l}\omega)\cdot\Phi(0) . \quad (16)$$

In Cohen and Daubechies's approach for $M = 2$ [12], and Chan, et al [1] for positive integer *M*, the infinite product in (16), if it exists, is used to define the Fourier Transform of the scaling function in the *M*-band wavelets. In the biorthogonal case, we also have a similar recursion for the dual scaling function:

$$\widetilde{\Phi}(\xi) = \widetilde{G}(\frac{\omega}{M})\widetilde{\Phi}(\frac{\xi}{M}) = \prod_{l=1}^{\infty} \widetilde{G}(M^{-l}\omega)\cdot\widetilde{\Phi}(0) . \quad (17)$$

Note, the FT of (19) leads to a convolution instead of a product of the two Fourier transforms. For the more general case of biorthogonal rational subdivision scheme, the dual scaling function $\widetilde{\phi}^{(s)}(x)$ will be generated by the mirror image of the synthesis filters of $G(z)$, $\widetilde{g}(n) \xrightarrow{Z} \widetilde{G}(z)$. It satisfies the recursion:

$$\widetilde{\phi}^{(s)}(x) = \sum_{k\in Z} \widetilde{g}(qk - ps)\widetilde{\phi}^{(s)}((p/q)x) . \quad (18)$$

In *M*-band wavelets, the spaces are generated by

$$\phi^{(l,k)}(x) = M^{-l/2}\phi(M^{-l}x - k)$$
$$\text{and} \quad \widetilde{\phi}^{(l,k)}(x) = M^{-l/2}\widetilde{\phi}(M^{-l}x - k) . \quad (19)$$

Denote the space generated by $\phi^{(l,k)}(x)$, $k \in Z$, as V_l , because of the dilation equation, we have $V_{l+1} \subset V_l$. Similarly, define the space generated by $\widetilde{\phi}^{(l,k)}(x)$, $k \in Z$, as $\widetilde{V}_l$. We have $\widetilde{V}_{l+1} \subset \widetilde{V}_l$. This is called the nested subspaces generated by the scaling functions. And $\widetilde{V}_{-\infty} = L_2(R)$. The gap between V_{l+1} and V_l is filled in by the space $W_{l+1}^{(i)}$ generated by $\psi_i^{(l+1,k)}(x)$, $k \in Z$. That is, $V_{l+1} \oplus \sum_{i=1}^{M-1} W_{l+1}^{(i)} = V_l$. Similar relations apply to the dual. In biorthogonal wavelets, $W_{l+1}^{(i)}$ and V_{l+1} are not orthogonal to each other, but they are all orthogonal to their duals (hence the name biorthogonal). The sum $W_{l+1}^{(i)}$ is the gap between:

$$V_l = span\{\phi^{(l,s)}(x) = \phi^{(s)}((p/q)^{-l}x) \quad (20)$$
$$\text{and} \quad V_{l+1} = span\{\phi^{(l+1,s)}(x) = \phi^{(s)}((p/q)^{-(l+1)}x) .$$

And those of its dual $\widetilde{V}_l = span\{\widetilde{\phi}^{(l,s)}(x) = \widetilde{\phi}^{(s)}((p/q)^{-l}x)$ and $\widetilde{V}_{l+1} = span\{\widetilde{\phi}^{(l+1,s)}(x) = \widetilde{\phi}^{(s)}((p/q)^{-(l+1)}x)$.

In the subdivision scheme constructed by biorthogonal RN CMFB, the scaling functions are combination of the *M*-band scaling function and some of the wavelet functions. Although they satisfy the dilation equation in the form of (16), after recombination, the new scaling function, like the rational subdivision scheme only obeys (15). Depending on how the remaining channels are treated, the space can exhibit a wide variety of structure. If the rest of the channels are not merged, then we have: $V_{l+1} \oplus \sum_{i=1}^{L-1} W_{l+1}^{(i)} = V_l$, with L being the number of the nonuniform bands of the rational filter bank. And, the property of $W_{l+1}^{(i)}$ is inherent from the original *M*-band wavelets. If the rest of the channels are also merged, then it is expected that the shift-invariant property like the new scaling function of the rational subdivision scheme will very likely be lost. A set of wavelet functions can also be constructed from $\phi^{(s)}(x)$ and $\widetilde{\phi}^{(s)}(x)$

$$\psi_i^{(s)}(x) = \sum_{k\in Z} g_i(qk - ps)\phi_i^{(s)}((p/q)x) ,$$
$$\text{and} \quad \widetilde{\psi}_i^{(s)}(x) = \sum_{k\in Z} \widetilde{g}_i(qk - ps)\widetilde{\phi}_i^{(s)}((p/q)x) , \quad (21)$$

where $g_i(n)$ and $\widetilde{g}_i(n)$ are the analysis and mirror synthesis filters of the merged filter bank. It was shown that [2] the PR property of the FB will induce the biorthonormality of $\phi^{(s)}(x)$, $\widetilde{\phi}^{(s)}(x)$, $\psi_i^{(s)}(x)$, and $\widetilde{\psi}_i^{(s)}(x)$. It is also possible to expand a function into a pseudo wavelet series to form a multiresolution analysis. However, whether the set of functions span the whole L^2 space is still an open question.

V. EXAMPLE

Fig. 4 shows an example RN cosine modulated wavelet FB with sampling factors (¾, ¼) and its corresponding limit scaling and wavelet functions with s=0 (Fig. 4(c)-(f)). The frequency responses of the 3-channel and 4-channel CMFBs are shown in Fig. 4(a) and (b). The lengths of them are respectively 30 and 40. Combining the first three channels in the original 4-channel CMFB by the synthesis filters of the 3-channel CMFB gives a 2-band (¾, ¼) nonuniform FB. It can be seen that the limit functions are very smooth.

IV. CONCLUSION

A rational subdivision scheme based on cosine-modulated wavelets is presented. It is based on a kind of nonuniform filter banks called recombination nonuniform filterbanks (RN FB). It has been shown that if the component FBs in a RNFB are wavelet FBs, then the necessary condition for convergence to limit functions in the subdivision scheme is also satisfied, which considerably simplifies the design of different rational subdivision schemes. An efficient RNFB, called RN cosine modulated FBs (CMFB), constructed from uniform CMFBs and TMUX are proposed together with its design procedure. Very smooth limit functions can be readily generated from this rational subdivision scheme. Another interesting property of using RNFB is that they can be recombined dynamically to yield multiresolution analysis with different sampling factors.

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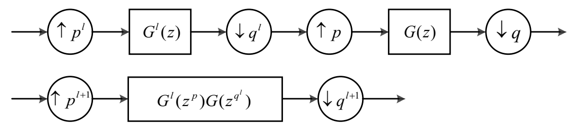

Fig. 1. Rational sub-division scheme.

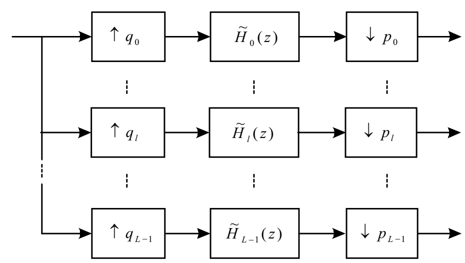

Fig. 2. Direct structure of nonuniform filter bank. p_l and q_l coprime numbers.

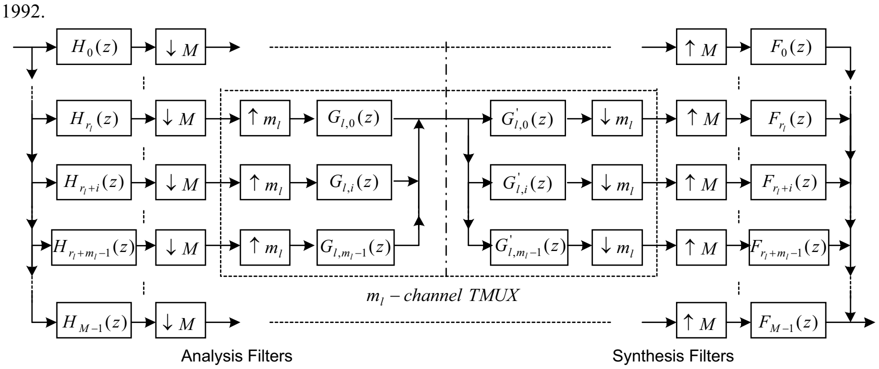

Fig. 3. Structure of recombination nonuniform filter bank.

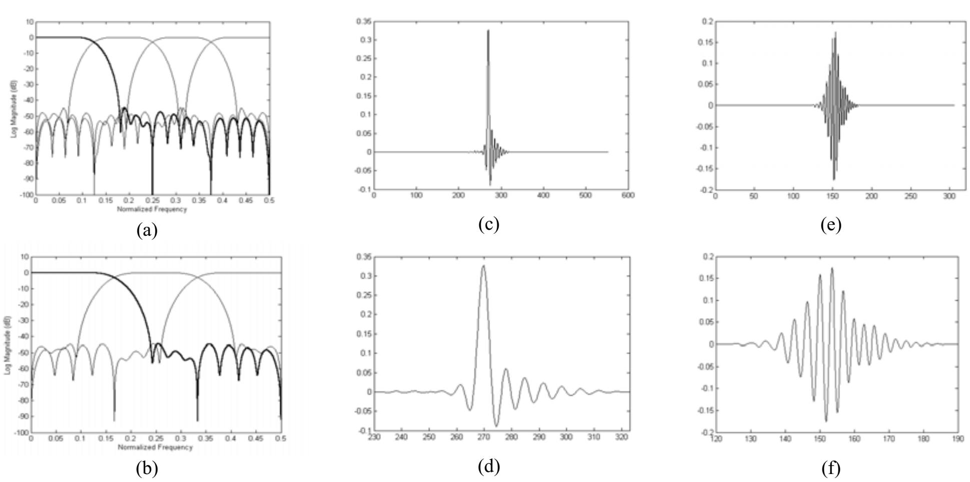

Fig. 4. (a) 4-channel and (b) 3-channel CMFBs; (c) $\phi^{(0)}(x)$ and (d) its enlarged part; (e) $\psi^{(0)}(x)$ and its enlarged part.