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Control of Uncertain Bilinear Systems using Linear Controllers: Stability Region Estimation and Controller Design *

Shoudong Huang
Department of Engineering
Australian National University
Canberra, ACT 0200, Australia
shoudong.huang@anu.edu.au

James Lam
Department of Mechanical Engineering
University of Hong Kong
Pokfulam Road, Hong Kong
james.lam@hku.hk

Abstract

This paper studies the problem of stability region estimation and controller design for uncertain bilinear systems when linear controllers are used. Iterative linear matrix inequality (ILMI) algorithms are presented to estimate the closed-loop stability region and design the controllers. No tuning of parameters is needed in the design methods. The design aims to optimize between the size of the stability region, damping of the state variables, and the feedback gain.

1. Introduction

Bilinear system is a special kind of nonlinear systems which could represent a variety of important physical processes [8]. A great amount of literature related to the control problems of such systems has been developed over the past decades. Among them, some results were concerned with bilinear systems with only multiplicative control [2, 9]. For bilinear systems with both additive and multiplicative control inputs, some control designs, such as bang-bang control [7] or optimal control [1], obtain global asymptotic stability under the assumption that the open-loop system is either stable or neutrally stable. When the open-loop system is unstable, it is difficult to obtain global asymptotic stability except when independent additive and multiplicative control inputs exist [4].

Though much attention has been paid to the study of bilinear systems, little research work was devoted to the control problems for bilinear systems with uncertainties. Recently, the quadratic stabilization problem of uncertain bilinear systems was studied in [10] and a sufficient condition for the system to be quadratically stabilizable by "quadratic" feedback control was presented.

In some practical control system designs, local closed-loop stability may be enough and linear controllers are preferred. In [5], Gutman suggested that a nonlinear controller be used firstly and forced the trajectory of a bilinear system sufficiently near the origin, then switch to a linear controller such that the closed-loop system is asymptotically stable. Hence the study of local stabilization using linear controller and the estimation of the closed-loop stability region are useful in practice. Such a problem had been studied in [3] but the results in there are not easy to be applied because it needs tuning of parameters. Moreover, [3] did not considered the robustness of the controller to model uncertainties.

This paper considers the stability region estimation and controller design for uncertain bilinear systems, where the controls act additively and multiplicatively simultaneously. The uncertainties are assumed to have a polytopic form. No assumption on the open-loop stability is made and linear state feedback control is studied. As compared with [3], the results in this paper has the following advantage. Firstly, we estimate the derivative of the Lyapunov function in a different way, and our results are less conservative than that in [3]. Secondly, our results are presented in the form of iterative linear matrix inequality (ILMI) algorithms. It needs no tuning of parameters and can be efficiently solved numerically using LMI toolbox in MATLAB. Thirdly, three design specifications, closed-loop system stability region, feedback gain, and damping of the state variables, are all taken into account in the controller design. Three design algorithms are presented to optimize one specification when the other two specifications are given. Finally, our results are suitable for bilinear systems with polytopic uncertainties.

This paper is organized as follows. Section 2 gives the problems statement. The stability region estimation method and controller design methods are presented in Sections 3 and 4, respectively. Two examples are given in Section 5 to illustrate our methods. Section 6 concludes the paper.

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2. Problem Statement

The following notations are used in this paper. $\partial\Omega$ denotes the boundary of a set $\Omega \subset \mathbb{R}^n$. e_i denotes the ith standard basis of $\mathbb{R}^n$. $(P)_{ii}$ denotes the element in the ith column and the ith row of a matrix P. $\sigma(A)$ denotes the spectrum of matrix A. For a real symmetric matrix M, $M > (\geq) 0$ means that M is positive (semi-)definite. If not explicitly stated, I and 0 denote the identity matrix and the zero matrix of appropriate dimensions, respectively. Besides, all matrices are assumed to have compatible dimensions.

Consider the bilinear system

$$\mathcal{S}: \quad \dot{x} = A(t)x + \sum_{i=1}^{n} x_i N_i(t)u + B(t)u \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $x_i(t) \in \mathbb{R}$ is the i-th element of $x(t)$, $u(t) \in \mathbb{R}^m$ is the control input. Suppose the system under consideration has polytopic uncertainties. That is, the system matrices belong to matrix polytopes

$$A(t) \in \mathbf{Co}\{A_1, \ldots, A_{n_A}\}, \tag{2}$$

$$B(t) \in \mathbf{Co}\{B_1, \ldots, B_{n_B}\}, \tag{3}$$

$$N_i(t) \in \mathbf{Co}\{N_{i1}, \ldots, N_{in_{N_i}}\}, \; i = 1, \ldots, n. \tag{4}$$

where **Co** denotes the convex hull.

Consider a linear state feedback

$$u = Kx,$$

the closed-loop system is

$$\mathcal{S}_c: \quad \dot{x} = [A(t) + B(t)K]\,x + \sum_{i=1}^{n} x_i N_i(t) K x \tag{5}$$

The first problem considered in this paper is to estimate the stability region of $\mathcal{S}_c$ for a given K. The second problem is to design K such that $\mathcal{S}_c$ satisfies certain specifications. The specifications considered in this paper include the size of the stability region, the damping of the state variables, and the magnitude of the feedback gain.

3. Estimation of Stability Region

Since there are uncertainties in system (5), we will consider its local quadratic stability. Consider the Lyapunov function $V(x) = x^T P x$ with $P > 0$.

The derivative of $V(x)$ along the trajectories of $\mathcal{S}_c$ is

$$\begin{aligned} \dot{V}(x) &= x^T \left\{ P\,[A(t) + B(t)K] + [A(t) + B(t)K]^T P \right\} x \\ &\quad + x^T \left[P \sum_{i=1}^{n} x_i N_i(t) K + K^T \left(\sum_{i=1}^{n} x_i N_i(t) \right)^T P \right] x \\ &= x^T \{ P[A(t) + B(t)K] + [A(t) + B(t)K]^T P \\ &\quad + \sum_{i=1}^{n} x_i \left[P N_i(t) K + K^T N_i^T(t) P \right] \} x. \end{aligned}$$

Define a set $\Omega \subset \mathbb{R}^n$ as follows. For any $x \in \mathbb{R}^n$, $x \in \Omega$ if and only if

$$\begin{aligned} & P\,[A(t) + B(t)K] + [A(t) + B(t)K]^T P \\ & + \sum_{i=1}^{n} x_i \left[P N_i(t) K + K^T N_i^T(t) P \right] < 0 \end{aligned}$$

for all $A(t), B(t)$ and $N_i(t)$ $(i = 1, \ldots, n)$ that satisfy (2), (3), and (4), respectively, where x_i is the i-th element of x.

Further define a set $\mathcal{D}_0 \subset \mathbb{R}^n$ as

$$\mathcal{D}_0 = \{ x \in \mathbb{R}^n : V(x) < V_0 \} \tag{6}$$

where

$$V_0 = \min_{x \in \partial\Omega} x^T P x.$$

From Lyapunov stability theorem, $\mathcal{D}_0$ is contained in the stability region of $\mathcal{S}_c$. The following lemma gives a method to compute $\mathcal{D}_0$ for given K and P.

Lemma 1 *Suppose K and $P > 0$ are given, and $r \in \mathbb{R} > 0$ is the solution of LMI problem*

$$\max \; r \; \textit{subject to}$$

$$\begin{aligned} & P\,(A_{l_A} + B_{l_B} K) + (A_{l_A} + B_{l_B} K)^T P \\ & + r \sum_{i=1}^{n} x_{0i}^j \sqrt{(P^{-1})_{ii}} \left(P N_{il_{N_i}} K + K^T N_{il_{N_i}}^T P \right) < 0 \end{aligned} \tag{7}$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B$, $l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, and $j = 1, \ldots, 2^n$, where $x_0^j = \left[x_{01}^j, \ldots, x_{0n}^j \right]^T$ $(j = 1, \ldots, 2^n)$ are the 2^n vectors whose elements x_{0i}^j can only take the value of 1 or -1, then an estimate of the stability region can be obtained by

$$\mathcal{D}_0 = \left\{ x \in \mathbb{R}^n : x^T P x < r^2 \right\}. \tag{8}$$

Proof. For any $r > 0$, since $|x_i| = r\sqrt{(P^{-1})_{ii}}$ $(i = 1, \ldots, n)$ are tangential to the ellipsoid $x^T P x = r^2$, we consider the polyhedron

$$\Omega_r = \left\{ x \in \mathbb{R}^n : |x_i| < r\sqrt{(P^{-1})_{ii}}, \quad i = 1, \ldots, n \right\} \tag{9}$$

that contains the ellipsoid $\{ x \in \mathbb{R}^n : x^T P x < r^2 \}$. Since Ω_r is convex,

$$\begin{aligned} & P\,[A(t) + B(t)K] + [A(t) + B(t)K]^T P \\ & + \sum_{i=1}^{n} x_i \left(P N_i(t) K + K^T N_i^T(t) P \right) < 0 \end{aligned} \tag{10}$$

for all x in Ω_r if and only if (10) holds for all the 2^n vertices of Ω_r, that is,

$$\begin{aligned} & P\,[A(t) + B(t)K] + [A(t) + B(t)K]^T P \\ & + r \sum_{i=1}^{n} x_{0i}^j \sqrt{(P^{-1})_{ii}} \left(P N_i(t) K + K^T N_i^T(t) P \right) < 0, \end{aligned} \tag{11}$$

for all $j = 1, \ldots, 2^n$.

Since $A(t), B(t)$ and $N_i(t)$ $(i = 1, \ldots, n)$ satisfy (2), (3) and (4), (11) is further equivalent to (7).

After the largest r that satisfies (7) is obtained, from the fact about quadratic functions (see e.g. Lemma 1 in

[6]), the estimated stability region $\mathcal{D}_0$ can be computed by (6) where

$$V_0 = \min_{1 \le i \le n} V_i$$

and

$$\begin{aligned} V_i &= \min\left\{x^T P x : x \in \mathbb{R}^n,\ |x_i| = r\sqrt{(P^{-1})_{ii}}\right\} \\ &= \min\left\{x^T P x : x \in \mathbb{R}^n,\ e_i^T x = r\sqrt{(P^{-1})_{ii}}\right\} \\ &= \frac{\left(r\sqrt{(P^{-1})_{ii}}\right)^2}{e_i^T P^{-1} e_i} \\ &= r^2. \end{aligned}$$

Hence (8) holds. The proof is completed. ∎

Now suppose K is given, we consider the problem to choose P such that $\mathcal{D}_0$ is the largest. The problem is to choose P and r satisfying (7) such that $\mathcal{D}_0$ in (8) is the largest.

For any scalar $\alpha > 0$, $\hat{P} = \alpha P$ implies $\hat{P}^{-1} = \frac{1}{\alpha} P^{-1}$ and $\sqrt{\left(\hat{P}^{-1}\right)_{ii}} = \frac{1}{\sqrt{\alpha}}\sqrt{(P^{-1})_{ii}}$. Since (7) is equivalent to

$$\alpha P\left(A_{l_A} + B_{l_B}K\right) + \left(A_{l_A} + B_{l_B}K\right)^T \alpha P + \sqrt{\alpha} r \sum_{i=1}^{n} x_{0i}^j \frac{1}{\sqrt{\alpha}} \sqrt{(P^{-1})_{ii}} \left(\alpha P N_{il_{N_i}} K + K^T N_{il_{N_i}}^T \alpha P\right) < 0$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B$, $l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, and $j = 1, \ldots, 2^n$, or equivalently,

$$\hat{P}\left(A_{l_A} + B_{l_B}K\right) + \left(A_{l_A} + B_{l_B}K\right)^T \hat{P} + \sqrt{\alpha} r \sum_{i=1}^{n} x_{0i}^j \sqrt{\left(\hat{P}^{-1}\right)_{ii}} \left(\hat{P} N_{il_{N_i}} K + K^T N_{il_{N_i}}^T \hat{P}\right) < 0,$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B$, $l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, and $j = 1, \ldots, 2^n$. Hence $\hat{P} = \alpha P$ and $\hat{r} = \sqrt{\alpha} r$ also satisfy (7). Notice that $\mathcal{D}_0$ is kept unchanged when substituting P and r by $\hat{P}$ and $\hat{r}$, respectively. Without loss of generality, we can suppose $P \le I$. In order to obtain a larger $\mathcal{D}_0$, we can maximize r (maximize the largest ball contained in $\mathcal{D}_0$). The problem becomes

$$\max r \text{ subject to (7) and}$$
$$P \le I$$

with variables r and P.

Denote $\lambda = \frac{1}{r}$, then we have the following theorem.

Theorem 1 *Suppose $P \in \mathbb{R}^{n \times n} > 0$ and $\lambda \in \mathbb{R} > 0$ are the solutions of the following optimization problem*

$$\begin{gathered} \min \lambda \quad \textit{subject to} \\ P \le I, \\ \sum_{i=1}^{n} x_{0i}^j \sqrt{(P^{-1})_{ii}} \left(P N_{il_{N_i}} K + K^T N_{il_{N_i}}^T P\right) \\ < \lambda\left[-P\left(A_{l_A} + B_{l_B}K\right) - \left(A_{l_A} + B_{l_B}K\right)^T P\right] \end{gathered} \qquad (12)$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B, l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, and $j = 1, \ldots, 2^n$, then an estimate of the closed-loop stability region is

$$\mathcal{D}_0 = \left\{x \in \mathbb{R}^n : x^T P x < \frac{1}{\lambda^2}\right\}, \qquad (13)$$

which contains the largest ball $\left\{x \in \mathbb{R}^n : x^T x < \frac{1}{\lambda^2}\right\}$.

Remark 1 *Depending on applications, we can also consider maximizing the volume of $\mathcal{D}_0$ instead of maximizing the largest ball contained in $\mathcal{D}_0$ and a similar result can be obtained.*

Generally speaking, it is not easy to solve the above optimization problem directly. However, when $\sqrt{(P^{-1})_{ii}}$ $(i = 1, \ldots, n)$ are fixed, the problem becomes a generalized eigenvalue minimization problem which can be solved by the LMI toolbox. The following iterative LMI (ILMI) algorithm is provided.

Algorithm ESTIMATION:

Step 1 Choose a small tolerance scalar $\eta > 0$. Let $P_0 = \varepsilon_0 I \in \mathbb{R}^{n \times n}$ for a sufficiently small ε_0 $(1 > \varepsilon_0 > 0)$.

Step 2 Compute $\sqrt{\left(P_0^{-1}\right)_{ii}}$ $(i = 1, \ldots, n)$.

Step 3 Solve LMI problem

$$\begin{gathered} \min \lambda \quad \text{subject to} \\ P \le I \\ P \ge P_0 \\ \sum_{i=1}^{n} x_{0i}^j \sqrt{\left(P_0^{-1}\right)_{ii}} \left(P N_{il_{N_i}} K + K^T N_{il_{N_i}}^T P\right) \\ < \lambda\left[-P\left(A_{l_A} + B_{l_B}K\right) - \left(A_{l_A} + B_{l_B}K\right)^T P\right], \end{gathered}$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B, l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, and $j = 1, \ldots, 2^n$, and obtain P and λ.

Step 4 If $\frac{\|P - P_0\|}{\|P_0\|} < \eta$, go to Step 6, else go to Step 5.

Step 5 Let $P_0 = P$, go to Step 2.

Step 6 The estimated stability region is given by (13).

Remark 2 *The LMI problem in Step 3 is feasible provided that there exists a $P > 0$ such that*

$$P\left(A_{l_A} + B_{l_B}K\right) + \left(A_{l_A} + B_{l_B}K\right)^T P < 0$$

for all $l_A = 1, \ldots, n_A$ and $l_B = 1, \ldots, n_B$ (P_0 can be chosen sufficiently small such that $P \ge P_0$). Since $P \ge P_0$ implies $\sqrt{\left(P_0^{-1}\right)_{ii}} \ge \sqrt{(P^{-1})_{ii}}$, the P and λ obtained in Step 3 satisfy (12). Moreover, once the LMI problem in Step 3 is feasible, it is also feasible for the next iterate defined in Step 5 because at least the new P_0 itself is a solution. Furthermore, the positive definite matrix sequences $\{P\}$ with $P \le I$ obtained in Algorithm ESTIMATION is non-decreasing (in the sense of positive semi-definiteness), the positive sequence $\{\lambda\}$ obtained in Algorithm ESTIMATION is non-increasing, thus $\{P\}$ and $\{\lambda\}$ will converge and the algorithm will converge.

4. Controller Design

Now we consider the controller design problems. Three design specifications will be considered: closed-loop system stability region, feedback gain and damping of the state variables. Certainly, we want to design a controller to obtain a larger stability region, a better damping of the state variables, and a smaller feedback gain. However, a design should compromise the above three specifications. In this section, we will consider the problems of optimizing one specification when the other two specifications are given.

First consider the problem of *maximizing the stability region*. Suppose a linear feedback controller $u = Kx$ should be designed such that

(i) for all $l_A = 1, \ldots, n_A$ and $l_B = 1, \ldots, n_B$, the real parts of the eigenvalues of $A_{l_A} + B_{l_B}K$ is less than $-\delta_0$ for a given $\delta_0 > 0$, that is, $\sigma(A_{l_A} + B_{l_B}K) \subset \mathbb{C}^{-\delta_0}$ where $\mathbb{C}^{-\delta_0} = \{s \in \mathbb{C} : \mathrm{Re}(s) < -\delta_0\}$ is a subset of the complex plane;

(ii) the feedback gain is constrained by $KK^T \leq k_0 I$ for a given $k_0 > 0$.

From Theorem 1, the problem of obtaining the largest stability region can be stated as

$$\min \lambda \quad \text{subject to}$$

$$\sum_{i=1}^{n} x_{0i}^{j} \sqrt{(P^{-1})_{ii}} \left(P N_{il_{N_i}} K + K^T N_{il_{N_i}}^T P \right) < \lambda \left[-P(A_{l_A} + B_{l_B}K) - (A_{l_A} + B_{l_B}K)^T P \right],$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B, l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, and $j = 1, \ldots, 2^n$,

$$P(A_{l_A} + B_{l_B}K) + (A_{l_A} + B_{l_B}K)^T P + 2\delta_0 P < 0, \quad (14)$$

for all $l_A = 1, \ldots, n_A$ and $l_B = 1, \ldots, n_B$, and

$$KK^T \leq k_0 I, \qquad P \leq I$$

where (14) implies that

$$\sigma(A_{l_A} + B_{l_B}K) \subset \mathbb{C}^{-\delta_0},$$

for all $l_A = 1, \ldots, n_A$ and $l_B = 1, \ldots, n_B$.

In the above problem, variables are K, P and λ. Denote $\tilde{P} = P^{-1}$, the problem is converted into

$$\min \lambda \quad \text{subject to}$$

$$\sum_{i=1}^{n} x_{0i}^{j} \sqrt{\left(\tilde{P}\right)_{ii}} \left(N_{il_{N_i}} K\tilde{P} + \tilde{P}K^T N_{il_{N_i}}^T \right) < \lambda \left[-(A_{l_A} + B_{l_B}K)\tilde{P} - \tilde{P}(A_{l_A} + B_{l_B}K)^T \right],$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B, l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, and $j = 1, \ldots, 2^n$,

$$(A_{l_A} + B_{l_B}K)\tilde{P} + \tilde{P}(A_{l_A} + B_{l_B}K)^T + 2\delta_0 \tilde{P} < 0,$$

for all $l_A = 1, \ldots, n_A$ and $l_B = 1, \ldots, n_B$,

$$KK^T \leq k_0 I, \qquad \tilde{P} \geq I.$$

Further denote $W = K\tilde{P}$, then $K = W\tilde{P}^{-1}$. Since $\tilde{P}^{-1} \leq I$, the constraint $KK^T \leq k_0 I$ can be substituted by $WW^T \leq k_0 I$. Hence the following theorem holds.

Theorem 2 *Suppose $\tilde{P} \in \mathbb{R}^{n\times n} > 0, W \in \mathbb{R}^{m \times n}$ and $\lambda \in \mathbb{R}$ are the solutions of the following optimization problem*

$$\min \lambda \quad \textit{subject to}$$

$$\sum_{i=1}^{n} x_{0i}^{j} \sqrt{\left(\tilde{P}\right)_{ii}} \left(N_{il_{N_i}} W + W^T N_{il_{N_i}}^T \right) < \lambda \left[-A_{l_A}\tilde{P} - \tilde{P}A_{l_A}^T - B_{l_B}W - W^T B_{l_B}^T \right],$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B, l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, and $j = 1, \ldots, 2^n$,

$$A_{l_A}\tilde{P} + \tilde{P}A_{l_A}^T + B_{l_B}W + W^T B_{l_B}^T + 2\delta_0\tilde{P} < 0, \quad (15)$$

for all $l_A = 1, \ldots, n_A$ and $l_B = 1, \ldots, n_B$,

$$\begin{bmatrix} k_0 I & W \\ W^T & I \end{bmatrix} \geq 0 \quad (16)$$

$$\tilde{P} \geq I \quad (17)$$

then the controller which can obtain the largest closed-loop system stability region is

$$u = Kx = W\tilde{P}^{-1}x. \quad (18)$$

Moreover, an estimate of the closed-loop system stability region is

$$\mathcal{D}_0 = \left\{ x \in \mathbb{R}^n : x^T \tilde{P}^{-1} x < \frac{1}{\lambda^2} \right\}, \quad (19)$$

which contains the ball $\left\{x \in \mathbb{R}^n : x^T x < \frac{1}{\lambda^2}\right\}$.

The optimization problem in Theorem 2 can be solved using the following algorithm, which is similar to Algorithm ESTIMATION.

Algorithm DESIGN:

Step 1 Choose a small tolerance scalar $\eta > 0$. Let $\tilde{P}_0 = \rho_0 I \in \mathbb{R}^{n \times n}$ for a sufficiently large $\rho_0 > 1$.

Step 2 Compute $\sqrt{\left(\tilde{P}_0\right)_{ii}}$ $(i = 1, \ldots, n)$.

Step 3 Solve LMI problem

$$\min \lambda \quad \text{subject to (15), (16), (17) and}$$

$$\sum_{i=1}^{n} x_{0i}^{j} \sqrt{\left(\tilde{P}_0\right)_{ii}} \left(N_{il_{N_i}} W + W^T N_{il_{N_i}}^T \right) < \lambda \left[-A_{l_A}\tilde{P} - \tilde{P}A_{l_A}^T - B_{l_B}W - W^T B_{l_B}^T \right],$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B, l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, and $j = 1, \ldots, 2^n$,

$$\tilde{P} \leq \tilde{P}_0$$

to obtain $\tilde{P}, W$ and λ.

Step 4 If $\frac{\|\tilde{P} - \tilde{P}_0\|}{\|\tilde{P}_0\|} < \eta$, go to Step 6, else go to Step 5.

Step 5 Let $\tilde{P}_0 = \tilde{P}$, go to Step 2.

Step 6 The controller and the estimated stability region are obtained by (18) and (19).

Similar to Theorem 2, when considering the problems of maximizing the damping rate of the state variables and minimizing the feedback gain, the following results hold.

Theorem 3 *Among all the controllers satisfying*

$$KK^T \le k_0 I$$

and such that the closed-loop stability region contains $\left\{x \in \mathbb{R}^n : x^T x < \frac{1}{\lambda_0^2}\right\}$, *the controller in (18) has the maximal damping rate of the state variables,*

$$\sigma\left(A_{l_A} + B_{l_B} K\right) \subset \mathbb{C}^{-\frac{1}{\tilde{\delta}}}$$

for all $l_A = 1, \ldots, n_A$ *and* $l_B = 1, \ldots, n_B$, *where* $\tilde{P} \in \mathbb{R}^{n\times n} > 0, W \in \mathbb{R}^{m\times n}$ *and* $\tilde{\delta} \in \mathbb{R} > 0$ *are the solutions of the following optimization problem*

$$\min \tilde{\delta} \quad \textit{subject to}$$

$$\begin{aligned} \sum_{i=1}^{n} x_{0i}^{j} \sqrt{\left(\tilde{P}\right)_{ii}} \left(N_{il_{N_i}} W + W^T N_{il_{N_i}}^T\right) \\ < \lambda_0 \left[-A_{l_A}\tilde{P} - \tilde{P}A_{l_A}^T - B_{l_B} W - W^T B_{l_B}^T\right], \end{aligned}$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B$, $l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, *and* $j = 1, \ldots, 2^n$,

$$2\tilde{P} < \tilde{\delta}\left(-A_{l_A}\tilde{P} - \tilde{P}A_{l_A}^T - B_{l_B} W - W^T B_{l_B}^T\right),$$

for all $l_A = 1, \ldots, n_A$ *and* $l_B = 1, \ldots, n_B$,

$$\begin{bmatrix} k_0 I & W \\ W^T & I \end{bmatrix} \ge 0, \qquad \tilde{P} \ge I.$$

Theorem 4 *Among all the controllers satisfying*

$$\sigma\left(A_{l_A} + B_{l_B} K\right) \subset \mathbb{C}^{-\delta_0}$$

for all $l_A = 1, \ldots, n_A$ *and* $l_B = 1, \ldots, n_B$, *and such that the closed-loop stability region contains*

$$\left\{x \in \mathbb{R}^n : x^T x < \frac{1}{\lambda_0^2}\right\},$$

the controller in (18) has the smallest feedback gain, $KK^T \le kI$, *where* $\tilde{P} \in \mathbb{R}^{n\times n} > 0, W \in \mathbb{R}^{m\times n}$ *and* $k \in \mathbb{R} > 0$ *are the solutions of the following optimization problem*

$$\min k \quad \textit{subject to}$$

$$\begin{aligned} \sum_{i=1}^{n} x_{0i}^{j} \sqrt{\left(\tilde{P}\right)_{ii}} \left(N_{il_{N_i}} W + W^T N_{il_{N_i}}^T\right) \\ < \lambda_0 \left[-A_{l_A}\tilde{P} - \tilde{P}A_{l_A}^T - B_{l_B} W - W^T B_{l_B}^T\right], \end{aligned}$$

for all $l_A = 1, \ldots, n_A$, $l_B = 1, \ldots, n_B$, $l_{N_i} = 1, \ldots, n_{N_i}$ $(i = 1, \ldots n)$, *and* $j = 1, \ldots, 2^n$,

$$A_{l_A}\tilde{P} + \tilde{P}A_{l_A}^T + B_{l_B} W + W^T B_{l_B}^T + 2\delta_0 \tilde{P} < 0,$$

for all $l_A = 1, \ldots, n_A$ *and* $l_B = 1, \ldots, n_B$,

$$\begin{bmatrix} kI & W \\ W^T & I \end{bmatrix} \ge 0, \qquad \tilde{P} \ge I.$$

Remark 3 *When* $\sqrt{\left(\tilde{P}\right)_{ii}}$ $(i = 1, \ldots, n)$ *are fixed, the optimization problems in Theorems 3 and 4 become a generalized eigenvalue minimization problem and a minimization of a linear objective under LMI constraints, respectively, which can be solved by the LMI toolbox in MATLAB. Hence, two ILMI algorithms similar to Algorithm DESIGN can be established to design a controller such that a better damping of state variables or a smaller feedback gain can be obtained.*

5. Examples

5.1 Example 1

Consider the example used in [3]. The system is expressed by (1) where

$$A(t) = A,\ B(t) = B,\ N_1(t) = N_1,\ N_2(t) = N_2$$

and

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, N_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, N_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

We first estimate the closed-loop stability region for the following three controllers given by [3]:

$K_1 = [-3.19, 0.41]$, $K_2 = [-4.35, 0.44]$, $K_3 = [-12.87, 0.47]$.

Using Algorithm ESTIMATION, we obtain the following stability region estimates:

$$\mathcal{D}_1 = \left\{x \in \mathbb{R}^n : x^T \begin{bmatrix} 0.9753 & 0.1382 \\ 0.1382 & 0.2249 \end{bmatrix} x < 0.0690\right\},$$

$$\mathcal{D}_2 = \left\{x \in \mathbb{R}^n : x^T \begin{bmatrix} 0.9809 & 0.1220 \\ 0.1220 & 0.2200 \end{bmatrix} x < 0.0627\right\},$$

and

$$\mathcal{D}_3 = \left\{x \in \mathbb{R}^n : x^T \begin{bmatrix} 0.9983 & 0.0364 \\ 0.0364 & 0.2074 \end{bmatrix} x < 0.0509\right\}$$

respectively. The regions are compared with those obtained in [3] in the three figures. It can be seen that our estimates are much better than those in [3].

5.2 Example 2

Consider an open-loop unstable system with two-dimensional control input (Example 11 in [10]). The system is expressed by (1) where

$$A(t) = \mathrm{Co}\{A_1, A_2\},\ B(t) = B,\ N_1(t) = N_1,\ N_2(t) = N_2$$

and

$$A_1 = \begin{bmatrix} \frac{1}{6} & 1 \\ 0 & \frac{1}{6} \end{bmatrix}, A_2 = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix},$$

$$B = \begin{bmatrix} -3 & 2 \\ -2 & -2 \end{bmatrix}, N_1 = \begin{bmatrix} 5 & 4 \\ 2 & 5 \end{bmatrix}, N_2 = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}.$$

We consider the controller design problems. For $\delta_0 = 1$ and $k_0 = 5$, using Algorithm DESIGN we obtain $\lambda = 1.0323$ and the controller

$$u = \tilde{K}_1 x = \begin{bmatrix} 1.0332 & 1.5965 \\ -1.3532 & -0.3955 \end{bmatrix} x,$$

the estimation of stability region is

$$\mathcal{D}_0 = \left\{x \in \mathbb{R}^n : x^T \begin{bmatrix} 0.9993 & -0.0004 \\ -0.0004 & 0.9998 \end{bmatrix} x < 0.9385\right\}.$$

For $\lambda_0 = 1$ and $k_0 = 5$, using Theorem 3 and an ILMI algorithm, we obtain $\delta = 0.9632$ and the controller

$$u = \tilde{K}_2 x = \begin{bmatrix} 1.0373 & 1.5850 \\ -1.3394 & -0.4013 \end{bmatrix} x.$$

For $\lambda_0 = 1$ and $\delta_0 = 1$, using Theorem 4 and an ILMI algorithm, we obtain $k = 5.2624$ and the controller

$$u = \tilde{K}_3 x = \begin{bmatrix} 1.0705 & 1.6310 \\ -1.3821 & -0.4126 \end{bmatrix} x.$$

The results are summarized in the following table.

controller	λ	$\frac{1}{\lambda}$	δ	k
$\tilde{K}_1$	1.0323	0.9687	1	5
$\tilde{K}_2$	1	1	0.9632	5
$\tilde{K}_3$	1	1	1	5.2624

6. Conclusion

This paper studied the stability region estimation and linear controller design problems for uncertain bilinear systems. Iterative linear matrix inequality (ILMI) algorithms were presented to estimate the stability region and design the controllers. Three closed-loop specifications, closed-loop system stability region, feedback gain and damping of the state variables were considered in the controller design problems. The controllers designed are such that one specification is enhanced while the other two specifications are constrained at a certain level. Examples show that our stability region estimation method is less conservative than that in [3], and our controller design methods are more effective.

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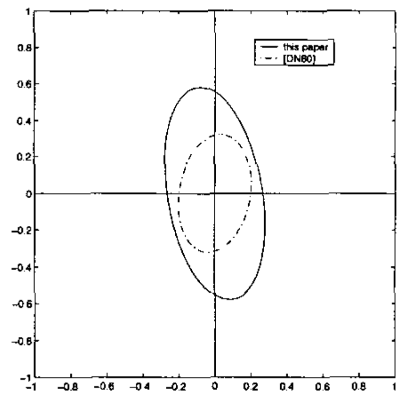
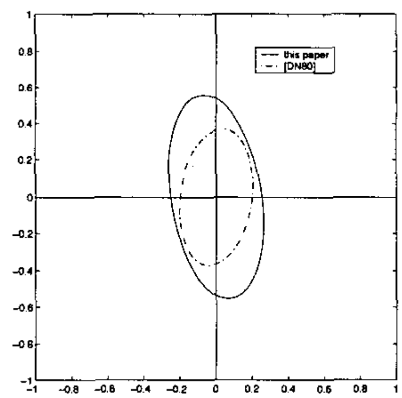
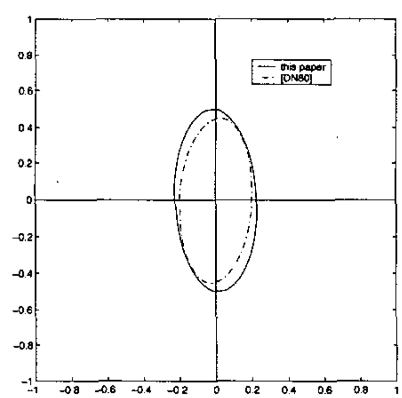


$K_1 = [-3.19, 0.41]$



$K_2 = [-4.35, 0.44]$



$K_3 = [-12.87, 0.47]$