Projective manifolds dominated by abelian varieties

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In [La], Lazarsfeld showed that if there exists a surjective morphism from the projective space $\mathbf{P}_n$ to a projective manifold X, then $X \cong \mathbf{P}_n$. A natural generalization of this would be the study of projective manifolds which can be the target of a surjective morphism from homogeneous spaces.

In [HM2], the authors studied the question for a rational homogeneous space S of Picard number 1. From the condition on the Picard number, a non-constant morphism $f : S \to X$ must be a finite morphism. The main result of [HM2] says that if $f : S \to X$ is a finite morphism with nonempty ramification onto a projective manifold, then $X \cong \mathbf{P}_n$.

Debarre studied the question for a simple abelian variety A ([De]). Since A is simple, a non-constant morphism $f : A \to X$ must be a finite morphism. His main result says that if $f : A \to X$ is a finite morphism with nonempty ramification from a simple abelian variety onto a projective manifold, then $X \cong \mathbf{P}_n$. He used the fact that curves in a simple abelian variety have ample normal bundles, making it possible to apply Mori's characterization of $\mathbf{P}_n$ ([Mo]) as in the argument in Lazarsfeld's proof for $\mathbf{P}_n \to X$.

In this article, we study the same question for an arbitrary abelian variety A. Note that if $f : A \to B$ is a surjective morphism with connected fibers onto a normal variety B, then vector fields of A descend to B as derivations of $f_*\mathcal{O}_A = \mathcal{O}_B$ and B must be a quotient abelian variety. By Stein factorization, any surjective morphism $f : A \to X$ to a projective manifold can be factored through a finite morphism on some quotient abelian variety of A. So to study target manifolds X we may restrict to the case of finite morphisms. For X of Picard number 1 we prove

Theorem 1 *Let $f : A \to X$ be a finite morphism with nonempty ramification from an abelian variety onto a projective manifold X of Picard number 1. Then, $X \cong \mathbf{P}_n$.*

When X is not assumed to be of Picard number 1, and A is not simple, then X may be more complicated than $\mathbf{P}_n$, e.g., X can be a bundle of projective spaces over some abelian variety. Our Main Theorem says that these are essentially the only possibilities.

Main Theorem *Let $f : A \to X$ be a finite morphism with nonempty ramification from an abelian variety onto a projective manifold X. Then X is a holomorphic bundle of projective spaces over a projective algebraic manifold Y of smaller dimension and there exists a finite morphism*

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from an abelian variety onto Y. Thus there exist a finite sequence of surjective morphisms between projective manifolds $\psi_i : X_{i-1} \to X_i, 1 \leq i \leq N$ with $X_0 = X$ such that each ψ_i is a holomorphic $\mathbf{P}_{k_i}$-bundle for some integer $k_i, 1 \leq i \leq N$ and X_N has a finite unramified cover by an abelian variety.

As is well-knwon, X_N in the above statement need not be an abelian variety (e.g. [Ue, p.208]).

As opposed to the case of simple abelian varieties, the fundamental difficulty of our problem arises from the fact that the normal sheaf of an algebraic curve on A need not be ample. (This difficulty arises even if X is of Picard number 1.) Deformation of minimal rational curves on X then leads to multi-valued distributions. We combine deformation theory of rational curves on X with topological considerations to handle multi-valuedness of such distributions. In addition to providing the key step to our solution of the Main Theorem, our treatment of multi-valued distributions furnishes an ingredient pertinent to the general theory of distributions defined by minimal rational curves, a systematic study of which just started in [HM1].

1 Varieties of minimal rational tangents on projective manifolds dominated by abelian varieties

In this section, we will prove some basic results about the deformation theory of rational curves on projective manifolds dominated by abelian varieties. To start with,

Proposition 1 *Let $f : A \to X$ be a finite morphism with nonempty ramification from an abelian variety onto a projective manifold X. Then X is uniruled.*

Proof. By the adjunction formula, $K_A = f^*K_X + R$ where R is the ramification divisor. By assumption, $R \neq \emptyset$. Choose a generic curve $C \subset A$ which meets R. Then $f(C)$ is a curve through a generic point on X, with $K_X \cdot f(C) < 0$. By [MM], this implies that X is uniruled. □

Let X be a uniruled projective manifold. Among components of the Hilbert scheme of rational curves on X whose members cover X, we choose one component $\mathcal{K}$ of minimal degree. $\mathcal{K}$ will be called a **minimal rational component**. A member of $\mathcal{K}$ will be called a $\mathcal{K}$-curve. Let $\mathcal{K}_x \subset \mathcal{K}$ be the subscheme consisting of $\mathcal{K}$-curves passing through a generic point $x \in X$. By the minimality of the degree, $\mathcal{K}_x$ is complete. Generic members of $\mathcal{K}_x$ are smooth at x, defining a rational map $\Phi_x : \mathcal{K}_x \to \mathbf{P}T_x(X)$ by associating the tangent vector at x to each curve smooth at x. The strict image of Φ_x will be denoted by $\mathcal{C}_x$, and called the **variety of minimal rational tangents**. The closure of the union of $\mathcal{C}_x$'s over generic points of X forms a subvariety $\mathcal{C} \subset \mathbf{P}T(X)$, called the **bundle of varieties of minimal rational tangetns**. $\mathcal{C}$ is irreducible, otherwise the collection of members of $\mathcal{K}$ tangent to a component of $\mathcal{C}$ will violate the irreducibility of $\mathcal{K}$. From Mori's bend-and-break trick the following is well-known ([Ko, IV.2.9]).

(*) For a generic $\mathcal{K}$-curve C, the normalization $\nu : \mathbf{P}_1 \to C$ is an immersion and $\nu^*T(X) = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$ for some nonnegative integer p which is equal to the dimension of $\mathcal{K}_x$ and $\mathcal{C}_x$ for a generic $x \in X$. It follows that for any positive-dimensional family of rational curves belonging to $\mathcal{K}_x$, the members of this family do not share a common point other than x.

We recall three lemmas about abelian varieties. Recall that when $S \subset A$ is a connected subvariety (not necessarily irreducible) of the abelian variety A, the smallest abelian subvariety of A containing all the differences $s - s'$, where $s, s' \in S$, is called the *toroidal hull* of S, and denoted by $< S >$. Note that for any $s, s' \in S$, S is contained in $s+ < S >= s'+ < S >$. For the proof of the following lemma, see [Za].

Lemma 1 ([Za], p.28, Lemma 3.2) *Let $S \subset A = \mathbf{C}^n/\Lambda$ be a connected subvariety whose tangent spaces to smooth points are contained in a vector subspace $\mathbf{C}^m \subset \mathbf{C}^n$. Then the toroidal hull of S has dimension $\leq m$.*

The next lemma is a simple consequence of the fact that we cannot have a positive-dimensional family of abelian subvarieties.

Lemma 2 *Given any irreducible family of subvarieties of an abelian variety, the toroidal hulls of generic members are identical.*

A direct consequence of these two lemmas is

Lemma 3 *Let $C_t \subset A$ be a p-dimensional irreducible family of curves on an abelian variety passing through a common point $a \in A$. If the union of C_t's covers a $(p+1)$-dimensional constructible subset in A and the subspace of $H^0(C_t, T^*(A))$ consisting of elements annihilating tangent vectors to C_t has dimension $\geq n-1-p$ for a generic member C_t, then the closure of the union of these curves is a translate of a $(p+1)$-dimensional abelian subvariety which is the toroidal hull of a generic C_t.*

Lemma 3 shows that the varieties of minimal rational tangents on our X are union of linear subspaces:

Proposition 2 *Let $f : A \to X$ be a finite morphism from an abelian variety onto a uniruled projective manifold. Let $\mathcal{K}$ be as above and let $\mathcal{C}_x$ be the variety of minimal rational tangents at a generic point $x \in X$, which has dimension p. Then there exist finitely many distinct $(p+1)$-dimensional abelian subvarieties $A_1, \ldots, A_m$ of A with the following property. Let $\mathcal{A}^i \subset T(A)$ be the distribution defined by the translates of A_i. Then for each generic $x \in X$,*

$$\mathcal{C}_x \;=\; df_{a_j}(\mathbf{P}\mathcal{A}^1 \cup \cdots \cup \mathbf{P}\mathcal{A}^m), \quad 1 \leq j \leq d$$

$$= df_{a_1}(\mathbf{P}\mathcal{A}^i) \cup \cdots \cup df_{a_d}(\mathbf{P}\mathcal{A}^i), \quad 1 \leq i \leq m$$

where $f^{-1}(x) = \{a_1, \ldots, a_d\}$. In particular, $\mathcal{C}_x$ is the union of m linear subspaces of dimension p, defining a multi-valued distribution on a Zariski dense open subset of X. A local integral submanifold of these distributions at $x \in X$ is given by the image of a translate of A_i. This local integral submanifold contains an open subset of the variety obtained by the union of members of an irreducible component of $\mathcal{K}_x$.

Proof. Choose a generic point $a \in A$ outside the ramification locus of f. Let

$$\mathcal{C}_x = \mathcal{C}_{x,1} \cup \cdots \cup \mathcal{C}_{x,m}$$

be the decomposition into irreducible components of the variety of minimal rational tangents at $x = f(a)$. Each component $\mathcal{C}_{x,i}$ is the strict image of an irreducible component $\mathcal{K}_{x,i}$ of $\mathcal{K}_x$ under $\Phi_x : \mathcal{K}_x \to \mathbf{P}T_x(X)$. For a rational curve $C_i \subset X$ corresponding to a generic point of $\mathcal{K}_{x,i}$, let $C_i' \subset A$ be an irreducible component of $f^{-1}(C_i)$ through a. Note that elements of $H^0(C_i, T^*(X))$ annihilates the tangent vectors to C_i and $h^0(C_i, T^*(X)) = n - 1 - p$ from (*). The pull-back of elements of $H^0(C_i, T^*(X))$ to $H^0(C_i', T^*(A))$ gives a subspace of dimension $\geq n - 1 - p$, annihilating tangent vectors of C_i'. By Lemma 3, the closure of the union of all such choices of C_i' is $a+ < C_i' >$. It follows that $df_a^{-1}(\mathcal{C}_{x,i}) = \mathbf{P}T_a(a+ < C_i' >)$. Putting $A_i =< C_i' >$, we get the first equality of the proposition. As for the second equality, the inclusion

$$df_{a_1}(\mathbf{P}\mathcal{A}^i) \cup \cdots \cup df_{a_d}(\mathbf{P}\mathcal{A}^i) \subset \mathcal{C}_x$$

follows from the first equality. Suppose this inclusion is not an equality for generic x. Then the closure of the union of $df_a(\mathbf{P}\mathcal{A}^i)$ for generic $a \in A$ form a subvariety of the bundle of varieties of minimal rational tangents $\mathcal{C}$, which has the same dimension as $\mathcal{C}$, contradicting the irreducibility of $\mathcal{C}$. The statement about the integral submanifold of the distribution follows from the way A_i is defined from Lemma 3. □

We have the following regularity of the integral subvarieties of this multi-valued distribution.

Proposition 3 *Let $a \in A$ be a generic point and A_i be as in Proposition 2. Let $Z = f(a+A_i)$ be the image of the translate of A_i. Then the normalization map $h : \hat{Z} \to Z$ is an immersion outside a set of codimension ≥ 2 on Z.*

Proof. Let $B \subset X$ be the branch divisor of $f : A \to X$. Certainly Z is smooth outside B. Let $D \subset \hat{Z}$ be a component of $h^{-1}(B \cap Z)$. We want to show that h is unramified at a generic point z of D.

From Proposition 2, Z is the closure of the union of members of an irreducible component $\mathcal{K}_{x,i}$ of $\mathcal{K}_x$ for a generic $x \in Z$. We know that $\mathcal{K}_{x,i}$ for a generic x is a complete family all members of which are irreducible. Let $\hat{\mathcal{K}}_{x,i}$ be the family obtained by pulling back $\mathcal{K}_{x,i}$ by h. $\hat{\mathcal{K}}_{x,i}$ is a complete family consisting of irreducible rational curves on $\hat{Z}$. A generic member C of $\hat{\mathcal{K}}_{x,i}$ lies on the smooth part of $\hat{Z}$, otherwise we have a 1-dimensional subfamily of curves passing through two distinct points, $h^{-1}(x)$ and a singular point of $\hat{Z}$, a contradiction to (*). Thus we can think of the tangent bundle $T(\hat{Z})$ in a neighborhood of a generic member C. Since $\hat{\mathcal{K}}_{x,i}$ gives deformations of C fixing the point $h^{-1}(x)$ covering the whole $\hat{Z}$, the tangent bundle $T(\hat{Z})$ restricted to a generic member C must be an ample vector bundle ([Ko, II.3.10.1]). Since C is of minimal degree among rational curves on Z through x, we have ([Ko, IV.2.9])

$$T(\hat{Z})|_C = \mathcal{O}(2) \oplus \mathcal{O}(1)^p.$$

For any point z of D, we can find a member of $\hat{\mathcal{K}}_{x,i}$ which passes through z, because members of $\hat{\mathcal{K}}_{x,i}$ cover Z. By the irreducibility of the family, all members of $\hat{\mathcal{K}}_{x,i}$ intersect D. It follows that for a generic point z of D, we can find a generic member C of $\hat{\mathcal{K}}_{x,i}$ passing through z. From the splitting type of $T(\hat{Z})|_C$ we have p-dimensional deformations C_s of C passing through z. The tangent vectors to these deformations cover a p-dimensional open subset in $\mathbf{P}T_z(\hat{Z})$.

Suppose h is ramified at z. Then $h(C_s)$ is a p-dimensional family of deformations of $h(C)$ whose tangent vectors at $h(z)$ can span only $(p-1)$-dimensional subset in $\mathbf{P}T_{h(z)}(X)$. This means that $h(C)$ has a deformation fixing the point $h(z)$ to second order and its Kodaira-Spencer class gives a section of the normal bundle of $h(C)$ vanishing to second order, a contradiction to the splitting type of $T(X)$ on $h(C)$ in (*). □

2 When the varieties of minimal rational tangents are nondegenerate

In this section, we will show that if $\mathcal{C}_x$ is nondegenerate in $\mathbf{P}T_x(X)$ then $X \cong \mathbf{P}_n$.

Proposition 4 *In the situation of Proposition 2, suppose $\mathcal{C}_x$ is nondegenerate in $\mathbf{P}T_x(X)$ for a generic $x \in X$, namely, it is not contained in any hyperplane of $\mathbf{P}T_x(X)$. Then $m = 1$ and $p = n-1$.*

Proof. First we claim that X is rationally connected. Otherwise, we have the maximal rational fibration of X over a positive-dimensional variety such that all rational curves through a very general point of X are contained in the fiber through the point ([Ko, IV.5.2]). This implies that $\mathcal{C}_x$ should be contained in the tangent space of the fiber through x, a contradiction to the nondegeneracy assumption. Since a rationally connected manifold is simply connected, we will prove $m = 1$ by constructing an unramified covering of degree m over $X - E$ for some subvariety E of codimension at least 2.

From Proposition 3, in an analytic neighborhood U of a generic point x of the branch divisor B,

$$U \cap (f(a+A_1) \cup \cdots \cup f(a+A_m)) \;=\; Z_1 \cup \cdots \cup Z_l$$

for some $l \geq m$, where a is any point of $f^{-1}(x)$ contained in the ramification divisor R and $Z_j, 1 \leq j \leq l$, are submanifolds of dimension $p+1$. We want to show that $T_x(Z_1) \cup \cdots \cup T_x(Z_l)$ has at least m distinct components in $T_x(X)$.

Among the m distinct abelian subvarieties $A_1, \ldots, A_m$, we may assume that $a+A_1, \ldots, a+A_r$ are contained in R while $a+A_{r+1}, \ldots, a+A_m$ are not contained in R, for some $0 \leq r \leq m$.

From the genericity of $x \in B$ and $a \in R$, we may assume that $f|_R$ is unramified at a and the kernel of df_a is a 1-dimensional complement to $T_a(R)$ in $T_a(A)$. The tangent spaces $T_a(a+A_1), \ldots, T_a(a+A_r)$ lie in $T_a(R)$. We claim that each of $T_a(a+A_{r+1}), \ldots, T_a(a+A_m)$ contains the kernel of df_a, i.e., $f|_{A_j+a}$ is ramified at a for all $r+1 \leq j \leq m$. As in the proof of Proposition 3, there exists a $\mathcal{K}$-curve C on $f(a+A_j)$ which passes through x and is transversal to B at x. Let C' be an irreducible component of $f^{-1}(C)$ lying on $a+A_j$. If C' is not tangent to the kernel of df_a, then the tangent of $f(C')$ would be contained in the image of df. This means that C is tangent to B, contradictory to our choice of C. It follows that the kernel of df at a is contained in $T_a(a+A_j)$.

Thus $T_a(R) \cap T_a(a+A_j), r+1 \leq j \leq m$ constitute a set of $m-r$ distinct subspaces of dimension p in $T_a(R)$. Since $f(a+A_j)$ is immersed at x and df_a sends $T_a(R)$ isomorphically to $T_x(B)$, we see that $T_x(f(a+A_j)), r+1 \leq j \leq m$ constitute a set of at least $m-r$ distinct subspaces of dimension $p+1$ in $T_x(X)$, which are not contained in $T_x(B)$. On the other hand, $T_x(f(a+A_i)), 1 \leq i \leq r$ constitute a set of r distinct subspaces of dimension $p+1$ in $T_x(B)$. It follows that $T_x(Z_1) \cup \cdots \cup T_x(Z_l)$ has at least m distinct components.

Let $\mathbf{Gr}T(X)$ be the Grassmann bundle of $(p+1)$-dimensional subspaces of $T(X)$. For a generic $y \in X$, $\mathcal{C}_y \subset \mathbf{P}T_y(X)$ consists of m linear subspaces of dimension p, defining m distinct points in $\mathbf{Gr}T_y(X)$. We denote this set of m distinct points by $\mathcal{S}_y$. The closure of the union of the $\mathcal{S}_y$'s gives a subvariety $\mathcal{S} \subset \mathbf{Gr}T(X)$. $\mathcal{S}$ is irreducible because $\mathcal{C}$ is irreducible. The natural projection $\mathcal{S} \to X$ is generically m-to-1. From the above discussion, $\mathcal{S}$ has at least m distinct points over $x \in B$. Thus it is m-to-1 over x, too. Applying the above argument to generic points of all components of the branch divisor of f, we see that $\mathcal{S} \to X$ is m-to-1 outside a set of codimension ≥ 2 in X. Thus $m=1$ from the simple connectedness of X. We conclude that $\mathcal{C}_x = \mathbf{P}T_x(X)$ and $p = n-1$ from the nondegeneracy of $\mathcal{C}_x$. □

Proposition 5 *Under the assumption of Proposition 4, X is isomorphic to $\mathbf{P}_n$.*

Proof. From Proposition 4, X is a uniruled projective manifold with a minimal rational component $\mathcal{K}$ having $p = n-1$. By [Mk] (2.4), if X is different from $\mathbf{P}_n$, then there exists a hypersurface $\mathcal{H}_x \subset \mathcal{K}_x$ for a generic point $x \in X$, so that a generic member of $\mathcal{H}_x$ is a rational curve C for which the pull-back of $T(X)$ to its normalization splits as $\mathcal{O}(2)^2 \oplus \mathcal{O}(1)^{n-3} \oplus \mathcal{O}$. Let Z'_x be the hypersurface in X covered by the members of $\mathcal{H}_x$. Applying Lemma 3 to the inverse images of members of $\mathcal{H}_x$ under f as in the proof of Proposition 2, we see that there exists an abelian hypersurface $H \subset A$ so that the Z'_x's are just images of translates of H. By the genericity of x, we can assume that Z'_x is smooth at x. Since Z'_x is covered by the members of $\mathcal{H}_x$, $T_x(Z'_x)$ corresponds to the positive part $\mathcal{O}(2)^2 \oplus \mathcal{O}(1)^{n-3}$ of the splitting of $T(X)$ on the normalization of C. In particular, elements of $H^0(C, T^*(X))$ annihilate $T_x(Z'_x)$.

We may replace A by an unramified finite cover and assume that $A = E \times H$ for some elliptic curve E. Let $C_t, t \in \Delta = \{t \in \mathbf{C}, |t| < 1\}$ be a family of curves belonging to $\mathcal{K}_x$ so that C_t is generic in $\mathcal{K}_x$ for $t \neq 0$ while C_0 is generic in $\mathcal{H}_x$. For $a \in f^{-1}(x)$ let C'_t be a component of $f^{-1}(C_t)$ through a. Then the toroidal hull of $C'_t, t \neq 0$ is A, while the toroidal hull of C'_0 is H. Considering small values of t, we may assume that C'_t form a flat family for $t \neq 0$ converging to $C'_0 \cup C^1 \cup \cdots \cup C^r$ for some irreducible curves $C^j, 1 \leq j \leq r$ in A. Clearly, $C'_0 \cup C^1 \cup \cdots \cup C^r$ is a connected curve and $f(C^j) = C_0$ for all j. Since $C'_t, t \neq 0$ surjects to E under the natural projection $E \times H \to E$, one of C^j's must surject to E.

Let $\omega \in H^0(C_0, T^*(X))$ be a nonzero element. Pulling back by $f^*|_{C'_0} : f^*(T(X)|_{C_0}) \to T^*(A)|_{C'_0}$, we get a non-zero element $(f^*|_{C'_0})\omega$ in $H^0(C'_0, T^*(A))$, which annihilates $T(H)$ because ω annihilates $T(Z'_x)$. From $H^0(C'_0, T^*(A)) = H^0(A, T^*(A))$, $(f^*|_{C'_0})\omega$ must be the restriction of a parallel 1-form $\Omega \in H^0(A, T^*(X))$ annihilating $T(H)$. From the connectedness of $C'_0 \cup C^1 \cup \cdots \cup C^r$, there exists some C^j, say C^1, which intersects C'_0 at some point b. Then $(f^*|_{C^1})\omega$ is again the restriction of a parallel 1-form $\Omega_1 \in H^0(A, T^*(A))$. Since Ω and Ω_1 agrees at b, $\Omega_1 = \Omega$. By the same argument applied to each component of the connected curve $C'_0 \cup C^1 \cup \cdots \cup C^r$, $(f^*|_{C^j})\omega$ must be the restriction of Ω for all j. Since $f(C^j) = C_0$, $(f^*|_{C^j})\omega$ annihilates the tangent vectors to C^j. This implies that the toroidal hull of C^j lies in H for all j and C^j cannot surject to E for any j, a contradiction. □

3 Structure of the distribution defined by the linear span of varieties of minimal rational tangents

When the varieties of minimal rational tangents are degenerate, we can use their linear span to define a distribution on X. We will show that this distribution defines the structure of a bundle of projective spaces as stated in the Main Theorem. We will say that a meromorphic distribution is integrable if it is integrable over the open set where it defines a genuine holomorphic distribution.

Proposition 6 *In the situation of Proposition 2, let* $W_x \subset T_x(X)$ *be the linear span of* $\mathcal{C}_x$ *for a generic* $x \in X$. *Let* W *be the meromorphic distribution on* X *defined by* W_x *at generic* $x \in X$. *There exists an abelian subvariety* $A' \subset A$ *so that the images of translates of* A' *are integral subvarieties of the distribution* W.

Proof. $df^{-1}W$ corresponds to the distribution $\mathcal{W}$ on A defined by the translates of the linear span of $A_1 \cup \cdots \cup A_m$. Since $\mathcal{W}$ is integrable, W is integrable because f is unramified at generic point. By [HM1] Proposition 11, the closure of each leaf of W is a subvariety in X. The inverse images of these subvarieties in A are the integral subvarieties of $\mathcal{W}$ and must be the translates of an abelian subvariety $A' \subset A$, where A' is the toroidal hull of $A_1 \cup \cdots \cup A_m$. Thus integral subvarieties of W are just images of the translates of A'. □

Proposition 7 *In the situation of Proposition 6, there exists a fibration* $\psi : X \to Y$ *over a normal variety* Y *such that the underlying reduced variety of each fiber of* ψ *is the image of a translate of the abelian subvariety* $A' \subset A$ *under* f. *Furthermore, let* $\mathcal{K}^y$ *be the subscheme of* $\mathcal{K}$ *corresponding to curves contained in the fiber* $X_y = \psi^{-1}(y)$ *over a generic point* $y \in Y$. *Then there exists a unique component* $\mathcal{K}'$ *of* $\mathcal{K}^y$ *whose members cover* X_y. *In particular,* $\mathcal{K}'$ *is a minimal rational component for* X_y *and its variety of minimal rational tangent at a generic point* $x \in X_y$ *coincides with* $\mathcal{C}_x$.

Proof. Consider an irreducible subscheme Y of the Chow scheme of X whose generic points parametrizes the integral subvarieties of the distribution W in Proposition 6. Let $\phi : \mathcal{F} \to X, \psi : \mathcal{F} \to Y$ be the universal family morphisms associated to Y ([Ko, I.3]). By taking reduced structures of the schemes involved and then taking normalization, we may assume that $\mathcal{F}$ and Y are normal. Since f is finite, there exist only a finite number of subvarieties through each point of X, which are the images of the translates of A'. It follows that ϕ is a finite morphism. On the other hand, parametrizing the leaves of a foliation, ϕ must be birational. Thus ϕ is an isomorphism and we get $\psi : X \to Y$. The underlying reduced varieties of the fibers of ψ are set-theoretic images of translates of A'.

From the definition of W, all members of $\mathcal{K}$ are contained in the fibers of ψ. Choose a component $\mathcal{K}'$ of $\mathcal{K}^y$ whose members cover X_y. For the uniqueness of $\mathcal{K}'$, it suffices to show that the variety of minimal rational tangents $\mathcal{C}'_x$ associated to $\mathcal{K}'$ coincides with $\mathcal{C}_x$. By Proposition 2, $\mathcal{C}_x = df_{a_1}(\mathbf{P}\mathcal{A}^i) \cup \cdots \cup df_{a_d}(\mathbf{P}\mathcal{A}^i)$ where $f^{-1}(x) = \{a_1, \ldots, a_d\}$, for the toroidal hull A_i of any component of the inverse image of a generic curve in $\mathcal{K}$. By the genericity of y, this holds for the toroidal hull A_i of any component of the inverse image of a generic curve in $\mathcal{K}'$. On the other hand, applying Proposition 2 to $f : a_1 + A' \to X_y$ and $\mathcal{K}'$, we see that $\mathcal{C}'_x = df_{a_1}(\mathbf{P}\mathcal{A}^i) \cup \cdots \cup df_{a_d}(\mathbf{P}\mathcal{A}^i)$, which is exactly $\mathcal{C}_x$. □

For the minimal rational component $\mathcal{K}'$ for X_y for generic $y \in Y$ in the above proposition,

$\mathcal{C}_x \subset \mathbf{P}T_x(X_y)$ is linearly nondegenerate at generic $x \in X_y$. From Proposition 5, we see that a generic fiber of $\psi : X \to Y$ is $\mathbf{P}_{p+1}$. This proves Theorem 1. Now by the next Proposition, we can see that Y is smooth and ψ is a $\mathbf{P}_{p+1}$-bundle (using, e.g., [GR, p.212]).

Proposition 8 *Let X be a projective algebraic manifold and $\psi : X \to Y$ be a surjective morphism onto a normal variety. Assume that the underlying reduced variety of each fiber of ψ is irreducible of dimension k. If a generic fiber is isomorphic to $\mathbf{P}_k$, then each fiber is isomorphic to $\mathbf{P}_k$ and Y is smooth.*

Proof. Let $o \in Y$ be any point and V be the underlying reduced subvariety of X_o. At a generic point $x \in V$, choose a local submanifold $S \subset X$ of codimension k transversal to V at x. Then $\psi|_S : S \to Y$ is a finite holomorphic map. If it is 1-to-1, it is biholomorphic over its image from the normality of Y. This implies that $o \in Y$ is a smooth point and X_o is Cohen-Macaulay. Moreover considering the intersection number $S \cdot X_y$, X_o is generically reduced. It follows that X_o is a reduced variety. On the other hand if $\psi|_S$ is not 1-to-1 for any choice of S, then X_o cannot be generically reduced and it is a multiple fiber of ψ. We will prove Proposition 8 in three steps.

Step 1 Generically reduced fiber over a smooth point

Suppose a fiber X_o at a smooth point $o \in Y$ is generically reduced. Then Y is smooth at o and ψ is flat, because morphisms between smooth varieties having equi-dimensional fibers are flat. Moreover X_o is a reduced Cohen-Macaulay variety. From the reducedness of X_o, ψ is a smooth morphism in a neighborhood of a generic point of X_o. Thus there exists an open neighborhood $U \subset Y$ of o and an open subset $U' \subset X$ in classical topology so that $U' \cong U \times \Delta^k$ where Δ^k is the polydisc in $\mathbf{C}^k$ and ψ corresponds to the projection $U \times \Delta^k \to U$. We can find k disjoint sections $\sigma_i : U \to U', 1 \leq i \leq k$, of ψ such that $\sigma_1(y), \ldots, \sigma_k(y)$ are in general position in $X_y \cong \mathbf{P}_k$ for each $y \in U - \{o\}$. The linear span of the $\sigma_i(y)$'s in $X_y, y \neq o$, determines a unique hyperplane L_y on $X_y \cong \mathbf{P}_k$. Taking the closure of the union of L_y's, we get a hypersurface L in $\psi^{-1}(U)$ whose restriction to $X_y, y \neq o$, is a hyperplane. L defines a line bundle on X_o with $L^k \cdot X_o = 1$ and $\dim(H^0(X_o, L)) \geq k+1$ from the flatness of ψ. Thus $X_o \cong \mathbf{P}_k$ by [Fu, I.1.1].

Step 2 Multiple fibers along a hypersurface of Y

Suppose ψ has multiple fibers along a hypersurface D in Y with generic multiplicity $r > 1$. Let o be a generic point of D which is a smooth point of Y. Let V be the underlying variety of X_o. At a generic point $x \in V$, choose a local submanifold $S \subset X$ of codimension k transversal to V at x. Then $\psi|_S$ is ramified over D. We can choose a coordinate system $(s_1, \ldots, s_{n-k})$ on S and $(y_1, \ldots, y_{n-k})$ on a smooth open neighborhood $U = \psi(S) \subset Y$ of o, so that $\psi|_S$ is given by $y_1 = s_1^r, y_2 = s_2, \ldots, y_{n-k} = s_{n-k}$. There exists a proper subvariety $W \subset V$ such that at any point of $V - W$, there exists a germ g of holomorphic functions on X satisfying $g^r = \psi^* y_1$.

Let $\tilde{\psi} : \mathcal{H}' \to S$ be the pull-back of $(\psi^{-1}(U) - W) \to U$ by $\psi|_S$, defined by $\mathcal{H}' := \{(s,z) \in S \times (\psi^{-1}(U) - W), \psi(s) = \psi(z)\}$. $\mathcal{H}'$ is locally defined by the equations

$$s_1^r - \psi^* y_1 = 0, \quad s_2 - \psi^* y_2 = 0, \quad \ldots, \quad s_{n-k} - \psi^* y_{n-k} = 0.$$

Since we are excluding W,

$$s_1^r - \psi^* y_1 \;=\; s_1^r - g^r \;=\; \prod_{\zeta^r=1}(s_1 - \zeta g).$$

So the normalization $\mathcal{H}$ of $\mathcal{H}'$ is smooth and the natural morphism $\rho : \mathcal{H} \to (\psi^{-1}(U) - W)$ is an unramified r-to-1 cover of $\psi^{-1}(U) - W$.

Since W is of codimension ≥ 2 in $\psi^{-1}(U)$, $\pi_1(\psi^{-1}(U)) = \pi_1(\psi^{-1}(U) - W)$. So there exists a complex manifold $\bar{\mathcal{H}}$ and an unramified r-to-1 covering $\bar{\mathcal{H}} \to \psi^{-1}(U)$ extending ρ. The morphism $\tilde{\psi} : \mathcal{H} \to S$ extends to $\bar{\psi} : \bar{\mathcal{H}} \to S$ by Hartogs. By construction, $\bar{\psi}$ has no multiple fiber and generic fiber is $\mathbf{P}_k$. Thus $\bar{\psi}$ is a $\mathbf{P}_k$-bundle over S by Step 1. But the fiber of $\bar{\psi}$ over $x \in S$ must be a finite unramified cover of V while $\mathbf{P}_k$ cannot be a non-trivial unramified cover of a variety, a contradiction.

Step 3 From Step 1 and Step 2, we may assume that $\psi : X \to Y$ is a $\mathbf{P}_k$-bundle outside a subvariety $E \subset Y$ of codimension ≥ 2.

Let $o \in E$ and V be the underlying subvariety of X_o. At a generic point $x \in V$, we choose a local submanifold $S \subset X$ of codimension k transversal to V at x. If the morphism $\psi|_S : S \to Y$ is 1-to-1, then X_o is generically reduced and $o \in Y$ is smooth. So $X_o \cong \mathbf{P}_k$ by Step 1. Let $\psi|_S$ be l-to-1 for some $l > 1$.

S is transversal to the underlying subvariety of any fiber X_y for y close to o. Thus S is transversal to X_y for $y \notin E$, y close to o. So $\psi|_S : S \to U = \psi(S)$ is unramified outside E. Since $\psi^{-1}(U - E) \to (U - E)$ is a $\mathbf{P}_k$-bundle, we have $\pi_1(\psi^{-1}(U - E)) = \pi_1(U - E)$. On the other hand, $\pi_1(\psi^{-1}(U - E)) = \pi_1(\psi^{-1}(U))$ since $\psi^{-1}(U)$ is smooth and $\psi^{-1}(E)$ is of codimension ≥ 2 in $\psi^{-1}(U)$. Thus the unramified covering $\psi|_{S-\psi^{-1}(E)} : (S - \psi^{-1}(E)) \to (U - E)$ induces an unramified covering $\rho : Q \to \psi^{-1}(U)$ where Q is a complex manifold and $Q' := \rho^{-1}(\psi^{-1}(U - E))$ is a $\mathbf{P}_k$-bundle over $S - \psi^{-1}(E)$. By Hartogs, we have a natural morphism $\mu : Q \to S$. The underlying reduced variety of each fiber of μ is irreducible because ρ is unramified and each fiber of $\psi^{-1}(U)$ is irreducible. Moreover μ admits a section arising from $\rho^{-1}(S)$. Thus by Step 1, Q is a $\mathbf{P}_k$-bundle over S while the fiber over $x \in S$ must be an unramified cover of V, a contradiction as in Step 2. □

By above we have a $\mathbf{P}_{p+1}$-bundle structure $\psi : X \to Y$ so that each fiber is the image of a translate of an abelian subvariety $A' \subset A$ under f. To finish the proof of Main Theorem, it remains to show that Y is the image of a finite morphism from an abelian variety. By taking a finite unramified covering, we may write $A = A' \times A"$ for some abelian subvariety $A"$. Since $f(A")$ intersects any fiber of ψ at only finitely many points, $\psi \circ f$ gives a finite morphism from an abelian variety $A"$ onto Y and we are done.

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