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Reliable Controller Design for Nonlinear Systems

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Abstract

This paper addresses the reliable H_∞ control problems for affine nonlinear systems. Based on the Hamilton-Jacobi inequality approach developed in the H_∞ control problems for affine nonlinear systems, a method for the design of reliable nonlinear control systems is presented. The resulting nonlinear control systems are reliable in that they provide guaranteed local asymptotic stability and H_∞ performance not only when all control components are operational, but also in case of some component outages within a prespecified subset of control components.

1 Introduction

In recent years, considerable attention has been paid to the design problems of reliable linear control systems achieving various reliability goals, and some design methods have been given by several authors [3,9,12-14]. In particular, Veillette, Medanic and Perkins [12] present a methodology for the design of reliable linear control systems by means of the algebraic Riccati equation approach from linear H_∞ control theory, such that the resulting designs guaranteed closed-loop stability and H_∞ performance not only when all control components are operating, but also in case of some admissible control component outages.

In the area of nonlinear H_∞ control, some important advances have been made by several authors [1,4-6,8-10,11]. In particular, in [11] it was shown that the solution of the H_∞ control problem via state feedback can be determined from the solution of a Hamilton-Jacobi equation (or inequality), which is the nonlinear version of the Riccati equation for the corresponding linear H_∞ control problem in [2]. The solution of the problem in the case of measurement feedback has also been given in terms of the solutions of a pair of Hamilton-Jacobi inequalities in [1,5,8]. For the computational method to find Taylor series approximations to the solutions of the Hamilton-Jacobi inequalities, the reader is referred to [7] and [11]. The purpose of this paper is to investigate the reliable H_∞ control problem for affine nonlinear systems by using the Hamilton-Jacobi inequality approach.

2 Problem formulation

Consider an affine nonlinear system Σ described by equations of the form

$$\dot{x} = f(x) + g_1(x)w_0 + \sum_{j=1}^{m} g_{2j}(x)u_j \qquad (1)$$

$$y_i = h_{2i}(x) + w_i, \quad i = 1,2,\ldots,q \qquad (2)$$

$$z = \begin{bmatrix} h_1(x) \\ u_1 \\ \vdots \\ u_m \end{bmatrix} \qquad (3)$$

where x is a state vector defined on a neighbourhood X of the origin in R^n, $u = [u_1\ u_2 \ldots\ u_m]^T \in R^m$ denotes the control input, $w_r = [w_0^T\ w_1 \ldots\ w_q]^T \in R^r$ the disturbance input, $z \in R^s$ the output to be regulated, $y = [y_1\ y_2 \ldots\ y_q]^T \in R^q$ the measured output, $f(x), g_1(x), h_1(x), g_{2j}(x) (j = 1,\ldots,m)$ and $h_{2i}(x) (i = 1,\ldots,q)$ are known smooth mappings defined in a neighbourhood of the origin in R^n, and $f(0) = 0,\ h_1(0) = 0$ and $h_{2i}(x) = 0\ (i = 1,\ldots,q)$. Denote

$$g_2(x) = [\ g_{21}(x) \quad g_{22}(x) \quad \ldots \quad g_{2m}(x)\ ] \qquad (4)$$

$$h_2(x) = [\ h_{21}(x) \quad h_{22}(x) \quad \ldots \quad h_{2q}(x)3\ ]^T \qquad (5)$$

Let $\Omega_a \subset \{1,2,\ldots,m\}$ and $\Omega_s \subset \{1,2,\ldots,q\}$ correspond to a selected subset of actuators susceptible to outages and a selected subset of sensors susceptible to outages, respectively. Then, the problem considered in this paper is as follows:

Given the system Σ described by equations (1)–(3) and a positive constant γ, find a controller K with the fol-

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lowing form

$$\begin{aligned} \dot{\xi} &= a(\xi) + b(\xi)y \\ u(\xi) &= c(\xi) \end{aligned} \quad (6)$$

where $\xi \in R^v$, such that for actuator outages corresponding to any $\omega_a \subset \Omega_a$, and sensor outages corresponding to any $\omega_s \subset \Omega_s$, the resulting closed-loop system is locally asymptotically stable, and has a local L_2 gain which is less than or equal to γ.

For $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$, introduce the decomposition

$$\begin{aligned} g_2(x) &= g_{2\omega_a}(x) + g_{2\bar{\omega}_a}(x) \\ u &= u_{\omega_a} + u_{\bar{\omega}_a} \\ h_2(x) &= h_{2\omega_s}(x) + h_{2\bar{\omega}_s}(x) \\ y &= y_{\omega_s} + y_{\bar{\omega}_s} \\ w &= \left[\begin{array}{ccc} w_1 & \dots & w_q \end{array}\right]^T = w_{\omega_s} + w_{\bar{\omega}_s} \\ b(x) &= \left[\begin{array}{cccc} b_1(x) & b_2(x) & \dots & b_q(x) \end{array}\right] \\ &= b_{\omega_s}(x) + b_{\bar{\omega}_s}(x) \end{aligned}$$

where

$$g_{2\omega_a}(x) = [\ \delta_{\omega_a}(1)g_{21}(x) \quad \delta_{\omega_a}(2)g_{22}(x) \quad \dots \quad \delta_{\omega_a}(m)g_{2m}(x)] \quad (7)$$

$$u_{\omega_a} = [\ \delta_{\omega_a}(1)u_1 \quad \delta_{\omega_a}(2)u_2 \quad \dots \quad \delta_{\omega_a}(m)u_m\ ]^T \quad (8)$$

$$h_{2\omega_s}(x) = [\ \delta_{\omega_s}(1)h_{21}(x) \quad \delta_{\omega_s}(2)h_{22}(x) \quad \dots \quad \delta_{\omega_s}(m)h_{2q}(x)]^T \quad (9)$$

$$y_{\omega_s} = [\ \delta_{\omega_s}(1)y_1 \quad \delta_{\omega_s}(2)y_2 \quad \dots \quad \delta_{\omega_s}(q)y_q\ ]^T \quad (10)$$

$$w_{\omega_s}(x) = [\ \delta_{\omega_s}(1)w_1 \quad \delta_{\omega_s}(2)w_2 \quad \dots \quad \delta_{\omega_s}(q)w_q\ ]^T \quad (11)$$

$$b_{\omega_s}(x) = [\ \delta_{\omega_s}(1)b_1(x) \quad \delta_{\omega_s}(2)b_2(x) \quad \dots \quad \delta_{\omega_s}(q)b_q(x)] \quad (12)$$

with δ_{ω_a} and δ_{ω_s} defined as follows:

$$\delta_{\omega_a}(i) = \begin{cases} 1, & \text{if } i \in \omega_a \\ 0, & \text{if } i \notin \omega_a \end{cases}$$

$$\delta_{\omega_s}(i) = \begin{cases} 1, & \text{if } i \in \omega_s \\ 0, & \text{if } i \notin \omega_s \end{cases}$$

Applying the controller K of (6) to the system Σ, when actuator and sensor outages corresponding to $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$, occur, the resulting closed-loop system $\Sigma_{\omega_a,\omega_s}$ is given by

$$\begin{aligned} \dot{x} &= f(x) + g_{2\bar{\omega}_a}(x)c_{\bar{\omega}_a}(\xi) + g_1(x)w_0 && (13) \\ \dot{\xi} &= a(\xi) + b_{\bar{\omega}_s}(\xi)y_{\bar{\omega}_s} \\ &= a(\xi) + b_{\bar{\omega}_s}(\xi)h_{2\bar{\omega}_s}(x) + b_{\bar{\omega}_s}(\xi)w_{\bar{\omega}_s} && (14) \\ z_{\bar{\omega}_a} &= \begin{bmatrix} h_1(x) \\ c_{\bar{\omega}_a}(\xi) \end{bmatrix} && (15) \end{aligned}$$

The goal is to select the functions $a(\xi)$, $b(\xi)$ and $c(\xi)$ such that for any $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$, the system $\Sigma_{\omega_a,\omega_s}$ is locally asymptoticallystable, and is locally dissipative with respect to the supply rate $s(w_{r\bar{\omega}_s}, z_{\bar{\omega}_a}) = \gamma^2\|w_{r\bar{\omega}_s}\|^2 - \|z_{\bar{\omega}_a}\|^2$, where

$$w_{r\bar{\omega}_s} = [\ w_0^T \quad w_{\bar{\omega}_s}^T\ ]^T \quad (16)$$

Next section will present a design procedure for the reliable controller design problem.

The following two inequalities are obvious, and will be used in the sequel.

$$g_{2\omega_a}(x)g_{2\omega_a}^T(x) \le g_{2\Omega_a}(x)g_{2\Omega_a}^T(x) \text{ for } \omega_a \subset \Omega a \quad (17)$$

$$h_{2\omega_s}(x)h_{2\omega_s}^T(x) \le h_{2\Omega_s}(x)h_{2\Omega_s}^T(x) \text{ for } \omega_s \subset \Omega s \quad (18)$$

3 Main results

In order to describe the main result of the section, we first recall a notion of detectability.

Definition 3.1 [4]: *Suppose $f(0) = 0$ and $h(0) = 0$. The pair $\{f, h\}$ is said to be* locally detectable *if there exists a neighbourhood U of the point $x = 0$ such that, if $x(t)$ is any integral curve of $\dot{x} = f(x)$ satisfying $x(0) \in U$, then $h(x(t))$ is defined for all $t \ge 0$ and $h(x(t)) = 0$ for all $t \ge 0$ implies $\lim_{t\to\infty} x(t) = 0$.*

Define the Hamiltonians $H_s(x,p)$ and $H_0(x,p)$ as follows

$$H_s(x,p) = p^T f(x) + h_1^T(x)h_1(x) + \gamma^2 h_{2\Omega_s}^T(x)h_{2\Omega_s}(x) + \frac{1}{4}p^T\left(\frac{1}{\gamma 2}g_1(x)g_1^T(x) - g_{2\bar{\Omega}_a}(x)g_{2\bar{\Omega}_a}^T(x)\right)p \quad (19)$$

$$H_0(x,p) = p^T f(x) + \frac{1}{4\gamma^2}p^T g_1(x)g_1^T(x)p + \frac{1}{4}p^T g_{2\Omega_a}(x) \times g_{2\Omega_a}^T(x)ph_1^T(x)h_1(x) - \gamma^2 h_{2\bar{\Omega}_s}^T(x)h_{2\bar{\Omega}_s}(x) \quad (20)$$

Then the following theorem presents a sufficient condition for the solvability of the reliable controller design problem.

Theorem 3.2 *Consider the system Σ described by equations (1)–(3) and suppose the following:*

(i) the pair $\{f, h_1\}$ is locally detectable.

(ii) there exists some C^2 function $\psi(x) \ge 0$ with $\psi(0) = 0$ such that

(a) there exists a C^3 positive definite function $V(x)$, locally defined in a neighbourhood of $x = 0$ and vanishing at $x = 0$, which satisfies the Hamilton-Jacobi equation

$$H_s(x, V_x^T) + \psi(x) = 0 \quad (21)$$

(b) there exists a C^3 positive definite function $U(x)$, locally defined in a neighbourhood of $x = 0$ and vanishing at $x = 0$, which satisfies the Hamilton-Jacobi inequality

$$H_0(x, U_x^T) + \psi(x) \leq 0 \qquad (22)$$

and such that $H_0(x, U_x^T) + \psi(x)$ has nonsingular Hessian matrix at $x = 0$.

(c) $U(x) - V(x)$ is positive definite, and

$$(U_x - V_x)L(x) = 2\gamma^2 h_2^T(x) \qquad (23)$$

has a solution $L(x)$.

where V_x and U_x are the Jacobian matrices of V and U, respectively.
Then, the controller K of (6) with

$$\begin{aligned} a(\xi) &= f(\xi) + \frac{1}{2\gamma^2} g_1(\xi) g_1^T(\xi) V_x^T(\xi) \\ &\quad - \frac{1}{2} g_{2\bar{\Omega}_a}(\xi) g_{2\bar{\Omega}_a}^T(\xi) V_x^T(\xi) - L(\xi) h_2(\xi) \qquad (24) \\ b(\xi) &= L(\xi) \qquad (25) \\ c(\xi) &= -\frac{1}{2} g_2^T(\xi) V_x^T(\xi) \qquad (26) \end{aligned}$$

is a solution of the reliable controller design problem for the system Σ of (1)-(3).

The following preliminaries are required in the proof of Theorem 3.2.

For the system Σ described by equations (1)–(3), consider an extended system Σ_e given by

$$\begin{aligned} \dot{x} &= f(x) + [g_1(x) \;\; \gamma g_{2\Omega_a}(x)]\bar{w}_0 + g_2(x)u \qquad (27) \\ y &= h_2(x) + w \qquad (28) \\ \bar{z} &= \begin{bmatrix} h_1(x) \\ \gamma h_{2\Omega_s}(x) \\ u \end{bmatrix} \qquad (29) \end{aligned}$$

Applying the controller K of (6) to the system Σ_e, then the resulting closed-loop system Σ_{ce} is as follow

$$\begin{aligned} \dot{x}_e &= \bar{f}_e(x_e) + \bar{g}_e(x_e)\bar{w} \qquad (30) \\ \bar{z} &= \begin{bmatrix} h_1(x) \\ \gamma h_{2\Omega_s}(x) \\ c(\xi) \end{bmatrix} \qquad (31) \end{aligned}$$

where $x_e = [x^T \;\; \xi^T]^T$, $\bar{w} = [\bar{w}_0^T \;\; w^T]^T$,

$$\begin{aligned} \bar{f}_e(x_e) &= \begin{bmatrix} f(x) + g_2(x)c(\xi) \\ a(\xi) + b(\xi)h_2(x) \end{bmatrix} \\ \bar{g}_e(x_e) &= \begin{bmatrix} [g_1(x) \;\; \gamma g_{2\Omega_a}(x)] & 0 \\ 0 & b(\xi) \end{bmatrix} \end{aligned}$$

The closed-loop system $\Sigma_{\omega_a,\omega_s}$ of (13)–(15) can be written as

$$\begin{aligned} \dot{x}_e &= f_{as}(x_e) + g_{as}(x_e) w_{r\bar{\omega}_s} \qquad (32) \\ z_{\bar{\omega}_a} &= \begin{bmatrix} h_1(x) \\ c_{\bar{\omega}_a}(\xi) \end{bmatrix} \qquad (33) \end{aligned}$$

where $w_{r\bar{\omega}_s}$ is given by (16),

$$\begin{aligned} f_{as}(x_e) &= \begin{bmatrix} f(x) + g_{2\bar{\omega}_a}(x) c_{\bar{\omega}_a}(\xi) \\ a(\xi) + b_{\bar{\omega}_s}(\xi) h_{2\bar{\omega}_s}(x) \end{bmatrix} \\ g_{as}(x_e) &= \begin{bmatrix} g_1(x) & 0 \\ 0 & b_{\bar{\omega}_s}(\xi) \end{bmatrix} \end{aligned}$$

Let $X(x_e)$ be a C^1 function defined in a neighbourhood of $(0, 0)$, and denote

$$\begin{aligned} J_{ce}(X, \Sigma_{ce}) &\triangleq X_{x_e} \bar{f}_e(x_e) + \bar{z}^T \bar{z} \\ &\quad + \frac{1}{4\gamma^2} X_{x_e} \bar{g}_e(x_e) \bar{g}_e^T(x_e) X_{x_e}^T \qquad (34) \\ J_{as}(X, \Sigma_{\omega_a,\omega_s}) &\triangleq X_{x_e} f_{as}(x_e) + z_{\bar{\omega}_a}^T z_{\bar{\omega}_a} \\ &\quad + \frac{1}{4\gamma^2} X_{x_e} g_{as}(x_e) g_{as}^T(x_e) X_{x_e}^T \qquad (35) \end{aligned}$$

Then, we have the following lemmas.

Lemma 3.3 *For any $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$, the following inequality holds*

$$J_{as}(X, \Sigma_{\omega_a,\omega_s}) \leq J_{ce}(X, \Sigma_{ce}) \qquad (36)$$

Proof: By equations (30), (31), (32) and (33), we have

$$\begin{aligned} X_{x_e} f_{as}(x_e) &= X_{x_e} \bar{f}_e(x_e) - X_{x_e} \begin{bmatrix} g_{2\omega_a}(x) c_{\omega_a}(\xi) \\ b_{\omega_s}(\xi) h_{2\omega_s}(x) \end{bmatrix} \\ &= X_{x_e} \bar{f}_e(x_e) - X_{x_e} \begin{bmatrix} g_{2\omega_a}(x) c_{\omega_a}(\xi) \\ 0 \end{bmatrix} \\ &\quad - X_{x_e} \begin{bmatrix} 0 \\ b_{\omega_s}(\xi) h_{2\omega_s}(x) \end{bmatrix} \\ &\leq X_{x_e} \bar{f}_e(x_e) + \frac{1}{4} X_{x_e} \begin{bmatrix} g_{2\omega_a}(x) g_{2\omega_a}^T(x) & 0 \\ 0 & 0 \end{bmatrix} X_{x_e}^T \\ &\quad + c_{\omega_a}^T(\xi) c_{\omega_a}(\xi) + \gamma^2 h_{2\omega_s}^T(x) h_{2\omega_s}(x) \\ &\quad + \frac{1}{4\gamma^2} X_{x_e} \begin{bmatrix} 0 & 0 \\ 0 & b_{\omega_s}(\xi) b_{\omega_s}^T(\xi) \end{bmatrix} X_{x_e}^T \qquad (37) \end{aligned}$$

$$\begin{aligned} z_{\bar{\omega}_a}^T z_{\bar{\omega}_a} &= h_1^T(x) h_1(x) + c_{\bar{\omega}_a}^T(\xi) c_{\bar{\omega}_a}(\xi) \\ &\quad h_1^T(x) h_1(x) + c^T(\xi) c(\xi) - c_{\omega_a}^T(\xi) c_{\omega_a}(\xi) \qquad (38) \end{aligned}$$

$$\begin{aligned} &X_{x_e} g_{as}(x_e) g_{as}^T(x_e) X_{x_e}^T \\ &= X_{x_e} \begin{bmatrix} g_1(x) g_1^T(x) & 0 \\ 0 & b_{\bar{\omega}_s}(\xi) b_{\bar{\omega}_s}^T(\xi) \end{bmatrix} X_{x_e}^T \\ &= X_{x_e} \begin{bmatrix} g_1(x) g_1^T(x) & 0 \\ 0 & b(\xi) b^T(\xi) \end{bmatrix} X_{x_e}^T \\ &\quad - X_{x_e} \begin{bmatrix} 0 & 0 \\ 0 & b_{\omega_s}(\xi) b_{\omega_s}^T(\xi) \end{bmatrix} X_{x_e}^T \qquad (39) \end{aligned}$$

Combining equations (37)–(39), (31), and inequalities (17) and (18), it follows that

$$J_{as}(X,\Sigma_{\omega_a,\omega_s}) \le X_{x_e}\bar{f}_e(x_e) + h_1^T(x)h_1(x) + c^T(\xi)c(\xi)$$

$$+\gamma^2 h_{2\omega_s}^T(x)h_{2\omega_s}(x) + \frac{1}{4\gamma^2}X_{x_e}\times$$

$$\begin{bmatrix} g_1(x)g_1^T(x)+\gamma^2 g_{2\omega_a}(x)g_{2\omega_a}^T(x) & 0 \\ 0 & b(\xi)b^T(\xi)\end{bmatrix} X_{x_e}^T$$

$$\le X_{x_e}\bar{f}_e(x_e) + \bar{z}^T\bar{z} + \frac{1}{4\gamma^2}X_{x_e}\times$$

$$\begin{bmatrix} g_1(x)g_1^T(x)+\gamma^2 g_{2\Omega_a}(x)g_{2\Omega_a}^T(x) & 0 \\ 0 & b(\xi)b^T(\xi)\end{bmatrix} X_{x_e}^T$$

$$= J_{ce}(X,\Sigma_{ce}).$$

◁

Lemma 3.4 *Under the assumptions of Theorem 3.2, the equilibrium $x = 0$ of the system*

$$\dot{x} = f(x) + \frac{1}{2\gamma^2}g_1(x)g_1^T(x)V_x^T(x) - L(x)h_2(x) \quad (40)$$

is locally asymptotically stable.

Proof. Let $Q(x) = U(x) - V(x)$,

$$H_w(x,Q_x^T) = Q_x(f(x) + \frac{1}{2\gamma^2}g_1(x)g_1^T(x)V_x^T + \frac{1}{2}g_{2\Omega_a}(x)$$

$$\times g_{2\Omega_a}^T(x)V_x^T) - Q_xL(x)h_2(x)$$

$$+\frac{1}{4\gamma^2}Q_x[g_1(x)g_1^T(x) + \gamma^2 g_{2\Omega_a}(x)g_{2\Omega_a}^T(x)]Q_x^T$$

$$+c^T(x)c(x) + \frac{1}{4\gamma^2}Q_xL(x)L^T(x)Q_x^T \quad (41)$$

Then, by equations (19)–(23) and (26), it follows

$$\begin{aligned} H_w(x,Q_x^T) &= H_0(x,U_x^T) - H_s(x,V_x^T) \\ &= H_0(x,U_x^T) + \psi(x) \le 0 \quad (42)\end{aligned}$$

By computing directly, we have

$$H_w(x,Q_x^T) \ge Q_x(f(x)+\frac{1}{2\gamma^2}g_1(x)g_1^T(x)V_x^T - L(x)h_2(x))$$

$$+c^T(x)c(x) - \frac{1}{4}V_xg_{2\Omega_a}(x)g_{2\Omega_a}^T(x)V_x^T$$

$$+\frac{1}{4\gamma^2}Q_xg_1(x)g_1^T(x)Q_x^T + \frac{1}{4\gamma^2}Q_xL(x)L^T(x)Q_x^T$$

$$\ge Q_x(f(x) + \frac{1}{2\gamma^2}g_1(x)g_1^T(x)V_x^T - L(x)h_2(x))$$

which further implies from (42) and the condition under which $H_0(x,U_x^T) + \psi(x)$ has nonsingular Hessian matrix at $x = 0$ that the system (40) is locally asymptotically stable. ◁

Lemma 3.5 *Under the assumptions of Theorem 3.2, let $Q(x) = U(x) - V(x)$, $X(x_e) = V(x) + Q(x-\xi)$, then there exists a neighbourhood of $(x,\xi) = (0,0)$ in which the following inequality holds:*

$$J_{ce}(X,\Sigma_{ce}) \le 0 \quad (43)$$

Proof. In the extended system Σ_e described by equations (27)–(29), let $\bar{g}_1(x) = [g_1(x)\ \ \gamma g_{2\Omega_a}(x)]$ and $\bar{h}_1(x) = \begin{bmatrix} h_1(x) \\ \gamma h_{2\Omega_s}(x)\end{bmatrix}$. Then, from equation (19), we have

$$V_xf(x) + \bar{h}_1^T(x)\bar{h}_1(x) + \frac{1}{4}V_x(\frac{1}{\gamma^2}\bar{g}_1(x)\bar{g}_1^T(x) - g_2g_2^T(x))V_x^T$$

$$= H_s(x,V_x^T) \quad (44)$$

Denote $c_1(x) = \frac{1}{2\gamma^2}\bar{g}_1^T(x)V_x^T$ and $\bar{f}(x) = f(x) + \bar{g}(x)c_1(x)$. By equations (23), (41) and (42), it follows

$$Q_x\bar{f}(x) + c_1^T(x)c_1(x) - \gamma^2h_2^T(x)h_2(x)$$

$$+\frac{1}{4\gamma^2}Q_x\bar{g}_1(x)\bar{g}_1^T(x)Q_x^T$$

$$= H_w(x,Q_x^T) = H_0(x,U_x^T) + \psi(x) \quad (45)$$

Then, from the assumptions of Theorem 3.2, equations (44) and (45), and the proof of Theorem 3.1 in [5], it follows that the inequality (43) holds in a neighbourhood of $(x,\xi) = (0,0)$. ◁

Proof of Theorem 3.2: By Lemma 3.3, Lemma 3.5, and Theorem 2 in [11], it follows that for any $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$, the system $\Sigma_{\omega_a,\omega_s}$ of (13)–(15) or (32)–(33) is locally dissipative with respect to the supply rate $s(w_{r\bar{\omega}_s}, z_{\bar{\omega}_a}) = \gamma^2\|w_{r\bar{\omega}_s}\|^2 - \|z_{\bar{\omega}_a}\|^2$.

In the following, we show that the system $\Sigma_{\omega_a,\omega_s}$ is locally asymptotically stable.

From $J_{as}(X,\Sigma_{\omega_a,\omega_s}) \le 0$ and $w_{r\bar{\omega}_s} = 0$, it follows

$$\frac{dX(x_e(t))}{dt} = X_{x_e}f_{as}(x_e(t)) \le -z_{\bar{\omega}_a}^Tz_{\bar{\omega}_a}$$

$$= -\|h_1(x(t))\|^2 - \|c_{\bar{\omega}_a}(\xi)\|^2$$

This proves that the system $\Sigma_{\omega_a,\omega_s}$ is stable at the equilibrium $(x,\xi) = (0,0)$, and any trajectory satisfying

$$\frac{dX(x_e(t))}{dt} = 0$$

is necessarily a trajectory of

$$\dot{x} = f(x) + g_{2\bar{\omega}_a}(x)c_{\bar{\omega}_a}(\xi)$$

such that $x(t)$ is bounded and $h_1(x(t)) = 0, c_{\bar{\omega}_a}(\xi(t)) = 0$, which further follows from assumption (i) that

$\lim_{t\to\infty} x(t) = 0$. Thus, the ω–limit set of such a trajectory is a subset of

$$M = \{(x,\xi) : \ x = 0, \ c_{\bar{\omega}_a}(\xi(t)) = 0\}$$

By equation (24), and $\bar{\Omega}_a \subset \bar{\omega}_a$, any initial condition on this ω–limit set yields a trajectory in which $x(t) = 0$ for all $t \geq 0$, while $\xi(t)$ is a trajectory of

$$\begin{aligned}
\dot{\xi} &= a(\xi) + b_{\bar{\omega}_s}(\xi)h_{2\bar{\omega}_s}(x) \\
&= f(\xi) + \frac{1}{2\gamma^2}g_1(\xi)g_1^T(\xi)V_x^T(\xi) \\
&\quad -\frac{1}{2}g_{2\bar{\Omega}_a}(\xi)g_{2\bar{\Omega}_a}^T(\xi)V_x^T(\xi) - L(\xi)h_2(\xi) \\
&= f(\xi) + \frac{1}{2\gamma^2}g_1(\xi)g_1^T(\xi)V_x^T(\xi) - L(\xi)h_2(\xi) \\
&\quad -\frac{1}{2}g_{2\bar{\Omega}_a}(\xi)c_{\bar{\Omega}_a}^T(\xi) \\
&= f(\xi) + \frac{1}{2\gamma^2}g_1(\xi)g_1^T(\xi)V_x^T(\xi) - L(\xi)h_2(\xi)
\end{aligned}$$

By Lemma 3.4, it follows that $\lim_{t\to\infty} \xi(t) = 0$. Thus, by the invariance principle, the system $\Sigma_{\omega_a,\omega_s}$ is locally asymptotically stable. $\triangleleft$.

In the case of a linear system

$$\begin{aligned}
\dot{x} &= Ax + Gw_0 + Bu && (46) \\
y &= Cx + w && (47) \\
z &= \begin{bmatrix} Hx \\ u \end{bmatrix} && (48)
\end{aligned}$$

a solution of the corresponding reliable controller design problem is given by the following corollary.

Corollary 3.6 *Consider the linear system described by equations (46)-(48) and suppose the following:*

(i) the pair (A, H) is detectable;

(ii) the following algebraic Riccati equation and inequality

$$A^TX + XA - XB_{\bar{\Omega}_a}B_{\bar{\Omega}_a}^TX + \frac{1}{\gamma^2}XGG^TX$$

$$+H^TH + \gamma^2C_{\Omega_s}^TC_{\Omega_s} = 0 \qquad (49)$$

$$A^TY + YA + YB_{\Omega_a}B_{\Omega_a}^TY + \frac{1}{\gamma^2}YGG^TY$$

$$+H^TH - \gamma^2C_{\bar{\Omega}_s}^TC_{\bar{\Omega}_s} < 0 \qquad (50)$$

have positive definite solutions X and Y, respectively, and $Y > X$, where the matrices B_{Ω_a}, $B_{\bar{\Omega}_a}$, C_{Ω_s}, and $C_{\bar{\Omega}_a}$ have meanings similar to those of $g_{2\Omega_a}(x)$, $g_{2\bar{\Omega}_a}(x)$, $h_{2\Omega_a}(x)$, and $h_{2\bar{\Omega}_a}(x)$ in (19) and (20).

Denote

$$\begin{aligned}
G_+ &= [\ G \quad \gamma B_{\Omega_a}\ ], \qquad K_{d+} = \frac{1}{\gamma^2}G_+^TX \\
K &= -B^TX \\
L &= \gamma^2(Y-X)^{-1}C^T
\end{aligned}$$

Then the controller

$$\begin{aligned}
\dot{\xi} &= (A + BK + G_+K_{d+} - LC)\xi + Ly && (51) \\
u &= K\xi && (52)
\end{aligned}$$

is a control law such that for actuator outages corresponding to any $\omega_a \subset \Omega_a$, and sensor outages corresponding to any $\omega_s \subset \Omega_s$, the resulting closed-loop system is asymptotically stable, and has an H_∞-norm bound γ.

Remark 3.7 Comparing with Theorems 4.1 and 4.2 in [12], Corollary 3.6 contains a condition under which the strictly Riccati inequality (50) has a positive definite solution, which is stronger than the condition in Theorem 4.2 in [12] under which the corresponding Riccati equation

$$A^TY + YA + YB_{\Omega_a}B_{\Omega_a}^TY + \frac{1}{\gamma^2}YGG^TY + H^TH$$

$$-\gamma^2C_{\bar{\Omega}_s}^TC_{\bar{\Omega}_s} = 0$$

has a positive definite solution, but the asymptotic stability of the controller given by equations (51) and (52) is not required in Corollary 3.6.

In the following, we present an example to illustrate the result of the paper.

Example 3.8 Let the considered system be described by equations (1)-(3), with $n = 1$, $m = 2$,q=1, $f(x) = 0$, $g_1(x) = x$, $g_{21}(x) = 1$, $g_{22}(x) = \frac{1}{2}$, $h_1(x) = x$, $h_2(x) = 2x$, $\Omega_a = \{2\}$, $\Omega_s = \emptyset$, $\gamma = 1$ and $\psi(x) = 0$. Then the Hamilton-Jacobi inequalities (21) and (22) take the form

$$\begin{aligned}
V_x^2(x^2-1) + 4x^2 &\leq 0 && (53) \\
U_x^2(\frac{x^2}{4} + \frac{1}{16}) - 3x^2 &\leq 0 && (54)
\end{aligned}$$

It is easy to show that for $|\, x \,| < \frac{1}{2}$, $V(x) = 1-(1-x^2)^{\frac{1}{2}}$ and $U(x) = (2.99)^{\frac{1}{2}}(x^2+\frac{1}{4})^{\frac{1}{2}} - \frac{1}{2}(2.99)^{\frac{1}{2}}$ satisfies the inequalities (53) and (54), and such that the Hessian matrix of the left-hand side of (54) is less than zero at $x = 0$. Thus, from Theorem 3.2, the controller

$$\begin{aligned}
\dot{\xi} &= (\xi^3 - \xi)(1-\xi^2)^{-\frac{1}{2}} \\
&\quad -\frac{4\xi(\xi^2+\frac{1}{4})^{\frac{1}{2}}(1-\xi^2)^{\frac{1}{2}}}{(2.99)^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}} - (\xi^2+\frac{1}{4})^{\frac{1}{4}}} \\
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= -\begin{bmatrix} \xi(1-\xi^2)^{-\frac{1}{2}} \\ \frac{1}{2}\xi(1-\xi^2)^{-\frac{1}{2}} \end{bmatrix}
\end{aligned}$$

is such that for the second actuator outage or operating, the resulting closed-loop system is locally asymptotically stable, and has a local L_2 gain which is less than or equal to 1.

4 Conclusions

This paper presents a solution of the reliable controller design problem for anaffine nonlinear system, and the solution of the problem is shown to be related to the existence of solutions of a Hamilton-Jacobi equation and a Hamilton-Jacobi inequality. The resulting nonlinear control systems are reliable in that they achieve asymptotic stability and H_∞ performance, not only when the system is operating properly, but also in case of some component outages within a prespecified subset of control components.

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