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# An Optimization Approach for Worst-case Fault Detection Observer Design *

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**Abstract**

This paper deals with the fault detection problem for linear system with unknown inputs. The $H_\infty$ norm and $H_-$ index are employed to measure the robustness to unknown inputs and the fault sensitivity, respectively. Furthermore, by using the pole assignment approach, the fault detection problem is transformed to an unconstrained optimization problem. With the aid of the gradient-based optimization approach, an explicit formula for designing the desirable observer gain is derived. On the other hand, the fault sensitivity over a finite frequency range can also be solved by the proposed method, in which case no constraint is required on $D$ being of full column rank for a system $(A, B, C, D)$. Numerical simulation has demonstrated the effectiveness of the present methodology.

## 1. Introduction

The research and application of robust fault detection in automated processes has received considerable attention during last decades. One of the popular approaches is to maximize the sensitivity due to faults meanwhile minimizing the sensitivity due to unknown inputs. In this sense, Ding and Frank [2] presented a performance index expressed as a ratio of sensitivities of the residuals due to the unknown inputs and the faults respectively. The design goal is to then construct an observer for fault detection with the performance index being minimized. This or similar idea is commonly used amongst a number of subsequent papers for model-based robust fault detection systems [4, 12, 14].

To solve this problem, some researchers use the $H_\infty$ optimization technique [3, 10]. However, the results are not ideal since it is only a best-case solution when $H_\infty$ norm is used to measure the fault sensitivity. Therefore, a different norm/index is needed to measure the fault sensitivity instead.

Similar to the $H_\infty$ norm optimization technique, the $H_-$ method has also gained much attention recently, which aims to study the worst-case fault sensitivity performance of a fault detection observer. An $H_-$ "*norm*" was defined in [4, 9] as the minimum *nonzero* singular value of the transfer function matrix from the fault to the residual output at the specific frequency of $\omega = 0$. In [1, 11], this definition was further extended from the single frequency $\omega = 0$ to a number of finite frequency ranges. It should be pointed out that these sensitivity measures are not truly worst-case measures due to the exclusion of possible zero singular values of the transfer function matrix. A truly worst-case fault sensitivity measure, $H_-$ *index*, was proposed in [6, 7] to include the possible zero singular values of the transfer function matrix. Specifically, the $H_-$ *index* was defined as the minimum singular value of the transfer function matrix over a given frequency range. Note that the constraint of "*non-zero singular value*" is absent here, which (together with the absence of the triangle inequality $\|a+b\|_- \le \|a\|_- + \|b\|_-$) makes the $H_-$ *index* no longer a norm (hence the term *index*). The frequency range can be either infinite (i.e. the entire frequency spectrum) or finite frequency intervals. Moreover, necessary and sufficient conditions in terms of LMIs have been obtained for the proposed $H_-$ *index* [7, 13]. However, only iterative LMI approach can be used to solve fault detection problem when robustness is also concerned. In this case, the solution may not be ideal since the advantage of the LMI approach is not fully utilized.

Following the above idea, a method for designing a fault detection observer is proposed in this paper. It is formulated as an optimization problem where $H_\infty$ norm is used to describe robustness and $H_-$ *index* to measure the fault sensitivity. Moreover, the observer poles are given as constraints. A gradient-based optimization approach is facilitated by using the explicit gradient expressions derived. Moreover, we also con-

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sider the fault sensitivity over a finite frequency range in which case the condition on the rank of $D$ is no longer required for a system $(A, B, C, D)$. Numerical simulation is used to illustrate the effectiveness of the results.

Throughout this paper, $\|\cdot\|_\infty$ is used to denote the $H_\infty$ norm . $\bar{\sigma}(\cdot)$ denotes the maximum singular value of a matrix while $\underline{\sigma}(\cdot)$ denotes the minimum singular value of a matrix. All matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions.

## 2. Problem Formulation

Consider the following linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_w w(t) + B_f f(t) \\ y(t) &= Cx(t) + D_w w(t) + D_f f(t) \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^m$ is the unknown input vector including modelling error, uncertain disturbance, process and measurement noises, $y(t) \in \mathbb{R}^r$ is the measurement vector, and $f(t) \in \mathbb{R}^p$ is the fault vector. Here, $A$, $B_w$, $B_f$, $C$, $D_w$ and $D_f$ are known constant matrices with appropriate dimensions. Moreover, $w(t)$ and $f(t)$ are assumed to be vector norm bounded and the pair $(C, A)$ is observable.

The state observer under consideration is of the form

$$\begin{aligned} \dot{\overline{x}}(t) &= A_L \overline{x}(t) + Ly(t) \\ \overline{y}(t) &= C\overline{x}(t) \end{aligned} \quad (2)$$

where $A_L = A - LC$ and $L$ is the observer gain matrix to be designed for achieving design requirements. Define the error state

$$e(t) = x(t) - \overline{x}(t)$$

then it follows from (1) and (2) that

$$\dot{e}(t) = A_L e(t) + (B_w - LD_w)w(t) + (B_f - LD_f)f(t) \quad (3)$$

The residual vector $r(t)$ is defined as

$$\begin{aligned} r(t) &= y(t) - \overline{y}(t) \\ &= Ce(t) + D_w w(t) + D_f f(t) \end{aligned}$$

The disturbance transfer function $H_{rw}(s)$ from $w(t)$ to $r(t)$ and the fault transfer function $H_{rf}(s)$ from $f(t)$ to $r(t)$ are obtained, respectively, as

$$H_{rw}(s) = C(sI - A_L)^{-1}(B_w - LD_w) + D_w \quad (4)$$

and

$$H_{rf}(s) = C(sI - A_L)^{-1}(B_f - LD_f) + D_f \quad (5)$$

For effective fault detection, the effect on the residual $r(t)$ due to unknown inputs $w(t)$ should be small while that due to faults $f(t)$ should be large. Obviously, the $H_\infty$ optimization techniques can be used to handle this disturbance attenuation problem. In the following, we introduce a notion to measure the effect due to faults.

**Definition 1** [7, 6]The $H_-$ index of a transfer function $G(s)$ over the frequency range $[0, \bar{\omega})$ is defined as

$$\|G(s)\|_-^{[0,\bar{\omega}]} = \inf_{\omega \in [0,\bar{\omega})} \underline{\sigma}[G(j\omega)] \quad (6)$$

where $\underline{\sigma}$ denotes the minimum singular value.

**Remark 1** *To indicate the dependency on the frequency range $[0, \bar{\omega})$, we write the $H_-$ index of $G(s)$ as $\|G(s)\|_-^{[0,\bar{\omega}]}$. However, when the frequency range is clear from the context, we simply write $\|G(s)\|_-$.*

Thus, the $H_-$ index of the fault transfer function $H_{rf}(s)$ can be used to describe a measurement of the worst case fault sensitivity. The ratio of sensitivities

$$\frac{\|H_{rw}(s)\|_\infty}{\|H_{rf}(s)\|_-^{[0,\bar{\omega}]}}$$

thus gives a 'noise-signal' measure in a robust fault detection context. Clearly, $\|H_{rw}(s)\|_\infty$ should be kept small so as to desensitize the influence of unknown inputs on the residual vector while $\|H_{rf}(s)\|_-^{[0,\bar{\omega}]}$ should be made large to enhance the sensitivity due to faults. In general, there is a trade-off between these two sensitivities.

In summary, based on the above motivations, we study the design problem of a fault detection observer for system (1) as follows:

**FDODP (Fault Detection Observer Design Problem):** For system (1) with the observer (2) and a spectrum of $[0, \bar{\omega})$, $0 \le \bar{\omega} \le \infty$, determine an observer gain matrix $L$ such that

P(i) The error system (3) is asymptotically stable.

P(ii) The fault detection 'noise-signal' ratio

$$\frac{\|H_{rw}(s)\|_\infty}{\|H_{rf}(s)\|_-^{[0,\bar{\omega}]}}$$

is minimized.

In the following, we use the pole assignment approach to transform problem **FDODP** to a minimization problem.

For the observer system matrix $A_L$, the observer gain $L$ can be chosen such that all eigenvalues are in the left half $s$-plane and distinct and $\text{spec}(A_L) \cap \text{spec}(A) = \emptyset$. The reason that the eigenvalues are chosen to be distinct is due to an eigenvalue sensitivity consideration (less susceptible to perturbation). This is always possible since $(C, A)$ is observable. Then there exists a real invertible $V$ such that

$$VA_L V^{-1} = \Lambda \quad (7)$$

where $\Lambda$ is a real pseudo-diagonal matrix with $\text{spec}(A_L) = \text{spec}(\Lambda)$. Specifically, we have

$$\Lambda = \text{diag}\left(\begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_{n'} & \beta_{n'} \\ -\beta_{n'} & \alpha_{n'} \end{pmatrix}, \gamma_1, \ldots, \gamma_{n-2n'}\right)$$

Note that the eigenvalues of $\Lambda$ are the desired observer eigenvalues, $\alpha_i \pm j\beta_i$, $i = 1, \ldots, n'$, and $\gamma_k$, $k = 1, \ldots, (n - 2n')$. Although $V$ is not an eigenvector matrix, there exists a unitary $U$ such that $VU$ is an eigenvector matrix of $A_L$. Nevertheless, $V$ defined via (7) is non-unique. By writing (7) as

$$VA - \Lambda V = MC, \quad L = V^{-1}M \qquad (8)$$

then for each $M \in \mathcal{M}$, a unique $V$ is obtained since $\text{spec}(\Lambda) \cap \text{spec}(A) = \emptyset$, where

$$\mathcal{M} := \{M \in \mathbb{R}^{n \times r} \mid V \text{ is invertible and satisfies } VA - \Lambda V = MC\}$$

and the set $\mathcal{M}$ is open and dense in $\mathbb{R}^{n \times r}$.

Obviously, condition P(ii) in problem **FDODP** can be expressed as

$$\min_L \frac{\|H_{rw}(s,L)\|_\infty}{\|H_{rf}(s,L)\|_-^{[0,\bar\omega]}}$$

For a given fixed set of desired observer poles characterized by $\Lambda$ and condition (8), problem **FDODP** can be re-formulated as

$$\inf_{M \in \mathcal{M}} \frac{\|H_{rw}(s,M)\|_\infty}{\|H_{rf}(s,M)\|_-^{[0,\bar\omega]}} \qquad \text{for } 0 \le \bar\omega \le \infty \qquad (9)$$

## 3. Gradient-based Optimization: Infinite Frequency Case

In this section, the problem **FDODP** will be considered over the full frequency spectrum, i.e. $\bar\omega = \infty$. Note that when $D_f$ does not satisfy the full column rank condition or even equals to zero, $\|H_{rf}(s)\|_-^{[0,\bar\omega]}$ is always zero for $\bar\omega = \infty$. Hereby the ratio $\frac{\|H_{rw}(s)\|_\infty}{\|H_{rf}(s)\|_-^{[0,\infty]}}$ does not make sense. Therefore, we assume $D_f$ is of full column rank in this section.

The minimization problem in (9) can be rewritten as

$$\inf_{M \in \mathcal{M}} \mathcal{J} \qquad (10)$$

with

$$\mathcal{J} := \frac{\|C(sI - A_L)^{-1}(B_w - LD_w) + D_w\|_\infty}{\|C(sI - A_L)^{-1}(B_f - LD_f) + D_f\|_-} \qquad (11)$$

To facility the solution of the minimization problem (10), we introduce the following lemma.

**Lemma 1** [5] *For a given $M_0$, if*

1. $\|H_{rw}(s,M_0)\|_\infty = \sup_{\omega \in \mathbb{R}^+} \bar\sigma(H_{rw}(j\omega, M_0))$ at $\omega_0 < \infty$ and $\bar\sigma(H_{rw}(j\omega_0, M_0)) > \bar\sigma(H_{rw}(j\omega, M_0))$, $\forall \omega \neq \omega_0$;
2. $\sigma_0 = \bar\sigma(H_{rw}(j\omega_0, M_0))$ is a distinct singular value of $H_{rw}(j\omega_0, M_0)$,

*Then,*

$$\left.\frac{\partial \|H_{rw}(s,M)\|_\infty}{\partial M}\right|_{M=M_0} = \left.\frac{\partial \bar\sigma(H_{rw}(j\omega,M))}{\partial M}\right|_{\substack{\omega=\omega_0 \\ M=M_0}}$$

Similar to Lemma 1, we have the following result.

**Lemma 2** *For a given $M_0$, if*

1. $\|H_{rf}(s,M_0)\|_- = \inf_{\omega \in \mathbb{R}^+} \underline{\sigma}(H_{rf}(j\omega, M_0)) > 0$ at $\omega_0 < \infty$ and $\underline{\sigma}(H_{rf}(j\omega_0, M_0)) < \underline{\sigma}(H_{rf}(j\omega, M_0))$, $\forall \omega \neq \omega_0$;
2. $\sigma_0 = \underline{\sigma}(H_{rf}(j\omega_0, M_0))$ is a distinct singular value of $H_{rf}(j\omega_0, M_0)$,

*Then,*

$$\left.\frac{\partial \|H_{rf}(s,M)\|_-}{\partial M}\right|_{M=M_0} = \left.\frac{\partial \underline{\sigma}(H_{rf}(j\omega,M))}{\partial M}\right|_{\substack{\omega=\omega_0 \\ M=M_0}}$$

Based on Lemma 2, (10) corresponds to an unconstrained minimization problem with (11) differentiable almost everywhere in the $\delta$ neighborhood of $M_0$. Thus, a gradient-based optimization procedure can be applied. The gradient of $\mathcal{J}$ with respect to $M$ is then summarized in the following proposition with proof omitted.

**Proposition 1** *Suppose the maximum singular value of $C(j\omega I - A_L)^{-1}(B_w - LD_w) + D_w$ and the minimum singular value of $C(j\omega I - A_L)^{-1}(B_f - LD_f) + D_f$ are distinct, where $D_f$ is assumed to be of full column rank. If*

$$\begin{aligned} & (C(j\omega I - A_L)^{-1}(B_w - LD_w) + D_w)v_1 \\ = \; & \bar\sigma(C(j\omega I - A_L)^{-1}(B_w - LD_w) + D_w)u_1 \\ & (C(j\omega I - A_L)^{-1}(B_f - LD_f) + D_f)v_2 \\ = \; & \underline{\sigma}(C(j\omega I - A_L)^{-1}(B_f - LD_f) + D_f)u_2 \end{aligned}$$

*where $(v_1, u_1)$, $(v_2, u_2)$ are the corresponding singular vector pairs (unit norm), then*

$$\begin{aligned} & \frac{\partial \mathcal{J}}{\partial M} \\ = \; & \frac{\partial \left(\frac{\|C(sI-A_L)^{-1}(B_w - LD_w) + D_w\|_\infty}{\|C(sI-A_L)^{-1}(B_f - LD_f) + D_f\|_-}\right)}{\partial M} \\ = \; & \frac{1}{\|C(sI - A_L)^{-1}(B_f - LD_f) + D_f\|_-}\{\text{Re}[CX_1 \\ & -C(j\omega_E I - A_L)^{-1}(B_w - LD_w) \\ & *v_1 u_1^T C(j\omega_E I - A_L)^{-1}V^{-1} \\ & -D_w v_1 u_1^T C(j\omega_E I - A_L)^{-1}V^{-1}]^T\} \\ & -\frac{\|C(sI - A_L)^{-1}(B_w - LD_w) + D_w\|_\infty}{\|C(sI - A_L)^{-1}(B_f - LD_f) + D_f\|_-^2}\{\text{Re}[CX_2 \\ & -C(j\omega_F I - A_L)^{-1}(B_f - LD_f) \\ & *v_2 u_2^T C(j\omega_F I - A_L)^{-1}V^{-1} \\ & -D_f v_2 u_2^T C(j\omega_F I - A_L)^{-1}V^{-1}]^T\} \end{aligned} \qquad (12)$$

*where*

$$\begin{aligned}
&\|C(sI-A_L)^{-1}(B_w-LD_w)+D_w\|_\infty \\
=\;& \bar{\sigma}(C(j\omega_E I-A_L)^{-1}(B_w-LD_w)+D_w), \\
&\|C(sI-A_L)^{-1}(B_f-LD_f)+D_f\|_- \\
=\;& \underline{\sigma}(C(j\omega_F I-A_L)^{-1}(B_f-LD_f)+D_f), \\
&AX_1-X_1\Lambda \\
=\;& LC(j\omega_E I-A_L)^{-1}(B_w-LD_w)v_1u_1^T C(j\omega_E I-A_L)^{-1} \\
&*V^{-1}+LD_w v_1 u_1^T C(j\omega_E I-A_L)^{-1}V^{-1}, \\
&AX_2-X_2\Lambda \\
=\;& LC(j\omega_F I-A_L)^{-1}(B_f-LD_f)v_2u_2^T C(j\omega_F I-A_L)^{-1} \\
&*V^{-1}+LD_f v_2 u_2^T C(j\omega_F I-A_L)^{-1}V^{-1}
\end{aligned}$$

Now, we summarize the process of obtaining the observer gain in the following schematic algorithm.

**Algorithm FDODP:** Given $A$, $B_w$, $B_f$, $C$, $D_w$, $D_f$ and $\Lambda$.

S1 Select an initial guess $M_0 \in \mathcal{M}$ and solve equation (8), the initial observer gain is given by $L_0 = V^{-1}M_0$. Then determine $\omega_{E_0}$ and $\omega_{F_0}$ such that

$$\begin{aligned}
&\|C(sI-A+L_0C)^{-1}(B_w-LD_w)+D_w\|_\infty \\
=\;& \bar{\sigma}(C(j\omega_{E_0} I-A+L_0C)^{-1}(B_w-LD_w)+D_w) \\
&\|C(sI-A+L_0C)^{-1}(B_f-LD_f)+D_f\|_- \\
=\;& \underline{\sigma}(C(j\omega_{F_0} I-A+L_0C)^{-1}(B_f-LD_f)+D_f)
\end{aligned}$$

S2 Solve minimization problem (10) based on the objective function (11) and its gradient function (12).

S3 Let $M_{opt}$ be the optimal solution obtained in S2. Solve equation (8), the required observer gain is given by $L_{opt} = V^{-1}M_{opt}$.

## 4. Finite Frequency Case

In the previous sections, $\|H_{rf}(s)\|_-^{[0,\bar{\omega}]}$ is considered over the full frequency spectrum, i.e. $\bar{\omega} = \infty$. However, in real applications, it would be preferred to consider the fault sensitivity within the lower frequency range including DC ($\omega = 0$), i.e. $\bar{\omega}$ is a finite number. Moreover, $\|H_{rf}(s)\|_-^{[0,+\infty]}$ is always zero when $D_f$ is not of full column rank. In this case, the result proposed in previous section is no longer applicable. Unfortunately, such full column rank constraint on $D_f$ is often not satisfied in practice. Therefore, it is necessary to consider the case of $\|H_{rf}(s)\|_-^{[0,\bar{\omega}]}$ over a finite frequency range in which case the constraint on $D_f$ can be avoided. In this section, the problem is considered over a finite frequency spectrum, i.e. $\bar{\omega}$ is a finite number and hereby $D_f$ is not assumed to be of full column rank.

In summary, for the finite frequency case, problem FDODP will be replaced by

$$\inf_{M\in\mathcal{M}} \frac{\|H_{rw}(s,M)\|_\infty}{\|H_{rf}(s,M)\|_-^{[0,\bar{\omega}]}} \qquad \text{for } 0 \le \bar{\omega} < \infty \qquad (13)$$

At first, we introduce the following lemma which is a variant of [11, Lemma 4].

**Lemma 3** [13] *Given $W(s)$ and $M(s)$ such that*

$$\sup_{\omega\in[0,\bar{\omega})} \bar{\sigma}[W(j\omega)] = \delta \;, \quad \|W(s)+M(s)\|_-^{[0,+\infty]} > \alpha \qquad (14)$$

*then*

$$\|M(s)\|_-^{[0,\bar{\omega}]} > \alpha - \delta$$

In this case, although the method proposed in previous sections cannot be directly applied to solve problem (13), we can solve the following problem instead:

$$\inf_{M\in\mathcal{M}} \frac{\|H_{rw}(s)\|_\infty}{\|W(s)+H_{rf}(s)\|_-^{[0,+\infty]}} \qquad (15)$$

where $W(s)$ is a given weighting transfer function such that $\|W(s)+H_{rf}(s)\|_-^{[0,+\infty]}$is nonzero.

**Remark 2** *Note that problem (15) is not equivalent to problem (13). An optimal solution $L$ obtained from problem (15) can make the ratio $\frac{\|H_{rw}(s)\|_\infty}{\|H_{rf}(s)\|_-^{[0,\bar{\omega}]}}$ small but not the smallest one.*

Suppose the state space realization of $W(s)$ is given by $(A_r, B_r, C_r, D_r)$. A realization of $W(s)+H_{rf}(s)$ is

$$\left(\begin{array}{cc|c} A_r & 0 & B_r \\ 0 & A-LC & B_f-LD_f \\ \hline C_r & C & D_r+D_f \end{array}\right)$$

Correspondingly, a similar proposition is given as follows without proof.

**Proposition 2** *Suppose the maximum singular value of $C(j\omega I-A_L)^{-1}(B_w-LD_w)+D_w$ and the minimum singular value of $W(j\omega)+C(j\omega I-A_L)^{-1}(B_f-LD_f)+D_f$ are distinct, where $W(s)$ is a given weighting transfer function such that $\|W(s)+H_{rf}(s)\|_-$ is nonzero. If*

$$\begin{aligned}
&(C(j\omega I-A_L)^{-1}(B_w-LD_w)+D_w)v_1 \\
=\;& \bar{\sigma}(C(j\omega I-A_L)^{-1}(B_w-LD_w)+D_w)u_1 \\
&(W(j\omega)+C(j\omega I-A_L)^{-1}(B_f-LD_f)+D_f)v_2 \\
=\;& \underline{\sigma}(W(j\omega)+C(j\omega I-A_L)^{-1}(B_f-LD_f)+D_f)u_2
\end{aligned}$$

*where $(v_1,u_1)$, $(v_2,u_2)$ are the corresponding singular vector pairs (unit norm), then*

$$\begin{aligned}
&\frac{\partial \mathcal{J}}{\partial M} \\
=\;& \frac{\partial\left(\frac{\|C(sI-A_L)^{-1}(B_w-LD_w)+D_w\|_\infty}{\|W(s)+C(sI-A_L)_f^{-1}(B_f-LD_f)+D_f\|_-}\right)}{\partial M} \\
=\;& \frac{1}{\|W(s)+C(sI-A_L)^{-1}(B_f-LD_f)+D_f\|_-}\{\mathrm{Re}[CX_1
\end{aligned}$$

$$
\begin{aligned}
& -C(j\omega_E I - A_L)^{-1}(B_w - LD_w) \\
& * v_1 u_1^T C(j\omega_E I - A_L)^{-1} V^{-1} \\
& -D_w v_1 u_1^T C(j\omega_E I - A_L)^{-1} V^{-1}]^T\} \\
& -\frac{\|C(sI - A_L)^{-1}(B_w - LD_w) + D_w\|_\infty}{\|W(s) + C(sI - A_L)^{-1}(B_f - LD_f) + D_f\|_-^2}\{\mathrm{Re}[CX_2 \\
& -C(j\omega_F I - A_L)^{-1}(B_f - LD_f) \\
& * v_2 u_2^T C(j\omega_F I - A_L)^{-1} V^{-1} \\
& -D_f v_2 u_2^T C(j\omega_F I - A_L)^{-1} V^{-1}]^T\}
\end{aligned}
$$

*where*

$$
\begin{aligned}
& \|C(sI - A_L)^{-1}(B_w - LD_w) + D_w\|_\infty \\
= \; & \bar{\sigma}(C(j\omega_E I - A_L)^{-1}(B_w - LD_w) + D_w), \\
& \|W(s) + C(sI - A_L)^{-1}(B_f - LD_f) + D_f\|_- \\
= \; & \underline{\sigma}(W(j\omega_F) + C(j\omega_F I - A_L)^{-1}(B_f - LD_f) + D_f), \\
& AX_1 - X_1\Lambda \\
= \; & LC(j\omega_E I - A_L)^{-1}(B_w - LD_w)v_1 u_1^T C(j\omega_E I - A_L)^{-1} \\
& * V^{-1} + LD_w v_1 u_1^T C(j\omega_E I - A_L)^{-1} V^{-1}, \\
& AX_2 - X_2\Lambda \\
= \; & LC(j\omega_F I - A_L)^{-1}(B_f - LD_f)v_2 u_2^T C(j\omega_F I - A_L)^{-1} \\
& * V^{-1} + LD_f v_2 u_2^T C(j\omega_F I - A_L)^{-1} V^{-1}
\end{aligned}
$$

**Remark 3** *It is obvious that $W(s)$ plays an important role in this case. Generally speaking, a better $W(s)$ should be chosen to satisfy: (i) $\sup_{\omega\in[0,\bar{\omega})} \bar{\sigma}[W(j\omega)]$ is small; (ii) $\inf_{\omega\in[\omega_2,+\infty)} \underline{\sigma}[W(j\omega)]$ is large for $\omega_2 > \bar{\omega}$; (iii) The transition frequency range $[\bar{\omega}, \omega_2]$ is narrow.*

## 5. Numerical Simulation

Consider the linearized longitudinal dynamics of a VTOL aircraft as proposed by Tripathi [8]. The continuous-time state-space description is

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + B_w w(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

where the states $x(t)$ are the horizontal velocity (knot), vertical velocity (knot), pitch rate (degree/s) and pitch angle (degree), respectively, and the actuator inputs $u(t)$ are the collective pitch control and the longitudinal pitch control respectively. In [8], the system parameters are given as follows:

$$
A = \begin{bmatrix} -9.9477 & -0.7476 & 0.2632 & 5.0337 \\ 52.1659 & 2.7452 & 5.5532 & -24.4221 \\ 26.0922 & 2.6361 & -4.1975 & -19.2774 \\ 0 & 0 & 1 & 0 \end{bmatrix},
$$

$$
B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},
$$

$$
C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}
$$

For system (1), the rest of the parameters are assumed to be:

$$
B_f = B, \quad D_w = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.1 \\ 0.3 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

It is obvious that $\|H_{rf}(s)\|_-^{[0,+\infty]}$ is always zero since $D_f$ is not of full column rank. Thus, the problem will be investigated over a finite frequency range $[0, 0.1)$ instead of the whole frequency range.

The choice of the target poles of observer (different from those of the original system: {-6.8271, -1.0112+1.5146i, -1.0112-1.5146i, -2.5506}) should reflect the response speed requirement and a faster speed generally lead to a larger feedback gain. Thus, a compromise between them should be reached. In our example, the poles of the diagnostic observer are designed at $-1, -2, -3, -4$. That is,

$$
\Lambda = \mathrm{diag}(\; -1, \quad -2, \quad -3, \quad -4 \;)
$$

With the aid of the techniques proposed in Section 4, a frequency weighting transfer matrix $W(s)$ is selected as

$$
W(s) = \begin{bmatrix} \frac{50(s+0.1)}{s+100} & 0 \\ 0 & 0 \\ 0 & \frac{50(s+0.1)}{s+100} \\ 0 & 0 \end{bmatrix}
$$

then a state space realization $(A_r, B_r, C_r, D_r)$ of $W(s)$ is given by

$$
\begin{aligned}
A_r &= \mathrm{diag}(\; -100, \quad -100, \quad -100, \quad -100 \;), \\
C_r &= \mathrm{diag}(\; -70.6753, \quad 0, \quad -70.6753, \quad 0 \;),
\end{aligned}
$$

$$
B_r = \begin{bmatrix} 70.6753 & 0 \\ 0 & 0 \\ 0 & 70.6753 \\ 0 & 0 \end{bmatrix}, \quad D_r = \begin{bmatrix} 50 & 0 \\ 0 & 0 \\ 0 & 50 \\ 0 & 0 \end{bmatrix}
$$

Here $D_f + D_r$ is of full column rank, $\sup_{\omega\in[0,\bar{\omega})} \bar{\sigma}[W(j\omega)] = 0.0707$ with $\bar{\omega} = 0.1$ and $\inf_{\omega\in[\omega_2,+\infty)} \underline{\sigma}[W(j\omega)] = 1.0011$ with $\omega_2 = 2$.

Using the algorithm FDODP developed in this paper, the optimization is initiated with a random initial value of $M$. The numerical simulation was carried out using MATLAB 6.1 (Control Toolbox 5.1, Optimization Toolbox 2.1.1) and an optimal observer gain $L_{opt}$ is obtained. For comparison, an observer gain $L_{place}$ which gives the same spectrum is obtained from the command place.m (the command has also taken the sensitivities of the eigenvalues into account). We have,

$$
L_{opt} = \begin{bmatrix} 4.3021 & -10.0144 & -3.5587 & 4.8599 \\ 6.3561 & -1.6791 & -0.9140 & -2.4219 \\ -21.1044 & 47.6843 & 17.6497 & -22.7378 \\ 2.9567 & -6.7268 & -2.7124 & 3.4869 \end{bmatrix}
$$

$$L_{place} = \begin{bmatrix} -8.9477 & -5.7813 & -4.7705 & 5.0337 \\ 52.1659 & 29.1673 & 29.9753 & -24.4221 \\ 26.0922 & 21.9135 & 18.0799 & -19.2774 \\ 0 & -4 & -3 & 4 \end{bmatrix}$$

The system is simulated with unknown input $w(t) = \begin{bmatrix} \sin(2t)e^{-0.05t} & \cos(2t)e^{-0.05t} \end{bmatrix}^T$. For an actuator fault $f(t)$ such that $f(t) = \begin{bmatrix} 1.2 & 0.8 \end{bmatrix}^T$, $t \geq 6s$ and $f(t) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ elsewhere, Figure 1 shows the evolution of the residual $r(t)$ responses due to observer gain $L_{opt}$ and $L_{place}$ respectively. In the case of $L = L_{opt}$, despite the influence of unknown input $w(t)$, a threshold at $\pm 0.3$ can easily be imposed on the residual signals to indicate the occurrence of fault at the time $t = 6.1s$. On the other hand, no reasonable threshold can be imposed to distinguish the influence between faults and unknown input in the case of $L = L_{place}$. In other words, the robust fault detection sensitivity in the case $L = L_{place}$ is comparatively smaller than in the case $L = L_{opt}$.

## 6. Conclusion

In this paper, we deal with a worst-case fault detection observer design problem. It is formulated as an optimization problem with the observer poles as constraints where $H_\infty$ norm is used to describe robustness to unknown inputs and $H_-$ *index* is used to measure the fault sensitivity. The gradient-based optimization approach is facilitated by the explicit gradient expressions derived. Moreover, we also consider the fault sensitivity over finite frequency range in which case the condition on the rank of $D$ is no longer required for a system $(A, B, C, D)$. Numerical simulation performed on the fault detection observer design of a VTOL aircraft is given to demonstrate the effectiveness of the present methodology.

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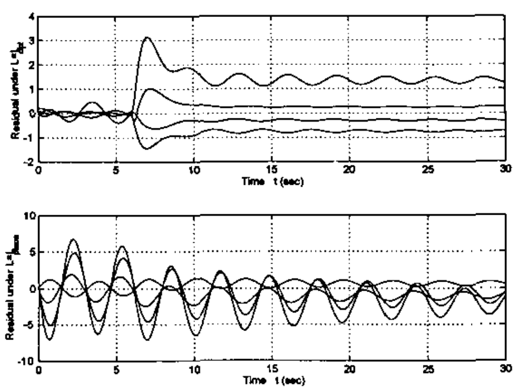

Figure 1. Comparison of two residual effects