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Design of Reliable Controllers for Symmetric Composite Systems: Primary Contingency Case

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Abstract

This paper discusses the reliable controller design problem for symmetric composite systems composed of several identical subsystems. A reliable controller design procedure is presented in terms of the solutions to the Algebraic Riccati Equations. The order of these AREs is much lower than that of the symmetric composite system. The resulting closed-loop system is reliable in that it provide guaranteed internal stability and H_∞ performance when all sensors and actuators are operational as well as when the sensors or actuators of a prescribed subsystem experiences an outage.

1 Introduction

Symmetric composite systems are systems consisting of identical subsystems with interconnections described by constant block-symmetric matrices. This class of systems occurs in very diverse areas such as electric power systems, industrial manipulators, computer networks, etc. [1] [2] [6] [8]. In recent years there has been a great interest in symmetric composite systems. Lunze first proposed the state model of symmetric composite systems, and investigated some of the fundamental properties of such systems [1]. In [6], a synthesis procedure for decentralized controllers for symmetric composite systems is presented and important characteristics of such systems are observed. For uncertain symmetric composite systems, a robust controller design procedure is given in [8].

Recently, some approaches to the design of reliable controllers that retain stability and H_∞ performance have been developed by several authors [3] - [5] [7]. In [7], Veillette *et al* presented a methodology for the design of reliable centralized and decentralized control systems by using the Algebraic Riccati Equation (ARE) approach The resulting controller guarantees closed-loop internal stability and H_∞ performance not only when all control components are operational, but also in case of some admissible control component outages. In [3], Medanic investigated the single contingency reliable design problem, and presented a technique for H_∞-norm bounding design that results in performance reliable with respect to the outage of any one sensor or any one actuator by introducing a redundant sensor and a redundant actuator.

In this paper, we study the reliable control problem for symmetric composite systems. By making use of the symmetric structures, the order of the AREs involved in the design process is drastically reduced. The paper is organized as follows. The mathematical description of symmetric composite systems and and problem formulation are given in Section 2, together with some technical preliminaries. In Section 3 a reliable controller design procedure is presented in terms of the solutions to the Algebraic Riccati Equations (AREs) whose order is much lower than that of the system. The controller is reliable with respect to the outage of sensors or actuators of a prescribed subsystem. Finally, some concluding remarks are given in Section 4.

2 Problem formulation and preliminaries

The symmetric composite system Σ under consideration consists of N $(N > 1)$ identical subsystems and the overall system is described by composite equations of the following form

$$\Sigma: \quad \dot{x} = Ax + Bu + Gw \tag{1}$$

$$y = Cx + w_0 \tag{2}$$

$$z = \begin{bmatrix} Hx \\ u \end{bmatrix} \tag{3}$$

where for $i = 1, \cdots, N$,

$$x = [x_1^T, \cdots, x_N^T]^T, \quad x_i \in \mathbf{R}^n$$

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$$u = [u_1^T, \cdots, u_N^T]^T, \quad u_i \in \mathbf{R}^m$$
$$y = [y_1^T, \cdots, y_N^T]^T, \quad y_i \in \mathbf{R}^p$$

Here u_i, x_i, and y_i are, respectively, the input, state, and output of the ith subsystem; $z \in \mathbf{R}^{rN+mN}$ is the output to be regulated; and w and w_0 are the square-integrable disturbances. The composite matrices $A \in \mathbf{R}^{nN\times nN}$, $B \in \mathbf{R}^{nN\times mN}$, $C \in \mathbf{R}^{pN\times nN}$, $G \in \mathbf{R}^{nN\times qN}$ and $H \in \mathbf{R}^{rN\times nN}$ all have block-symmetric structures. For example,

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{12} \\ A_{12} & A_{11} & \cdots & A_{12} \\ \vdots & \vdots & \ddots & \vdots \\ A_{12} & A_{12} & \cdots & A_{11} \end{bmatrix} \quad (4)$$

When an actuator outage occurs in, say, the first subsystem, the B matrix in (1) is replaced by

$$B_0 = \begin{bmatrix} 0 & B_{12} & \cdots & B_{12} \\ 0 & B_{11} & \cdots & B_{12} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & B_{12} & \cdots & B_{11} \end{bmatrix} \quad (5)$$

Similarly, when a sensor outage occurs in, say, the first subsystem, the C matrix in (2) is replaced by

$$C_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C_{12} & C_{11} & \cdots & C_{12} \\ \vdots & \vdots & \ddots & \vdots \\ C_{12} & C_{12} & \cdots & C_{11} \end{bmatrix} \quad (6)$$

By defining

$$B_1 = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{12} & 0 & \cdots & 0 \end{bmatrix} \quad (7)$$

$$C_1 = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{12} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (8)$$

we have the following decompositions for the B and C matrices.

$$B = B_0 + B_1, \quad C = C_0 + C_1 \quad (9)$$

Motivated by the controller structure by Veillette *et al.* in [7], we consider dynamic controllers of the following form.

$$\dot{\xi} = (A + BK)\xi + G_+\hat{w} + L(y - C\xi) \quad (10)$$
$$u = K\xi, \quad \xi \in \mathbf{R}^{nN} \quad (11)$$

where $dim\ \hat{w} = dim\ w + dim\ u$. Then the reliable control problem considered in this paper is defined as follows. Given the symmetric composite linearsystem $\mathbf{\Sigma}$ in (1) - (3), design a controller of the form (10)-(11) such that the resulting closed-loop system is asymptotically stable, and the H_∞-norm of the closed-loop transfer function matrix from $[w^T, w_0^T]^T$ to z is bounded by some prescribed $\gamma > 0$ when all sensors and actuators are operational as well as when the sensors and actuators of one, but only one, prescribed subsystem experience outages.

Due to the symmetry in the system, it is sufficent to consider only outages in the first subsystem. Before going to the next section to give the design procedure for the above reliable control problem, the following technical preliminaries are needed.

Suppose that $v_0, \cdots, v_{s-1}$ are the s roots of unity in complex plane. That is $v_j = exp(2\pi j\sqrt{-1}/s)$, $j = 0, 1, \cdots, s-1$. Let $m_j = [1, v_j, v_j^2, \cdots, v_j^{s-1}]^T$. For given positive integers p, q and s, denote

$$\lambda = \begin{cases} \frac{s-1}{2} & \text{for } s \text{ odd} \\ \frac{s}{2} & \text{for } s \text{ even} \end{cases}$$

and

$$T_R(p, q, s) = diag[I_p,\ R_s \otimes I_q] \quad (12)$$

where I_q denotes the $q \times q$ identity matrix, $\otimes$ denotes the Kronecker product and the matrix R_s is defined as follows

$$R_s = M_s U_s \quad (13)$$

with

$$M_s = \begin{cases} \frac{1}{\sqrt{s}}[m_0\ m_1\ m_{N-1}\ m_2\ m_{N-2}\ \cdots\ m_\lambda\ m_{s-\lambda}] & \text{for } s \text{ odd} \\ \frac{1}{\sqrt{s}}[m_0\ m_1\ m_{N-1}\ m_2\ m_{N-2}\ \cdots\ m_{\lambda-1}\ m_\lambda\ m_{s-\lambda+1}] & \text{for } s \text{ even} \end{cases}$$

$$U_s = \begin{cases} diag[1, \underbrace{V, \cdots, V}_{\lambda}] & \text{for } s \text{ odd} \\ diag[1, \underbrace{V, \cdots, V}_{\lambda-1}, 1] & \text{for } s \text{ even} \end{cases}$$

and

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{bmatrix}$$

It is straightforward to verify that R_s is a real orthogonal matrix. (In fact, R_s is also normal.) Hence, $T_R(p, q, s)$ is a real orthogonal matrix.

Lemma 1: For any given matrices $E_{00} \in \mathbf{R}^{q_0\times k_0}$, $E_{01} \in \mathbf{R}^{q_0\times k}$, $E_{10} \in \mathbf{R}^{q\times k_0}$, $E_{11} \in \mathbf{R}^{q\times k}$ and $E_{11} \in$

$\mathbf{R}^{q\times k}$, denote

$$E=\begin{bmatrix} E_{00} & E_{01} & \cdots & E_{01} & E_{01} \\ E_{10} & E_{11} & \cdots & E_{12} & E_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ E_{10} & E_{12} & \cdots & E_{11} & E_{12} \\ E_{10} & E_{12} & \cdots & E_{12} & E_{11} \end{bmatrix} \in \mathbf{R}^{(q_0+sq)\times(k_0+sk)}$$

$$\begin{aligned} E_p &= \begin{bmatrix} E_{00} & \sqrt{s}E_{01} \\ \sqrt{s}E_{10} & E_{11}+(s-1)E_{12} \end{bmatrix} \\ E_m &= E_{11}-E_{12} \end{aligned}$$

Then

$$T_R^T(q_0,q,s)\ E\ T_R(k_0,k,s) = diag[E_p,\ E_m,\cdots,\ E_m] \tag{14}$$

Proof: Note that

$$\begin{aligned} 1+v_j+v_j^2+\cdots+v_j^{s-1} &= \begin{cases} 0 & j=1,\cdots,s-1 \\ 1 & j=s \end{cases} \\ \sum_{i=0}^{s-1}(\bar{v}_j v_k)^i &= \begin{cases} 0 & j\neq k \\ s & j=k \end{cases} \end{aligned}$$

where $\bar{v}_j$ is the complex conjugate of v_j Denoting by E_1 the matrix E with the 1^{st} block row and the 1^{st} block column deleted, and $(.)^H$ the Hermitian transpose of a matrix, we have, for s odd

$$(diag[I_{q_0}\quad M_s\otimes I_q])^H E$$

$$= \left[\begin{array}{c|c} E_{00} & E_{01}\ E_{01}\ \cdots\ E_{01} \\ \hline \sqrt{s}E_{10} & \\ \frac{1}{\sqrt{s}}\sum_{i=0}^{s-1}\bar{v}_1^i E_{10} & (M_s\otimes I_q)^H E_1 \\ \vdots & \\ \frac{1}{\sqrt{s}}\sum_{i=0}^{s-1}\bar{v}_{s-\lambda}^i E_{10} & \end{array}\right]$$

$$= \left[\begin{array}{c|c} E_{00} & E_{01}\ E_{01}\ \cdots\ E_{01} \\ \hline \sqrt{s}E_{10} & \\ 0 & (M_s\otimes I_q)^H E_1 \\ \vdots & \\ 0 & \end{array}\right]$$

But

$$(M_s\otimes I_q)^H E_1$$

$$= (M_s\otimes I_q)^H \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{12} & E_{12} \\ E_{12} & E_{11} & \cdots & E_{12} & E_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ E_{12} & E_{12} & \cdots & E_{11} & E_{12} \\ E_{12} & E_{12} & \cdots & E_{12} & E_{11} \end{bmatrix}$$

who's j^{th} column (for $j=1,\cdots,s$) is

$$\frac{1}{\sqrt{s}}\begin{bmatrix} E_{11}+(s-1)E_{12} \\ \bar{v}_1^{j-1}E_{11}+(\sum_{i=0,\ i\neq j-1}^{s-1}\bar{v}_1^i)E_{12} \\ \bar{v}_{s-1}^{j-1}E_{11}+(\sum_{i=0,\ i\neq j-1}^{s-1}\bar{v}_{s-1}^i)E_{12} \\ \vdots \\ \bar{v}_\lambda^{j-1}E_{11}+(\sum_{i=0,\ i\neq j-1}^{s-1}\bar{v}_\lambda^i)E_{12} \\ \bar{v}_{s-\lambda}^{j-1}E_{11}+(\sum_{i=0,\ i\neq j-1}^{s-\lambda}\bar{v}_{s-1}^i)E_{12} \end{bmatrix}$$

$$= \frac{1}{\sqrt{s}}\begin{bmatrix} E_{11}+(s-1)E_{12} \\ \bar{v}_1^{j-1}(E_{11}-E_{12}) \\ \bar{v}_{s-1}^{j-1}(E_{11}-E_{12}) \\ \vdots \\ \bar{v}_\lambda^{j-1}(E_{11}-E_{12}) \\ \bar{v}_{s-\lambda}^{j-1}(E_{11}-E_{12}) \end{bmatrix} \tag{15}$$

Then

$$(diag[I_{q_0}, M_s\otimes I_q])^H E(diag[I_{q_0}, M_s\otimes I_k])$$

$$= \left[\begin{array}{c|c} E_{00} & ZZ \\ \hline \sqrt{s}E_{10} & \\ 0 & (M_s\otimes I_q)^H E_1 (M_s\otimes I_k) \\ \vdots & \\ 0 & \end{array}\right]$$

where

$$\begin{aligned} ZZ &= \begin{bmatrix} \sqrt{s}E_{01} & \frac{1}{\sqrt{s}}\sum_{i=0}^{s-1}\bar{v}_1^i E_{01} & \cdots & \frac{1}{\sqrt{s}}\sum_{i=0}^{s-1}\bar{v}_{s-\lambda}^i E_{01} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{s}E_{01} & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

By equation (15), it follows that

$$\begin{aligned} &(M_s\otimes I_q)^H E_1 (M_s\otimes I_k) \\ &= diag[E_{11}+(s-1)E_{12},\ \ E_{11}-E_{12},\ \ \cdots, \\ &\qquad E_{11}-E_{12}] \end{aligned}$$

Thus,

$$\begin{aligned} & T_R^T(q_0,q,s)ET_R(k_0,k,s) \\ = & (diag[I_{q_0},\ U_s\otimes I_q])^H[(diag[I_{q_0},\ M_s\otimes I_q])^H E \\ & \times diag[I_{k_0},\ M_s\otimes I_k]]diag[I_{k_0},\ U_s\otimes I_k] \\ = & diag[E_p,\ E_m,\ \cdots,\ E_m] \end{aligned}$$

For the case of s even, the proof is similar. □

The following notations will be used in the sequel.

$$\begin{aligned}
A_p &= \begin{bmatrix} A_{11} & \sqrt{N-1}A_{12} \\ \sqrt{N-1}A_{12} & A_{11}+(N-2)A_{12} \end{bmatrix} \\
B_p &= \begin{bmatrix} \sqrt{N-1}B_{12} \\ B_{11}+(N-2)B_{12} \end{bmatrix} \\
C_p &= [\sqrt{N-1}C_{12}\ \ C_{11}+(N-1)C_{12}] \\
B_{p0} &= \begin{bmatrix} B_{11} \\ \sqrt{N-1}B_{12} \end{bmatrix} \\
C_{p0} &= [C_{11}\ \sqrt{N-1}C_{12}] \\
G_p &= \begin{bmatrix} G_{11} & \sqrt{N-1}G_{12} \\ \sqrt{N-1}G_{12} & G_{11}+(N-2)G_{12} \end{bmatrix} \\
H_p &= \begin{bmatrix} H_{11} & \sqrt{N-1}H_{12} \\ \sqrt{N-1}H_{12} & H_{11}+(N-2)H_{12} \end{bmatrix} \\
A_{p1} &= A_{11}+(N-1)A_{12} \\
B_{p1} &= (N+1)(B_{11}+(N-1)B_{12}) \\
C_{p1} &= (N+1)(C_{11}+(N+1)C_{12}) \\
G_{p1} &= G_{11}+(N-1)G_{12} \\
H_{p1} &= H_{11}+(N-1)H_{12} \\
A_m &= A_{11}-A_{12},\quad B_m = B_{11}-B_{12} \\
C_m &= C_{11}-C_{12},\quad G_m = G_{11}-G_{12} \\
H_m &= H_{11}-H_{12}
\end{aligned}$$

3 Reliable controller design

In this section, we consider the reliable control problem for the linear system $\mathbf{\Sigma}$ of (1 - 3) in case of sensor and actuator outages in the first subsystem. First, we review some results in [7] on the reliable control of general linear systems. Note that in Theorems 4.1 and 4.2 of [7], instead of AREs (Algebraic Riccati Equalities), only ARI (Algebraic Riccati In-qualities) are required. We summarize Theorems 4.1 and 4.2 of [7] as the following theorem. Denote

$$\Delta_x \triangleq A^TX + XA - XB_0B_0^TX + \frac{1}{\gamma^2}XGG^TX + H^TH + \gamma^2C_1^TC_1 \qquad (16)$$

$$\Delta_y \triangleq AY + YA^T - YC_0^TC_0Y + \frac{1}{\gamma^2}YH^THY + GG^T + \gamma^2B_1B_1^T \qquad (17)$$

Theorem 1: [7] Given the linear system $\mathbf{\Sigma}$ in (1 - 3), suppose that (A,H) is a detectable pair, and $0 \le X \in \mathbf{R}^{nN\times nN}$ and $0 < Y \in \mathbf{R}^{nN\times nN}$ satisfy the AREs

$$\Delta_x = 0 \qquad (18)$$
$$\Delta_y = 0 \qquad (19)$$

respectively, with $\sigma_{max}\{YX\} < \gamma^2$ and $A+BK+G_+K_{d_+}$ and $A+BK+G_+K_{d_+}-LC$ Hurwitz where $\sigma(\cdot)$ is the singular value of a matrix and

$$K = -B^TX \qquad (20)$$
$$G_+ = [\ G\quad \gamma B_1\ ] \qquad (21)$$
$$K_{d_+} = \frac{1}{\gamma^2}G_+^TX \qquad (22)$$
$$L = (I-\gamma^2YX)^{-1}YC^T. \qquad (23)$$

Then the controller in (10) and (11) with

$$\hat{w} = K_{d+}\xi \qquad (24)$$

asymptotically stablizes the closed-loop system, and the H_∞ norm of the closed-loop transfer function matrix from $[w^T, w_0^T]^T$ to z is bounded by some prescribed $\gamma > 0$ when all sensors and actuators are operational as well as when the sensors and actuators of one, but only one, prescribed subsystem experience outages. □

Note that the orders of the two AREs in (18) and (19) are $nN \times nN$, and this poses enomous computation burden on the design process especially when N, the number of subsystems, is large. But the system $\mathbf{\Sigma}$ in (1 - 3) under consideration has very strong symmetry. We show in this section that the this high-order problem of the two AREs is equivalent to four AREs of much lower dimensions: two of order $n\times n$ and two of order $2n\times 2n$. This reduces greatly the computational complexity of this reliable control problem. Denote

$$\Delta_{xp} \triangleq A_p^TX_p + X_pA_p - X_pB_pB_p^TX_p + \frac{1}{\gamma^2}X_pG_pG_p^TX_p + H_p^TH_p + \gamma^2C_{p0}^TC_{p0} \qquad (25)$$

$$\Delta_{xm} \triangleq A_m^TX_m + X_mA_m - X_mB_mB_m^TX_m + \frac{1}{\gamma^2}X_mG_mG_m^TX_m + H_m^TH_m \qquad (26)$$

$$\Delta_{yp} \triangleq A_p^TY_p + Y_pA_p - Y_pC_pC_p^TY_p + \frac{1}{\gamma^2}Y_pH_pH_p^TY_p + G_pG_p^T + \gamma^2B_{p0}B_{p0}^T \qquad (27)$$

$$\Delta_{ym} \triangleq A_m^TY_m + Y_mA_m - Y_mC_mC_m^TY_m + \frac{1}{\gamma^2}Y_mH_mH_m^TY_mG_mG_m^T \qquad (28)$$

Theorem 2: Suppose that the pairs (A_{p1}, H_{p1}) and (A_m, H_m) are detectable, and γ is a positive constant. Suppose also that the following hold.

1. There exist $0 \le X_p \in \mathbf{R}^{2n\times 2n}$ and $0 \le X_m \in \mathbf{R}^{n\times n}$ such that

$$\begin{aligned} \Delta_{xp} &= 0 \quad (29)\\ \Delta_{xm} &= 0. \quad (30)\end{aligned}$$

2. There exist $0 \le Y_p \in \mathbf{R}^{2n\times 2n}$ and $0 \le Y_m \in \mathbf{R}^{n\times n}$ such that

$$\begin{aligned} \Delta_{yp} &= 0 \quad (31)\\ \Delta_{ym} &= 0. \quad (32)\end{aligned}$$

3. (a) The matrices $A_p - B_pB_p^TX_p + 1/\gamma^2G_pG_p^TX_p$ and $A_m - B_mB_m^TX_m + 1/\gamma^2G_mG_m^T$ are Hurwitz; (b)

$$max\{\sigma_{max}(Y_pX_p),\ \ \sigma_{max}(Y_mX_m)\} < \gamma^2; \quad (33)$$

(c) the two closed-loop matrices $A_p - B_pB_p^TX_p + 1/\gamma^2G_pG_p^TX_p - L_p\bar{C}_p$ and $A_m - B_mB_m^TX_m + 1/\gamma^2G_mG_m^TX_m - L_mC_m$ are Hurwitz, where

$$\begin{aligned} \bar{C}_p &= \begin{bmatrix} C_{11} & \sqrt{N-1}C_{12} \\ \sqrt{N-1}C_{12} & C_{11}+(N-1)C_{12} \end{bmatrix}\\ L_p &= (I-\gamma^2Y_pX_p)^{-1}Y_p\bar{C}_p^T\\ L_m &= (I-\gamma^2Y_mX_m)^{-1}Y_mC_m^T. \end{aligned}$$

Then there exists a controller of the form (7) and (8) such that the resulting closed-loop system is asymptotically stable and has the H_∞-norm bound not greater than γ when all sensors and actuators are operational as well as when the sensors and actuators of the first subsystem experience outages.

Furthermore, the construction of the controller is given as follows. Partition X_p and Y_p into $n\times n$ blocks

$$X_p = \begin{bmatrix} X_{p00} & X_{p01} \\ X_{p10} & X_{p11}\end{bmatrix}, \quad Y_p = \begin{bmatrix} Y_{p00} & Y_{p01} \\ Y_{p10} & Y_{p11}\end{bmatrix} \quad (34)$$

and let

$$X = \begin{bmatrix} X_{00} & X_{01} & \cdots & X_{01} & X_{01} \\ X_{10} & X_{11} & \cdots & X_{12} & X_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{10} & X_{12} & \cdots & X_{11} & X_{12} \\ X_{10} & X_{12} & \cdots & X_{12} & X_{11} \end{bmatrix} \quad (35)$$

$$Y = \begin{bmatrix} Y_{00} & Y_{01} & \cdots & Y_{01} & Y_{01} \\ Y_{10} & Y_{11} & \cdots & Y_{12} & Y_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{10} & Y_{12} & \cdots & Y_{11} & Y_{12} \\ Y_{10} & Y_{12} & \cdots & Y_{12} & Y_{11} \end{bmatrix} \quad (36)$$

where

$$\begin{aligned} X_{00} &= X_{p00}\ , \quad X_{01} = 1/\sqrt{N-1}X_{p01}\\ X_{10} &= 1/\sqrt{N-1}X_{p10}\\ X_{11} &= 1/(N-1)[X_{p11}+(N-2)X_m]\\ X_{12} &= 1/(N-1)[X_{p11}-X_m]\\ Y_{00} &= Y_{p00}\ , \quad Y_{01} = 1/\sqrt{N-1}Y_{p01}\\ Y_{10} &= 1/\sqrt{N-1}Y_{p10}\\ Y_{11} &= 1/(N-1)[Y_{p11}+(N-2)Y_m]\\ Y_{12} &= 1/(N-1)[Y_{p11}-Y_m]. \end{aligned}$$

Then, the controller of equation (7) and (8) is given by (20 - 23) of Theorem 1. □

The following lemma will be used in the proof of Theorem 2.

Lemma 2: Under the assumptions of Theorem 2, the following hold:

1.

$$\begin{aligned} \Delta_x &= 0, \quad (37)\\ \Delta_y &= 0; \quad (38)\end{aligned}$$

2. the matrix $A + BK + G_+K_{d_+} = A - B_0B_0^TX + \frac{1}{\gamma^2}GG^TX$ is Hurwitz;

3. $\sigma_{max}\{YX\} < \gamma^2$;

4. the controller given by (7) and (8) with (20 - 23) is open-loop stable. □

Proof: 1. By Lemma 1 and the orthogonality of the matrix $T_R(p,q,s)$, we have

$$\begin{aligned} & T_R^T(n,n,N-1)A^TXT_R(n,n,N-1)\\ =\ & T_R^T(n,n,N-1)A^TT_R(n,n,N-1)\\ & \times T_R^T(n,n,N-1)XT_R(n,n,N-1)\\ =\ & diag[A_p,\ A_m,\ \cdots,\ A_m]diag[X_p,\ X_m,\ \cdots,\ X_m]\\ =\ & diag[A_pX_p,\ A_mX_m,\ \cdots,\ A_mX_m]. \end{aligned}$$

Similarly,

$$\begin{aligned} & T_R^T(n,n,N-1)XAT_R(n,n,N-1)\\ =\ & diag[A_pX_p,\ A_mX_m,\ \cdots,\ A_mX_m] \end{aligned}$$

$$\begin{aligned} & T_R^T(n,n,N-1)XB_0B_0^TXT_R(n,n,N-1)\\ =\ & T_R^T(n,n,N-1)XT_R(n,n,N-1)\\ & \times T_R^T(n,n,N-1)B_0T_R(m,m,N-1)\\ & \times T_R^T(m,m,N-1)B_0^TT_R(n,n,N-1)\\ & \times T_R^T(n,n,N-1)XT_R(n,n,N-1)\\ =\ & diag[X_pB_pB_p^TX_p, X_mB_mB_m^TX_m,\cdots,\\ & X_mB_mB_m^TX_m] \end{aligned}$$

$$T_R^T(n,n,N-1)XGG^TXT_R(n,n,N-1) = diag[X_pG_pG_p^TX_p, X_mG_mG_m^TX_m, \cdots, X_mG_mG_m^TX_m]$$

$$T_R^T(n,n,N-1)H^THT_R(n,n,N-1) = diag[H_p^TH_p, H_m^TH_m, \cdots, H_m^TH_m]$$

$$T_R^T(n,n,N-1)C_1^TC_1T_R(n,n,N-1) = diag[C_{p0}^TC_{p0},\ 0,\ \cdots,\ 0].$$

From equalities (29) and (30), it follows

$$T_R^T(n,n,N-1)\ \Delta_x\ T_R(n,n,N-1) = diag[\Delta_{xp},\ \Delta_{xm},\ \cdots,\ \Delta_{xm}] = 0$$

By assumption 1 in Theorem 2, $\Delta_{xp} = 0$ and $\Delta_{xm} = 0$. Thus, the equality (37) holds. The proof for equality (38) is similar.

Conclusions 2, 3 and 4 follow from the assumption 3 of Theorem 2 and the similar arguments as above. □

Proof of Theorem 2: Note that $C = C_0 + C_1, B = B_0 + B_1$, then the proof is completed by using Theorem 1 and Lemma 2. □

Remark 1: Theorem 2 presents a reliable controller design procedure for symmetric composite systems in terms of the solutions of the algebraic Reccati equations (AREs). But the order of these AREs are much lower than that of Theorem 1. The resulting closed-loop system is reliable with respect to the outage of sensors and actuators of a prescribed sybsystems (the first subsystem, in this case). This result is also different from that given in [6] [8] where the solution of the ARE for a symmetric composite system is constructed from the solutions of two AREs of the same order as the subsystem. When only sensor outages, or only actuator outages are considered, simpler design procedures can be obtained from Theorem 2 and its proof. The details are omitted.

Remark 2: It should also be noted that the condition under which the pair (A_p, B_p) is stabilizable and the pair (A_p, C_p) is detectable, is a necessary condition for the equation (29) and (31) to have positive definite solutions. This condition requires that the unstable modes of the prescribed subsystem be controlled or detected by other subsystem inputs or outputs through the coupling between the subsystems. When the matrix $A_m = A_{11} - A_{12}$ is unstable, it can be shown by a method similar to that in the proof of Theorem 2, that the reliable controller design is impossible for the case in which the sensors and actuators of any two subsystems are susceptible to outages.

4 Conclusion

This paper treats the reliable control problems for symmetric composite systems composed of several identical subsystems. By taking advantage of the symmetric structure of the systems, a reliable controller design procedure is presented in terms of the solutions to the algebraic Riccati equations with lower-order. The resulting control systems are reliable in that they provide guaranteed asymptotic stability and H_∞ performance when all sensors and actuators are operational as well as when the sensors or actuators of a prescribed subsystem experience outages.

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