COMPARISON OF THE MAIN FORMS OF HALF-QUADRATIC REGULARIZATION

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ABSTRACT

We consider the reconstruction of images by minimizing regularized cost-functions. To accelerate the computation of the estimate, two forms of half-quadratic regularization, multiplicative and additive, are often used. The goal of this paper is to compare both theoretically and experimentally the efficiency of these two forms. We provide a theoretical and experimental analysis of the speed of convergence that they allow. We show that the multiplicative form gives rise to a better rate of convergence.

1. INTRODUCTION

We address image reconstruction where a sought image $\hat{x} \in R^p$ is estimated from degraded data $y \in \mathrm{R}^q$ by minimizing a cost function $J : \mathrm{R}^p \to \mathrm{R}$ which combines a quadratic data-fidelity term and a regularization term Φ via a parameter $\beta > 0$:

$$\hat{x} = \min_{x \in \mathrm{R}^p} J(x), \text{ where } J(x) = \|Ax - y\|^2 + \beta\Phi(x). \quad (1)$$

We shall assume that the observation operator $A \in \mathrm{R}^{q \times p}$ is known. We focus on regularization term Φ of the form

$$\Phi(x) = \sum_{i=1}^{r} \phi(g_i^T x), \quad (2)$$

where $\phi : \mathrm{R} \to \mathrm{R}$ is a potential function and g_i^T, for $i = 1, \dots, r$, are linear operators. Typically, $\{g_i^T x\}$ are first or second-order differences between neighboring pixels. If G is the $r \times p$ matrix whose ith row is g_i^T, for $i = 1, \dots, r$, a basic requirement is

$$\ker(A^T A) \cap \ker(G^T G) = \{0\}. \quad (3)$$

We suppose that ϕ is smooth and convex, and *edge-preserving*, i.e. $\phi(t) < t^2$ for $|t| \to \infty$. Such functions are for instance [1, 2, 3, 4]:

$$\begin{aligned} \phi(t) &= |t|^\alpha, \; 1 < \alpha \le 2, \\ \phi(t) &= \sqrt{\alpha + t^2}, \\ \phi(t) &= \begin{cases} t^2/2 & \text{if } |t| \le \alpha, \\ \alpha|t| - \alpha^2/2 & \text{if } |t| > \alpha. \end{cases} \end{aligned} \quad (4)$$

Cost-functions of this form are popular in various inverse problems such as denoising, deblurring, seismic imaging, tomography.

However, the resultant minimizers $\hat{x}$ are non-linear with respect to data y and their computation is costly, especially when A has many non-zero entries. In order to cope with numerical slowness, *half-quadratic* (HQ) reformulation of J has been pioneered, using two different ways, in [5] and [6]. The idea is to construct an *augmented cost function* $\mathcal{J} : \mathrm{R}^p \times \mathrm{R}^r \to \mathrm{R}$ which involves an auxiliary variable $s \in \mathrm{R}^r$, and two new functions, $Q : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$, where $Q(., s_i)$ is quadratic $\forall s_i \in \mathrm{R}$, and $\psi : \mathrm{R} \to \mathrm{R}$,

$$\mathcal{J}(x, s) = \|Ax - y\|^2 + \beta \sum_{i=1}^{r} Q(g_i^T x, s_i) + \beta \sum_{i=1}^{r} \psi(s_i), \quad (5)$$

$$\text{so that } \phi(t) = \min_{s \in \mathrm{R}} \{Q(t, s) + \psi(s)\}, \quad \forall t \in \mathrm{R}. \quad (6)$$

By (6), the global minimizer $(\hat{x}, \hat{s})$ of $\mathcal{J}$ yields the solution initially defined in (1), since $J(x) = \min_{s \in \mathrm{R}^r} \mathcal{J}(x, s)$, $\forall x \in \mathrm{R}^p$. In [5], Geman & Reynolds first considered a quadratic term Q of the *multiplicative form*,

$$Q(t, s) = t^2 s, \quad \text{for } t \in \mathrm{R}, \; s \in \mathrm{R}_+. \quad (7)$$

Later, Geman & Yang [6] proposed an *additive form* for Q:

$$Q(t, s) = (t - s)^2, \quad \text{for } t \in \mathrm{R}, \; s \in \mathrm{R}. \quad (8)$$

In both cases (7) and (8), the dual function ψ, which ensures (6), is obtained using the theory of convex conjugacy [7, 8].

The augmented cost-function $\mathcal{J}$ is minimized using an *alternating minimization scheme.* Let the solution obtained at iteration $(k-1)$ read $(x^{(k-1)}, s^{(k-1)})$. At the next iteration k we calculate

$$\begin{aligned} &s^{(k)} \text{ such that } \quad \mathcal{J}(x^{(k-1)}, s^{(k)}) \le \mathcal{J}(x^{(k-1)}, s), \quad \forall s \in \mathrm{R}^r, \\ &x^{(k)} \text{ such that } \quad \mathcal{J}(x^{(k)}, s^{(k)}) \le \mathcal{J}(x, s^{(k)}), \quad \forall x \in \mathrm{R}^p. \end{aligned}$$

These minimizations give rise to two *minimizer mappings*, $x \to \left[\sigma(g_1^T x), \dots, \sigma(g_r^T x)\right]^T$ with $\sigma : \mathrm{R} \to \mathrm{R}$, and $s \to \chi(s)$ with $\chi : \mathrm{R}^r \to \mathrm{R}^p$. The alternate minimization thus reads

$$s_i^{(k)} = \sigma(g_i^T x^{(k-1)}), \; \forall i = 1, \dots, r, \quad (9)$$

$$x^{(k)} = \chi(s^{(k)}). \quad (10)$$

For both forms, the functions σ and χ admit an explicit form.

These ideas has been pursued and deepened by many authors [2, 11, 3, 4, 9, 10]. Although the intuition that HQ regularization does indeed increase the speed of the minimization of regularized cost-functions of the form (1), this critical question has never been considered in a theoretical way. Moreover, the performance of both formulations (7) and (8) has never been compared. The goal of our paper is to fulfill this gap by characterizing both theoretically and experimentally the speed of convergence relevant to these two forms. The obtained results reveal that in general, the multiplicative form (7) allows to reach a better convergence rate. Furthermore, derived expressions allows to consider the convergence speed relevant to different potential functions ϕ. The proofs of the main theorems, as well as other details, can be found in [13].

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2. SOME FACTS ABOUT HQ REGULARIZATION

2.1. Multiplicative form

We consider potential functions ϕ such that

$$\begin{array}{ll} t \to \phi(\sqrt{t}) \text{ is concave on } \mathrm{R}_+, & \lim_{t\searrow 0} \phi'(t)/t < \infty, \\ t \to \phi(t) \text{ is convex on } \mathrm{R}, & \lim_{t\to\infty} \phi(t)/t^2 = 0, \\ \phi \text{ is twice differentiable on } \mathrm{R}, & \phi(t) = \phi(-t),\ \forall t \in \mathrm{R}. \end{array} \quad (11)$$

Then the expressions below are equivalent [5, 4, 10]:

$$\begin{array}{rcl} \phi(t) & = & \inf_{s\in \mathrm{R}} \left\{ st^2 + \psi(s) \right\}, \\ \psi(s) & = & \sup_{t\in \mathrm{R}} \left\{ \phi(t) - st^2 \right\}. \end{array} \quad (12)$$

Notice that ψ is convex and $\psi(s) = +\infty$ for $s < 0$; hence the infimum in (12) can be considered only for $s \geq 0$. The resultant augmented cost-function $\mathcal{J}$ is defined on $\mathrm{R}^p \times \mathrm{R}^r_+$ and reads

$$\mathcal{J}(x, s) = \|Ax - y\|^2 + \beta (Gx)^T \mathrm{diag}(s)\ Gx + \beta \sum_{i=1}^{r} \psi(s_i), \quad (13)$$

where $\mathrm{diag}(s)$ is a diagonal matrix whose diagonal elements are s_i, for $i = 1, \ldots, r$. The function σ, as given in (9), reads [4, 10]

$$\sigma(t) = \left\{ \begin{array}{ll} \dfrac{\phi'(t)}{2\,t} & \text{if } \ t \neq 0 \\ \zeta & \text{if } \ t = 0 \end{array} \right. \quad \text{where } \zeta := \lim_{\tau \searrow 0} \frac{\phi'(\tau)}{2\,\tau}, \quad (14)$$

where clearly $\sigma(t) \geq 0$, $\forall t \in \mathrm{R}$. The minimizer mapping χ, introduced in (10), satisfies $D_1 \mathcal{J}(\chi(s), s) = 0$, $\forall s \in \mathrm{R}^r$, and reads:

$$\begin{array}{rcl} \chi(s) & = & (H(s))^{-1} A^T y, \\ \text{where } \ H(s) & = & A^T A + \beta G^T \mathrm{diag}(s) G. \end{array} \quad (15)$$

Although $\mathcal{J}$ is convex with respect to x and to s separately, it is not globally convex in general. Nevertheless, when ϕ satisfies (11), it is shown in [10] that the sequence $(x^{(k)}, s^{(k)})$, generated by (9)-(10) and (14)-(15), converges to a point $(\hat{s}, \hat{x})$ as $k \to \infty$, such that $\hat{x}$ is the sought minimizer of J as given in (1)-(2).

2.2. Additive form

This form is considered under the condition that the function

$$t \to t^2/2 - \phi(t) \quad (16)$$

is convex, continuous and finite for every $t \in \mathrm{R}$. Then the following expressions are equivalent:

$$\begin{array}{rcl} \phi(t) & = & \inf_{s\in R} \left\{ \psi(s) + (t-s)^2/2 \right\}, \\ \psi(s) & = & \sup_{t\in R} \left\{ \phi(t) - (t-s)^2/2 \right\}. \end{array} \quad (17)$$

The condition (16) implies that $\phi'(t^-) \geq \phi'(t^+)$, for any $t \in \mathrm{R}$. Whenever ϕ is convex, it implies that ϕ is differentiable. The augmented cost-function now reads

$$\mathcal{J}(x, s) = \|Ax - y\|^2 + \frac{\beta}{2}\|Gx - s\|^2 + \beta \sum_{i=1}^{r} \psi(s_i). \quad (18)$$

The minimizer function σ reads [12, 3, 13]:

$$\sigma(t) = t - \phi'(t). \quad (19)$$

The minimizer function χ relevant to $\mathcal{J}(., s)$ reads

$$\begin{array}{rcl} \chi(s) & = & H^{-1}\left(2A^T y + \beta G^T s\right), \\ \text{where } \ H & = & 2A^T A + \beta\, G^T G. \end{array} \quad (20)$$

Here again, $\mathcal{J}$ is convex with respect to x and to s separately, in general it is not convex with respect to (x, s) jointly. In spite of this, if ϕ is twice differentiable and convex, and if the function in (16) is convex, it is seen [3, 13] that the sequence $(x^{(k)}, s^{(k)})$, given by (9)-(10) and (19)-(20), converges to $(\hat{s}, \hat{x})$ as $k \to \infty$ where $\hat{x}$ is the sought minimizer of J as given in (1)-(2).

3. CONVERGENCE RATE FOR MULTIPLICATIVE FORM

Our main result for multiplicative form is stated below.

Theorem 1 *For J of the form (1)-(2), where (3) and (11) are satisfied, consider $\mathcal{J}$ as defined in (12)-(13). Assume that ϕ satisfies*

$$\phi''(t)t \leq 2\phi'(t), \quad \forall t \in \mathrm{R}_+, \quad (21)$$

and $\phi''(t)/t$ is well defined when $t \searrow 0$. Suppose also that one of the following conditions is satisfied:

(i) $A^T A$ is invertible;

(ii) the inequality in (21) is strict *and $\phi''(t) > 0$ for all $t \in \mathrm{R}$.*

Consider the sequence $\{x_k\}_{k=0}^{\infty}$ generated by (9)-(10) and (14)-(15). Then there is a point $\hat{x} \in \mathrm{R}^p$, an integer $k_\infty > 0$ and a constant $\mu \in [0, 1)$ such that

$$\frac{\|\hat{x} - x^{(k)}\|}{\|\hat{x} - x^{(k-1)}\|} \leq \mu < 1, \quad \forall k \geq k_\infty, \quad (22)$$

i.e., the sequence $\{x_k : k \geq k_\infty\}$ converges linearly to $\hat{x}$.

Sketch of the proof. After some calculations, we get

$$\frac{\|\hat{x} - x^{(k)}\|}{\|\hat{x} - x^{(k-1)}\|} \leq \left\| M(x^{(k-1)}) \right\| + \left\| E(\hat{x} - x^{(k-1)}) \right\|, \quad (23)$$

where $E(x)$ is an error term which goes to 0 as $\|x\| \to 0$ and

$$\begin{array}{rcl} M(x) & = & \left(A^T A + \beta G^T \mathrm{diag}\left(\left[\sigma(g_i^T x)\right]_{i=1}^{r}\right) G\right)^{-1} \\ & & \times \quad \beta G^T \mathrm{diag}\left(\left[g_i^T x\, \sigma'(g_i^T x)\right]_{i=1}^{r}\right) G. \end{array}$$

The convergence rate is determined by $M(x)$. The essential point is that for every $x \in \mathrm{R}^p$, and for every $i \in \{1, \ldots, r\}$ such that $g_i^T x \neq 0$, we have

$$\|M(x)\| \leq \frac{\left|g_i^T x\, \sigma'(g_i^T x)\right|}{\sigma(g_i^T x)}, \quad (24)$$

where the inequality is strict when $A^T A$ is invertible. On the other hand, (21) amounts to

$$\left|\sigma'(t)t\right| \leq \sigma(t), \quad \forall t \in \mathrm{R},$$

which inequality is strict if $A^T A$ is not invertible. We conclude that $\|M(x)\| < 1$, for any $x \in \mathrm{R}^p$. Furthermore, for k large enough, all iterates $x^{(k)}$ are contained in a compact neighborhood K of the limit point $\hat{x}$. Since $x \to \|M(x)\|$ is continuous, we see that $\mu = \sup_{x\in K} \|M(x)\| < 1$. ⋄

4. SPEED OF CONVERGENCE FOR ADDITIVE FORM

There is an analogous result for the additive form.

Theorem 2 *Consider $\mathcal{J}$ as defined in (18) and (17), and suppose that (3) holds. Let ϕ be twice differentiable and satisfy*

$$0 \leq \phi''(t) < 1, \quad \forall t \in \mathrm{R}. \quad (25)$$

Suppose also that one of the following conditions is satisfied:

(i) $A^T A$ is invertible;

(ii) $\phi''(t) > 0$ for all $t \in \mathrm{R}$.

Consider the sequence $\{x_k\}_{k=0}^{\infty}$ generated by (9)-(10) and (19)-(20). Then there are $\hat{x} \in \mathrm{R}^p$, $k_\infty > 0$ and $\mu \in [0, 1)$ such that

$$\frac{\|\hat{x} - x^{(k)}\|}{\|\hat{x} - x^{(k-1)}\|} \leq \mu < 1, \quad \forall k \geq k_\infty, \quad (26)$$

i.e., the sequence $\{x_k : k \geq k_\infty\}$ converges linearly to $\hat{x}$.

Sketch of the proof. In this case, we obtain an inequality of the same form (23), with E an error term converging to zero. Now, the matrix M, which gives the convergence rate, reads

$$M(x) = \beta(2A^T A + \beta G^T G)^{-1} G^T \text{diag}\left(\left[1 - \phi''(g_i^T x)\right]_{i=1}^r\right) G.$$

It is easy to see that for every $x \in \mathbb{R}^p$ we have

$$\|M(x)\| \leq 1 - \phi''(g_i^T x), \ \forall i = 1, \ldots, r, \tag{27}$$

which inequality is strict if $A^T A$ is not invertible. This allows us to deduce that $\|M(x)\| < 1$ for every $x \in \mathbb{R}^p$. As previously, $\hat{x}$ has a compact neighborhood K such that $x^{(k)} \in K$ for all k large enough. The constant μ is deduced by the same reasoning. ⋄

5. MULTIPLICATIVE VERSUS ADDITIVE FORM

Based on (24), the convergence rate for the multiplicative form is essentially determined by the function

$$\mathcal{M}(t) = |\sigma'(t)t|/\sigma(t), \tag{28}$$

where σ is as given in (14). Similarly, (27) shows that the convergence rate for the additive form is essentially determined by

$$\mathcal{M}(t) = 1 - \phi''(t). \tag{29}$$

Notice that by (22) and (26), the convergence rate is better if $\mathcal{M}$ is smaller. We can say that $\mathcal{M}$ is a rate function. In Fig. 1 we present the function $\mathcal{M}$ relevant to (28) and (29) for two potential functions ϕ. In both cases, the function $\mathcal{M}$ relevant to (28) is *smaller* than the one relevant to (29): this suggests that multiplicative form needs less iterations than additive form in order to find $\hat{x}$.

Furthermore, the cost-per-iteration for the multiplicative form follows from (14)-(15), and the one for the additive form comes from (19)-(20). The comparison of these expressions clearly shows that the cost of each iteration under the additive form is much easier than under the multiplicative form.

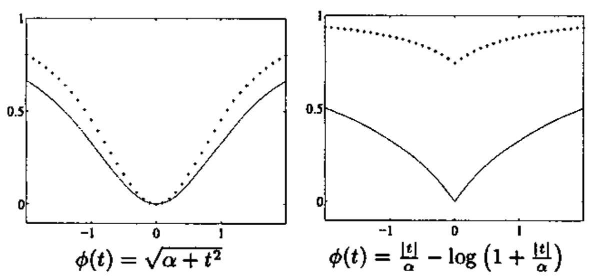


Fig. 1. The rate-function $\mathcal{M}$ for two potential functions. X-axis t, Y-axis $\mathcal{M}(t)$, multiplicative form (—), additive form (+++).

6. EXPERIMENTAL STUDY OF THE RATE OF CONVERGENCE

In this section, we consider restoring an one-dimensional signal and use this example to compare the additive and the multiplicative forms of the HQ regularization. In Figure 1 are displayed: the original signal, the observed blurred and noisy version, and the restored signal using a cost-function of the form (1)-(2) where ϕ is a Huber function (4). The stopping criterion of the HQ iterations is $\|x^{(k)} - x^{(k-1)}\| < 1 \times 10^{-3}$.

The numbers of HQ iterations required are listed in Table 1 for different values of α and β. It is obvious from these results that *the multiplicative form of the HQ regularization is more efficient than the additive form*, in terms of iterations required.

In the additive form of the HQ regularization, the matrix involved is fixed at each HQ iteration, and the right hand side involved is changing at each iteration and is affine in y and in s. It follows that efficient method such as fast cosine transform can be applied to solving such kind of linear systems. In contrast, in the multiplicative form of the HQ regularization, the matrix involved $H(s)$, given in (15), is changing at each iteration, and the right hand side involved is fixed and is given by $A^T y$. We can only apply the Gaussian elimination method to solve these linear systems. It is very expensive. In Fig. 2 is displayed the condition numbers of the matrices $H(s)$ during the iterations, for different values of the parameters α and β. In Table 2, we list the average of the condition numbers of $H(s)$ for HQ iterations. It is seen that the condition number of the matrix in the additive form is less than those in the multiplicative form. Since the condition numbers of the matrices in the multiplicative form are large, the number of inner iterations required to solve the corresponding linear systems are more when we apply iterative methods in the inner iterations.

Finally, we estimate the convergence speeds of both forms. In Figure 3, we show the ratios $\|x^{(k+1)} - \hat{x}\|/\|x^{(k)} - \hat{x}\|$ for the additive and the multiplicative HQ iterations. We observe that the convergence speed of the multiplicative HQ iterations is faster than that of the additive HQ iterations especially for the first few iterations. We also see that the convergence factor of the additive HQ iterations is very close to 1, especially when $x^{(k)}$ is close to $\hat{x}$. However, the convergence factor of the multiplicative HQ iterations is strictly below 1. These results explain why the multiplicative form converges faster than the additive form.

β	α	Additive	Multiplicative
5	1	> 1000	306
5	0.5	> 1000	361
5	0.1	> 1000	308
10	0.1	> 1000	447
1	0.1	937	162

Table 1. Number of HQ iterations required.

β	α	Additive	Multiplicative (average)
5	1	38.01	44.27
5	0.5	38.01	59.22
5	0.1	38.01	128.38
10	0.1	50.33	159.31
1	0.1	26.29	102.29

Table 2. Condition numbers of matrices involved in the additive and multiplicative forms.

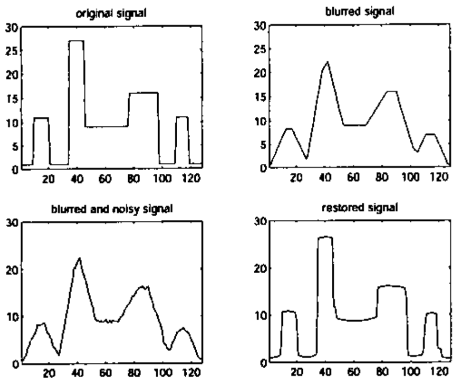
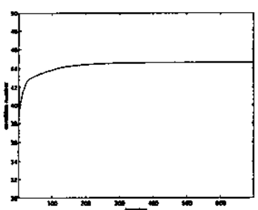
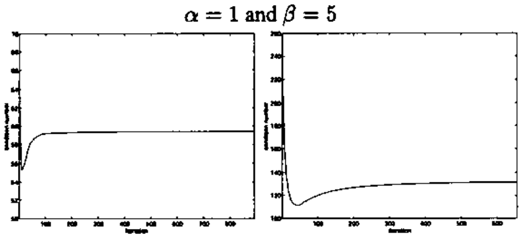


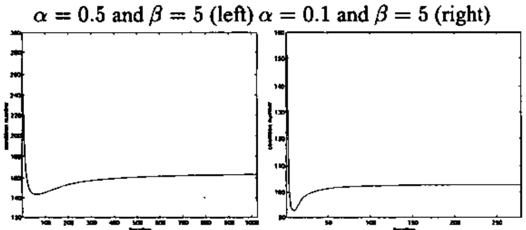
Fig. 1. Results of the test image when $\alpha = 0.1$ and $\beta = 5$.



$\alpha = 1$ and $\beta = 5$



$\alpha = 0.5$ and $\beta = 5$ (left) $\alpha = 0.1$ and $\beta = 5$ (right)



$\alpha = 0.1$ and $\beta = 10$ (left) $\alpha = 0.1$ and $\beta = 1$ (right)

Fig. 2. Condition numbers of matrices involved in the multiplicative form.

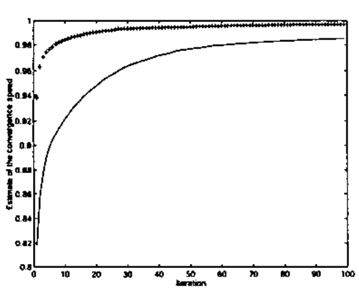


Fig. 3. Estimates of the convergence speeds for the additive (++++++) and the multiplicative (——) forms.

7. CONCLUDING REMARKS

We performed a both theoretical and numerical comparison of the two forms of HQ regularization, multiplicative and additive. The obtained results clearly stipulate that the multiplicative form is more attractive in terms of speed of convergence. In contrast, additive the form presents some possibilities to further improvement of the conditioning.

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