

UPPER BOUNDS ON N-DIMENSIONAL KLOOSTERMAN SUMS

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ABSTRACT. Let p^m be any prime power and $K_n(a, p^m)$ be the n -dimensional Kloosterman sum

$$K_n(a, p^m) = \sum_{x_1=1}^{p^m} \cdots \sum_{x_n=1}^{p^m} e_{p^m}(x_1 + \cdots + x_n + a \overline{x_1 x_2 \cdots x_n}),$$

where the x_i are restricted to values not divisible by p . Let m, n be positive integers with $m \geq 2$ and suppose that $p \nmid (n+1)$. We obtain the upper bound $|K_n(a, p^m)| \leq (n+1, p-1) p^{\frac{1}{2} \min(\gamma, m-2)} p^{mn/2}$, for odd p . For $p = 2$ we obtain the same bound, with an extra factor of 2 inserted.

1. INTRODUCTION

Let p be a prime, m, n be positive integers, a be any integer and $K_n(a, p^m)$ be the n -dimensional Kloosterman sum

$$(1.1) \quad K_n(a, p^m) = \sum_{\substack{x_1=1 \\ p \nmid x_1 x_2 \cdots x_n}}^{p^m} \cdots \sum_{x_n=1}^{p^m} e_{p^m}(x_1 + \cdots + x_n + a \overline{x_1 x_2 \cdots x_n}),$$

where the overline denotes multiplicative inverse (mod p^m). If $p|a$ it is easily seen that $K_n(a, p) = (-1)^n$ and that $K_n(a, p^m) = 0$ for $m \geq 2$ (see Theorem 3 of [9]), and so we may always assume that $p \nmid a$. Deligne [4], appealing to his deep work on the Weil conjectures established in the case $m = 1$ that

$$(1.2) \quad |K_n(a, p)| \leq (n+1)p^{n/2}.$$

There is also an elementary upper bound,

$$(1.3) \quad |K_n(a, p)| \leq p^{\frac{n+1}{2}},$$

due to Mordell [7] and Smith [9], which is sharper than (1.2) for $n > \sqrt{p}$. The reader is referred to the paper of Smith for a historical discussion on the estimation of the Kloosterman sum.

For a general value of $m \geq 1$ Smith [9] proved that for odd p we have

$$(1.4) \quad |K_n(a, p^m)| \leq (n+1)p^{\frac{nm}{2}}.$$

The same argument that was used by Smith to prove (1.3) can also be used to obtain the upper bound

$$(1.5) \quad |K_n(a, p^m)| \leq p^{\frac{(n+1)m}{2}}.$$

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Dabrowski and Fisher [3] (Example 1.17) obtained an extra savings in the special case that $p = n + 1$:

$$(1.6) \quad |K_n(a, p^m)| \leq \begin{cases} p^{\frac{nm}{2}}, & \text{if } m = 2; \\ p^{1/2} p^{\frac{nm}{2}}, & \text{if } m = 3 \text{ or } m \geq 5; \\ p \cdot p^{\frac{nm}{2}}, & \text{if } m = 4. \end{cases}$$

See also the work of Ye [10] for an application of (1.6).

Here, we take the work of Smith one step further and establish an upper bound that sharpens (1.4), (1.5) and (1.6).

Theorem 1.1. *Let p be a prime, n be a positive integer and suppose that $p^\gamma \parallel (n+1)$.*

(a) *If p is odd then for $m \geq 2$,*

$$(1.7) \quad |K_n(a, p^m)| \leq (n+1, p-1) p^{\frac{1}{2} \min(\gamma, m-2)} p^{nm/2}.$$

(b) *If $p = 2$ then for $m \geq 2$,*

$$(1.8) \quad |K_n(a, 2^m)| \leq 2 \cdot 2^{\frac{1}{2} \min(\gamma, m-2)} 2^{nm/2}.$$

In Proposition 2.1 we state a more precise version of this theorem for the case of odd p . When $(n+1, p-1)$ is bounded by a constant we deduce from (1.7) the upper bound

$$(1.9) \quad |K_n(a, p^m)| \ll \sqrt{n} p^{nm/2},$$

which is a best possible type of upper bound. Indeed, by Proposition 2.1, it follows that if $n+1 = p^\gamma$ then for any $m \geq \gamma + 2$ we have

$$|K_n(1, p^m)| = \sqrt{n+1} p^{nm/2}.$$

It is reasonable to conjecture that (1.9) holds in general, even when $m = 1$, but this is no doubt a very difficult problem.

2. PROOF OF THEOREM 1.1

We take up the case of odd p in this section and deal with $p = 2$ in section 3. The theorem is deduced from the following result of Smith [9].

Theorem 2.1. *Let p be an odd prime and a an integer not divisible by p .*

(i) *Suppose that m is even. Then by Theorem 4 of [9] we have*

$$(2.1) \quad K_n(a, p^m) = p^{nm/2} \sum_{\substack{u=1 \\ u^{n+1} \equiv a \pmod{p^{m/2}}} }^{p^{m/2}} e_{p^m}(nu + a\bar{u}^n).$$

(ii) *Suppose that $m \geq 3$ is odd. Let $\beta = (m-1)/2$. If $p \nmid (n+1)$ we have by Theorem 5 (i) of [9] that $K_n(a, p^m)$ is a sum of $(n+1, p-1)$ complex numbers of modulus $p^{nm/2}$ and so*

$$(2.2) \quad |K_n(a, p^m)| \leq (n+1, p-1) p^{nm/2}.$$

If $p \mid (n+1)$ then by Theorem 5 (ii) of [9],

$$(2.3) \quad K_n(a, p^m) = p^{(nm+1)/2\theta} \sum_{\substack{u=1 \\ u^{n+1} \equiv a \pmod{p^{\beta+1}}} }^{p^\beta} e_{p^m}(nu + a\bar{u}^n),$$

where θ is a complex number of modulus one.

(We note that the value of θ in Smith's paper should be corrected to read $\theta = \epsilon_{n-1}(p)\chi_p(gN(a))$.)

In the course of proving Theorem 1.1 we actually establish the more precise result,

Proposition 2.1. *Let n be a positive integer, p an odd prime and suppose that $p^\gamma \parallel (n+1)$.*

(i) *If a is not an $(n+1)$ -th power $(\bmod p^{\gamma+1})$ and $m \geq \gamma+2$ then $K_n(a, p^m) = 0$.*

(ii) *If a is an $(n+1)$ -th power $(\bmod p^{\gamma+1})$ and $m \geq 2$ then $K_n(a, p^m)$ is a sum of $(n+1, p-1)$ complex numbers, each of modulus $p^{nm/2} p^{\frac{1}{2} \min\{\gamma, m-2\}}$.*

The values of the complex numbers in part (ii) may be calculated explicitly using the method here together with the results of Smith [9]. This may give one hope of making a further savings in the constant $(n+1, p-1)$ on the right-hand side of (1.7).

We start with the following elementary lemma, which follows from the standard criterion for an element to be an $(n+1)$ -th power in a cyclic group.

Lemma 2.1. *Suppose that p is an odd prime with $p^\gamma \parallel (n+1)$ and $p \nmid a$. If a is an $(n+1)$ -th power $(\bmod p^{\gamma+1})$ then a is an $(n+1)$ -th power modulo any power of p .*

The next lemma is an easy application of the method of critical points for estimating exponential sums. If $f(x)$ is a polynomial over \mathbb{Z} and p^t is the largest power of p dividing all of the coefficients of $f'(x)$ then the set critical points \mathcal{A} associated with the sum $S := \sum_{x=1}^{p^m} e_{p^m}(f(x))$ is just the set of zeros of the congruence $p^{-t}f'(x) \equiv 0 \pmod{p}$. The basic result we need here is that if $m \geq t+2$ then

$$(2.4) \quad S = \sum_{\alpha \in \mathcal{A}} S_\alpha,$$

where S_α is the same sum as S with x restricted to the residue class $\alpha \pmod{p}$; see eg. Theorem 2.1 of [2] or Loh [6] or Ding [5]. Also, if α is a zero of multiplicity one then

$$(2.5) \quad |S_\alpha| = p^{\frac{m+t}{2}}.$$

When $p=2$ the same result holds provided that $m \geq t+3$.

Lemma 2.2. (a) *Let p be an odd prime, a, b be integers with $p \nmid b$ and $f(x)$ be a polynomial with integer coefficients. Then for $m \geq 1$ we have*

$$(2.6) \quad \left| \sum_{x=1}^{p^m} e_{p^m}(bx^2 + cx + pf(x)) \right| = p^{m/2}.$$

(b) *If $p=3$ and $3 \nmid b$ then for any $a, f(x)$ and $m \geq 1$ we have*

$$(2.7) \quad \left| \sum_{x=1}^{p^m} e_{p^m}(ax^3 + bx^2 + cx + pf(x)) \right| = p^{m/2}.$$

Proof. When $m=1$, the two sums are just quadratic Gauss sums (replacing x^3 with x in part (b).) For $m \geq 2$ the critical point congruence associated with each of the sums is just $2bx + c \equiv 0 \pmod{p}$, and thus there is a single critical point of multiplicity one. The result follows from (2.4) and (2.5). \square

The case of even m . We proceed now to the proof of the Theorem 1.1 and Proposition 2.1. Suppose first that m is even. Let $n + 1 = p^\gamma d$ with $p \nmid d$ and let U be the set of residues $u \pmod{p^{m/2}}$ satisfying $u^{n+1} \equiv a \pmod{p^{m/2}}$. Then by (2.1) we have the immediate upper bound

$$(2.8) \quad |K_n(a, p^m)| \leq p^{mn/2} |U| \leq p^{mn/2} (p^{\frac{m}{2}-1} (p-1), n+1).$$

If $\gamma = 0$ or $\gamma \geq m - 2$ then (1.7) follows immediately from (2.8). Also, if $\gamma = 0$ we see by (2.1) that $K_n(a, p^m) = 0$ if a is not an $(n+1)$ -th power \pmod{p} and that $K_n(a, p^m)$ is a sum of $(n+1, p-1)$ complex numbers of modulus $p^{nm/2}$ if a is an $(n+1)$ -th power \pmod{p} . Here we have used Lemma 2.1.

Suppose next that $1 \leq \gamma < \frac{m}{2}$. If a is not an $(n+1)$ -th power $\pmod{p^{\gamma+1}}$ then by (2.1) it follows that $K_n(a, p^m) = 0$. Suppose now that a is an $(n+1)$ -th power $\pmod{p^{\gamma+1}}$. Then by Lemma 2.1 a is an $(n+1)$ -th power modulo any power of p . We first note that (2.1) may be written in the manner

$$(2.9) \quad K_n(a, p^m) = p^{nm/2} \sum_{u \in U} e_{p^m}(nu + a\bar{u}^n),$$

since, for $u \in U$, the value of $nu + a\bar{u}^n \pmod{p^m}$ depends only on the value of $u \pmod{p^{m/2}}$. To see this, let

$$(2.10) \quad k := (\phi(p^m) - 1)n.$$

In particular,

$$(2.11) \quad p^{m-1} \mid (n+k).$$

Let u be any integer satisfying $u^{n+1} \equiv a \pmod{p^{m/2}}$, and let \bar{u} denote a multiplicative inverse of $u \pmod{p^m}$. Then

$$a\bar{u}^{n+1} p^{m/2} \equiv p^{m/2} \pmod{p^m},$$

and so

$$\begin{aligned} n(u + p^{m/2}) + a\overline{(u + p^{m/2})}^n & \\ & \equiv nu + np^{m/2} + a\bar{u}^n(1 + \bar{u}p^{m/2})^k \pmod{p^m} \\ & \equiv nu + np^{m/2} + a\bar{u}^n(1 + k\bar{u}p^{m/2} + \binom{k}{2}\bar{u}^2p^m + \dots) \pmod{p^m} \\ & \equiv nu + a\bar{u}^n \pmod{p^m}. \end{aligned}$$

We partition U into $(d, p-1)$ subsets as follows. Start by observing that the congruence $x^{n+1} \equiv a \pmod{p}$ has $(d, p-1)$ distinct solutions $\alpha_1, \dots, \alpha_{(d, p-1)} \pmod{p}$, each of which can be lifted to a solution $\pmod{p^m}$. We may assume that representatives have been chosen so that each α_i satisfies the congruence $x^{n+1} \equiv a \pmod{p^m}$. Put $j = \frac{m}{2} - \gamma$. In particular $j \geq 1$. For $i = 1, 2, \dots, (d, p-1)$, let

$$U_i = \{\alpha_i + p^j t : t = 1, 2, \dots, p^\gamma\}.$$

Then (viewing the elements of U , U_i as residue classes $\pmod{p^{m/2}}$) U is the disjoint union of the sets U_i and we have by (2.9)

$$K_n(a, p^m) = p^{nm/2} \sum_{i=1}^{(p-1, d)} S_i,$$

where

$$S_i = \sum_{t=1}^{p^\gamma} e_{p^m} \left(n(\alpha_i + p^j t) + a(\overline{\alpha_i + p^j t})^n \right).$$

In what follows let $\alpha = \alpha_i$ and $S_\alpha = S_i$ for a typical value i , and let k be as defined in (2.10). Thus

$$(2.12) \quad S_\alpha = \sum_{t=1}^{p^\gamma} e_{p^m}(f_\alpha(t)),$$

where

$$(2.13) \quad f_\alpha(t) := n(\alpha + p^j t) + a(\overline{\alpha + p^j t})^n.$$

The theorem will be proved if we can show that $|S_\alpha| = p^{\gamma/2}$.

Now for any integer t ,

$$(1 + \overline{\alpha p^j t})^n \equiv (1 + \overline{\alpha p^j t})^k \pmod{p^m}.$$

Also,

$$(2.14) \quad p^\gamma \parallel (k-1),$$

and $a\overline{\alpha}^n \equiv \alpha \pmod{p^m}$. Thus,

$$\begin{aligned} f_\alpha(t) &\equiv n\alpha + np^j t + \alpha(1 + \overline{\alpha p^j t})^n \pmod{p^m} \\ &\equiv n\alpha + np^j t + \alpha(1 + \overline{\alpha p^j t})^k \pmod{p^m} \\ &\equiv n\alpha + np^j t + \alpha(1 + k\overline{\alpha p^j t} + \binom{k}{2}\overline{\alpha}^2 p^{2j} t^2 + \binom{k}{3}\overline{\alpha}^3 p^{3j} t^3 + \dots) \pmod{p^m}. \end{aligned}$$

Using (2.11) we obtain

$$(2.15) \quad f_\alpha(t) \equiv (n+1)\alpha + \overline{\alpha} \binom{k}{2} p^{2j} t^2 + \overline{\alpha}^2 \binom{k}{3} p^{3j} t^3 + \dots := \sum_{r=0}^{\infty} a_r t^r, \pmod{p^m}$$

say. Put

$$(2.16) \quad \sigma_\alpha = \min_{r \geq 1} \text{ord}_p(a_r), \quad g_\alpha(t) = p^{-\sigma_\alpha} f_\alpha(t).$$

Now, since $p^\gamma \parallel (k-1)$ we have $\text{ord}_p(a_2) = 2j + \gamma$ and for any $r \geq 3$

$$\text{ord}_p(a_r) = \text{ord}_p \binom{k}{r} p^{rj} \geq \gamma + rj - \text{ord}_p(r!) > \gamma + rj - \frac{r}{p-1}$$

If $p > 3$ or $p = 3$ and $j > 1$ it follows that $\text{ord}_p(a_r) > 2j + \gamma$ for $r \geq 3$. Thus $\sigma_\alpha = 2j + \gamma$, and since $m - (2j + \gamma) = \gamma$, we can write

$$S_\alpha = \sum_{t=1}^{p^\gamma} e_{p^{m-\sigma_\alpha}}(g_\alpha(t)) = e_{p^m}((n+1)\alpha) \sum_{t=1}^{p^\gamma} e_{p^\gamma}(\overline{2\alpha k} \frac{k-1}{p^\gamma} t^2 + ph(t)),$$

for some polynomial $h(t) \in \mathbb{Z}[t]$. By Lemma 2.2 (a) we have $|S_\alpha| = p^{\gamma/2}$, and the result follows. If $p = 3$ and $j = 1$ then the same conclusion can be made using Lemma 2.2 (b).

Suppose finally that $\frac{m}{2} \leq \gamma$. If a is not an $(n+1)$ -th power $\pmod{p^{m/2}}$ then by (2.1), $K_n(a, p^m) = 0$. Suppose that a is an $(n+1)$ -th power $\pmod{p^{m/2}}$.

Let $\alpha_1, \dots, \alpha_{(d,p-1)}$ be distinct values (mod p), each satisfying the congruence $x^{n+1} \equiv a \pmod{p^{m/2}}$. Then U can be written as a disjoint union of the sets

$$U_i = \{\alpha_i + pt : t = 1, 2, \dots, p^{\frac{m}{2}-1}\},$$

and we have

$$K_n(a, p^m) = \sum_{i=1}^{(p-1, d)} S_i,$$

where $S_i := \sum_{u \in U_i} e_{p^m}(nu + a\bar{u}^n)$. Let $\alpha = \alpha_i$, $S_\alpha = S_i$ for a typical value i . Now, since $a\bar{\alpha}^n \equiv \alpha \pmod{p^{m/2}}$ we can write

$$a\bar{\alpha}^n = \alpha + \rho p^{m/2},$$

for some integer ρ . Then

$$S_\alpha = \sum_{t=1}^{p^{\frac{m}{2}-1}} e_{p^m}(f_\alpha(t)),$$

where

$$\begin{aligned} f_\alpha(t) &\equiv n\alpha + npt + (\alpha + \rho p^{m/2})(1 + \bar{\alpha}pt)^n \pmod{p^m} \\ &\equiv n\alpha + npt + (\alpha + \rho p^{m/2})(1 + \bar{\alpha}p^j t)^k \pmod{p^m} \\ (2.17) \quad &\equiv (n+1)\alpha + \rho p^{m/2} + k\rho\bar{\alpha}p^{\frac{m}{2}+1}t + (\alpha + \rho p^{\frac{m}{2}})\bar{\alpha}^2 \binom{k}{2} p^2 t^2 \\ &\quad + (\alpha + \rho p^{\frac{m}{2}})\bar{\alpha}^3 \binom{k}{3} p^3 t^3 + \dots \pmod{p^m} \\ &:= \sum_{r=0}^{\infty} a_r t^r. \end{aligned}$$

We consider two cases. If a is not an $(n+1)$ -th power (mod $p^{\gamma+1}$) then $\text{ord}_p(\rho) + m/2 < \gamma + 1$ and so

$$\text{ord}_p(a_1) = \text{ord}_p(\rho) + 1 + m/2 < \gamma + 2 = \text{ord}_p(a_2).$$

It follows that $\sigma_\alpha = \text{ord}_p(a_1)$ and that g_α is linear (mod p), where σ_α and g_α are as defined in (2.16). Thus for any $p > 2$ and $m \geq \gamma + 2$, $S_\alpha = 0$. If a is an $(n+1)$ -th power (mod $p^{\gamma+1}$) then $\sigma_\alpha = \gamma + 2$ and we may assume (by Lemma 2.1) that $\alpha^{n+1} \equiv a \pmod{p^m}$. If $\gamma \geq m - 2$ then we just get $|S_\alpha| = p^{\frac{m}{2}-1}$. If $p \geq 3$ and $\gamma < m - 2$ then

$$S_\alpha = e_{p^m}((n+1)\alpha)p^{\gamma+1-\frac{m}{2}} \sum_{t=1}^{p^{m-\gamma-2}} e_{p^{m-\gamma-2}}(2\bar{\alpha}k \frac{k-1}{p^\gamma} t^2 + ph(t)),$$

for some polynomial $h(t) \in \mathbb{Z}[t]$. It follows from Lemma 2.2 that $|S_\alpha| = p^{\gamma/2}$. The case $p = 3$ can be dealt with in a similar manner using Lemma 2.2 (b).

The case of odd m . Suppose now that $m \geq 3$ is odd, say $m = 2\beta + 1$. Again write $n+1 = p^\gamma d$ with $p \nmid d$. If $\gamma = 0$ then (1.7) is an immediate consequence of (2.2). Indeed, if a is not an $(n+1)$ -th power (mod p) then by Theorem 3 of [9], $K_n(a, p^m) = 0$, and if a is an $(n+1)$ -th power (mod p) then by Theorem 1.2 (ii), $K_n(a, p^m)$ is a sum of $(n+1, p-1)$ complex numbers of modulus $p^{nm/2}$.

Suppose that $\gamma \geq 1$. Set

$$U = \{u : 1 \leq u \leq p^\beta, u^{n+1} \equiv a \pmod{p^{\beta+1}}\}.$$

By (2.3) we have the immediate upper bound

$$(2.18) \quad |K_n(a, p^m)| \leq p^{(nm+1)/2} |U| = p^{(nm-1)/2} (p^\beta (p-1), n+1).$$

If $\gamma \geq m-2$ or $\gamma = 1$ then (1.7) follows from (2.18). Moreover, if $\gamma = 1$, then Proposition 2.1 also follows immediately from (2.3).

Suppose now that $2 \leq \gamma \leq \beta$. If a is not an $(n+1)$ -th power $(\text{mod } p^\gamma)$, then U is empty and $K_n(a, p^m) = 0$. If a is an $(n+1)$ -th power $(\text{mod } p^\gamma)$ then we proceed as above. Let $\alpha_1, \dots, \alpha_{(d,p-1)}$ be distinct $(n+1)$ -th roots of a chosen so that each α_i satisfies $\alpha_i^{n+1} \equiv a \pmod{p^m}$. The trick for dealing with the sum in (2.3) is to note that for $u \in U$, the value of $nu + a\bar{u}^n \pmod{p^m}$ depends only on the value of $u \pmod{p^\beta}$. Indeed, if $u^{n+1} \equiv a \pmod{p^\beta}$ then setting $k = n(p^{m-1}(p-1) - 1)$ we have

$$\begin{aligned} n(u + p^\beta) + a\overline{(u + p^\beta)}^n & \\ & \equiv n(u + p^\beta) + a(u + p^\beta)^k \pmod{p^m} \\ & \equiv nu + np^\beta + a(u^k + ku^{k-1}p^\beta) \pmod{p^m}, \quad (\text{since } p|(k-1)) \\ & \equiv nu + au^k + p^\beta n(1 - au^{k-1}) \pmod{p^m} \\ & \equiv nu + au^k + p^\beta n(1 - u^{n+k}) \pmod{p^m} \\ & \equiv nu + au^k \pmod{p^m}. \end{aligned}$$

Set $j = \frac{m+1}{2} - \gamma \geq 1$. For $i = 1, \dots, (d, p-1)$, let

$$U_i = \{\alpha_i + p^j t : t = 1, 2, \dots, p^{\gamma-1}\}.$$

Then viewing the sets U, U_i as residue classes $(\text{mod } p^\beta)$, we see that U is a disjoint union of the sets U_i . Thus by (2.3) we have

$$(2.19) \quad |K_n(a, p^m)| = p^{(nm+1)/2} \left| \sum_{i=1}^{(d,p-1)} S_i \right|,$$

where

$$S_i = \sum_{t=1}^{p^{\gamma-1}} e_{p^m}(n(\alpha_i + p^j t) + a\overline{(\alpha_i + p^j t)}^n).$$

Let $\alpha = \alpha_i, S_\alpha = S_i$. Now,

$$\begin{aligned} f_\alpha(t) & := n\alpha + np^j t + a\bar{\alpha}^n (\overline{1 + \bar{\alpha} p^j t})^n \\ & \equiv (n+1)\alpha + (n+k)p^j t + \overline{2\alpha}(k-1)kp^{2j}t^2 + \dots \end{aligned}$$

This time the multiplicity of p dividing the t^2 coefficient is $2j + \gamma$. Since $m - (2j + \gamma) = \gamma - 1$ it follows as above that $|S_\alpha| = p^{\frac{\gamma-1}{2}}$, and the theorem follows.

Finally, suppose that $\beta + 1 \leq \gamma$. If a is not an $(n+1)$ -th power $(\text{mod } p^{\beta+1})$ then U is empty and $K_n(a, p^m) = 0$. If a is an $(n+1)$ -th power $(\text{mod } p^{\beta+1})$ then we set

$$U_i = \{\alpha_i + pt : t = 1, 2, \dots, p^{\beta-1}\},$$

where the α_i are distinct $(n+1)$ -th roots of $a, (\text{mod } p)$. Then U (viewed as residue classes $(\text{mod } p^\beta)$) is the disjoint union of the sets U_i . Let $\alpha = \alpha_i$ and

$$S_\alpha = \sum_{t=1}^{\beta-1} e_{p^m}(f_\alpha(t)),$$

where

$$f_\alpha(t) := (n(\alpha + pt) + a(\overline{\alpha + pt})^n).$$

Write

$$a\overline{\alpha}^n = \alpha + p\rho^{\beta+1},$$

for some integer ρ . If a is not an $(n+1)$ -th power $(\bmod p^{\gamma+1})$ then $\text{ord}_p(\rho) + \beta + 1 < \gamma + 1$ and so

$$\text{ord}_p(a_1) < \gamma + 2 = \text{ord}_p(a_2) < \text{ord}_p(a_i), \quad \text{for } i \geq 3.$$

It follows that $\sigma_\alpha = \text{ord}_p(a_1)$, and that g_α is linear $(\bmod p)$. If $\sigma_\alpha \geq \gamma - 1$ then $|S_\alpha| = p^{\beta-1}$ and $p^{1/2}|S_\alpha| = p^{\frac{\beta}{2}-1}$. If $\sigma_\alpha < \gamma - 1$ then $S_\alpha = 0$. If a is an $(n+1)$ -th power $(\bmod p^{\gamma+1})$ then as above we obtain that $|S_\alpha| = p^{\frac{\beta}{2}}$.

3. THE CASE $p = 2$

Theorem 1.1 (b) is deduced from the following result of Smith [9], his Theorem 4 and Lemma 5 combined.

Theorem 3.1. *Let m, n be positive integers with $m \geq 2$ and suppose that $2^\gamma \parallel (n+1)$. Then*

(i) *If m is even then*

$$(3.1) \quad K_n(a, 2^m) = 2^{\frac{mn}{2}} \sum_{\substack{u=1 \\ u^{n+1} \equiv a \pmod{2^{m/2}}}^{2^{m/2}}} e_{2^m}(nu + a\overline{u}^n).$$

(ii) *If m is odd and $\gamma = 0$ then $|K_n(a, 2^m)| = 2^{mn/2}$.*

(iii) *If m is odd and $\gamma = 1$ then, letting $\beta = (m-1)/2$,*

$$(3.2) \quad |K_n(a, 2^m)| = 2^{\frac{mn+1}{2}} \left| \sum_{\substack{u=1 \\ 2^\beta \parallel (u^{n+1}-a)}}^{2^\beta} e_{2^m}(nu + a\overline{u}^n) \right|.$$

(iv) *If m is odd and $\gamma \geq 2$ then, letting $\beta = (m-1)/2$,*

$$(3.3) \quad |K_n(a, 2^m)| = 2^{\frac{mn+1}{2}} \left| \sum_{\substack{u=1 \\ 2^{\beta+1} \parallel (u^{n+1}-a)}}^{2^\beta} e_{2^m}(nu + a\overline{u}^n) \right|.$$

We need also the following lemmas. For any odd integer a and positive integers s, λ let $N_s(a, \lambda)$ denote the number of solutions of the congruence $u^s \equiv a \pmod{2^\lambda}$. The first lemma is elementary; see eg. [8] Corollary 2.44.

Lemma 3.1. *Suppose that a is odd and $\lambda \geq 1$.*

(i) *If s is odd then $N_s(a, \lambda) = 1$.*

(ii) *If s is even then $N_s(a, \lambda) = (2s, 2^{\lambda-1})$ if $a \equiv 1 \pmod{2(2s, 2^{\lambda-1})}$, and equal to zero otherwise.*

We deduce easily the following analogue of Lemma 2.1.

Lemma 3.2. *Let a be odd and suppose that $2^\gamma \parallel (n+1)$. If a is an $(n+1)$ -th power $(\bmod 2^{\gamma+2})$ then a is an $(n+1)$ -th power modulo any power of 2.*

Lemma 3.3. *Let a be an odd integer and $H(x)$ be any polynomial over \mathbb{Z} . Then for any $m \geq 1$ we have*

$$(3.4) \quad \left| \sum_{x=1}^{2^m} e_{2^m}(ax^2 + 2^2 H(x)) \right| \leq 2^{\frac{m+1}{2}},$$

and consequently

$$(3.5) \quad \left| \sum_{x=1}^{2^m} e_{2^{m+1}}(ax^2 + 2^2 H(x)) \right| \leq 2^{m/2}.$$

Proof. We note that if $x \equiv y \pmod{2^m}$ then

$$ax^2 + 2^2 H(x) \equiv ay^2 + 2^2 H(y) \pmod{2^{m+1}},$$

and thus (3.5) is an immediate consequence of (3.4). The critical point congruence for the sum in (3.4) is just $x \equiv 0 \pmod{2}$ and thus there is a single critical point, of multiplicity 1. The inequality in (3.4) then follows from (2.4) and (2.5) for $m \geq 4$. For $m = 1$ the inequality is trivial. For $m = 2$ it is well known that $\sum_{x=1}^4 e_4(ax^2) = 2(1 + i^a)$, while for $m = 3$ we have

$$\sum_{x=1}^8 e_8(ax^2 + 4H(x)) = 4(-1)^{H(1)} e_8(a).$$

□

Lemma 3.4. *Let a be any integer, b, c, d be odd integers and $H(x)$ be any polynomial over \mathbb{Z} . Then for any $m \geq 1$ we have*

$$(3.6) \quad \left| \sum_{x=1}^{2^m} e_{2^m}(ax + bx^2 + 2cx^3 + 2dx^4 + 2^2 H(x)) \right| \leq 2^{\frac{m+3}{2}}.$$

Consequently, if a is even then

$$(3.7) \quad \left| \sum_{x=1}^{2^m} e_{2^{m+1}}(ax + bx^2 + 2cx^3 + 2dx^4 + 2^2 H(x)) \right| \leq 2^{\frac{m+2}{2}}.$$

Proof. We note that the inequality in (3.6) is trivial for $m = 1, 2, 3$. If $4 \nmid a$ then there are no critical points associated with the sum and so the sum is zero for $m \geq 4$. If $4 \mid a$ then the critical point congruence for the sum in (3.6) is just $x(x+1) \equiv 0 \pmod{2}$, and thus there are two critical points, each of multiplicity one. The result follows from (2.4) and (2.5) for $m \geq 4$. □

We turn now to the proof of Theorem 1.1 when $p = 2$. Suppose first that $\gamma = 0$, that is, $n + 1$ is odd. Then for any λ , the congruence $u^{n+1} \equiv a \pmod{2^\lambda}$ has a unique solution. It follows from parts (i) and (ii) of Theorem 3.1 that $|K_n(a, 2^m)| = 2^{m/2}$. Henceforth we may assume that $\gamma \geq 1$.

The case of even m . Suppose first that m is even. By Lemma 3.1 and Theorem 3.1 (i) we have the immediate upper bound

$$(3.8) \quad |K_n(a, 2^m)| \leq 2^{\min(\gamma+1, \frac{m}{2}-1)} \cdot 2^{mn/2}.$$

The upper bound in (1.8) follows trivially if $\gamma \geq m - 4$. Thus we may assume that $1 \leq \gamma \leq m - 5$ and that $m \geq 6$. We first consider the case that $1 \leq \gamma \leq \frac{m}{2} - 2$. Then by Lemma 3.1, the congruence $u^{n+1} \equiv a \pmod{2^{m/2}}$ has either no solution,

in which case $|K_n(a, 2^m)| = 0$, or $2^{\gamma+1}$ solutions. Suppose that the latter holds. Then by Lemma 3.2 a is also an $(n+1)$ -th power (mod 2^m). Let α be a fixed value satisfying $\alpha^{n+1} \equiv a \pmod{2^m}$. Then the set of solutions of the congruence $u^{n+1} \equiv a \pmod{2^{m/2}}$ (regarded as residue classes (mod $2^{m/2}$)) may be written

$$\{2^{\frac{m}{2}-\gamma}t \pm \alpha : 1 \leq t \leq 2^\gamma\}.$$

Since the value of $nu + a\bar{u}^n \pmod{2^m}$ in the sum (3.1) depends only on the value of $u \pmod{2^{m/2}}$ we may write

$$(3.9) \quad K_n(a, 2^m) = 2^{mn/2}(S^+ + S^-),$$

where, setting $j = \frac{m}{2} - \gamma$,

$$(3.10) \quad S^+ = \sum_{t=1}^{2^\gamma} e_{2^m} \left(n(2^j t \pm \alpha) + a \overline{(2^j t \pm \alpha)}^n \right).$$

Set $k = (2^{m-1} - 1)n$ and $f(t) = n(2^j t + \alpha) + a \overline{(2^j t + \alpha)}^n$. Then, since $2^\gamma \mid (k-1)$ and $a\bar{\alpha}^n \equiv \alpha \pmod{2^m}$ we have for any value of t ,

$$(3.11) \quad f(t) \equiv n(2^j t + \alpha) + \alpha(1 + 2^j \bar{\alpha} t)^k \pmod{2^m}$$

$$(3.12) \quad \equiv (n+1)\alpha + a_2 2^{m-\gamma-1} t^2 + \sum_{r \geq 3} a_r t^r \pmod{2^m},$$

say, where $a_2 = k \frac{k-1}{2^\gamma} \bar{\alpha}$ and the coefficients a_r satisfy

$$(3.13) \quad \text{ord}_2(a_r) = \text{ord}_2 \binom{k}{r} + rj, \quad \text{for } r \geq 3.$$

Now $\text{ord}_2(a_3) = \gamma - 1 + 3j \geq 2j + \gamma + 1$ (since $j \geq 2$) and for $r \geq 4$,

$$\text{ord}_2(a_r) \geq \gamma + 1 - \text{ord}_2(r!) + rj > \gamma + 1 - r + rj > 2j + \gamma + 1 = m - \gamma + 1.$$

Therefore we may write

$$f(t) \equiv (n+1)\alpha + a_2 2^{m-\gamma-1} t^2 + 2^{m-\gamma+1} H(t) \pmod{2^m}$$

for some polynomial $H(t)$ with integer coefficients. It follows from Lemma 3.3 that,

$$|S^+| = \left| \sum_{t=1}^{2^\gamma} e_{2^m} (a_2 t^2 + 2^2 H(t)) \right| \leq 2^{\gamma/2}.$$

The same upper bound holds for S^- and the theorem follows.

Next we consider the case that $\frac{m}{2} - 1 \leq \gamma \leq m - 5$. Then, assuming that a is an $(n+1)$ -th power (mod $2^{m/2}$), we may write the set of solutions of the congruence $u^{n+1} \equiv a \pmod{2^{m/2}}$ as

$$\{2t + \alpha : 1 \leq t \leq 2^{\frac{m}{2}-1}\},$$

where α is a fixed value satisfying $\alpha^{n+1} \equiv a \pmod{2^{m/2}}$.

As with the case of odd p we write

$$a\bar{\alpha}^n = \alpha + \rho 2^{m/2},$$

for some integer ρ . Letting $f(t) = n(2t + \alpha) + a \overline{(2t + \alpha)}^n$, we have by Theorem 3.1 (i) that

$$K_n(a, 2^m) = 2^{mn/2} \sum_{t=1}^{\frac{m}{2}-1} e_{2^m}(f(t)).$$

Now, expanding $f(t)$ as above, we see that

$$f(t) \equiv a_0 + a_1 2^{\frac{m}{2}+1}t + a_2 2^{\gamma+1}t^2 + a_3 2^{\gamma+2}t^3 + a_4 2^{\gamma+2}t^4 + 2^{\gamma+3}H(t) \pmod{2^m},$$

for some integers a_r , $0 \leq r \leq 4$, with a_2, a_3, a_4 odd, and polynomial $H(t)$ over \mathbb{Z} . Let δ denote the multiplicity of 2 dividing the coefficient of t . Note, $\delta \geq \frac{m}{2} + 1$. If $\delta < \gamma + 1$ then $K_n(a, 2^m) = 0$. If $\delta \geq \gamma + 1$ then we obtain

$$|K_n(a, 2^m)| = 2^{mn/2} \left| \sum_{t=1}^{2^{\frac{m}{2}-1}} e_{2^{m-\gamma-1}}(a'_1 t + a_2 t^2 + 2a_3 t^3 + 2a_4 t^4 + 4H(t)) \right|,$$

for some integer a'_1 . If $\gamma = \frac{m}{2} - 1$ then a'_1 is even and so by inequality (3.7) of Lemma 3.4 we obtain

$$|K_n(a, 2^m)| \leq 2 \cdot 2^{\gamma/2} 2^{mn/2}.$$

If $\gamma \geq \frac{m}{2}$ then, by (3.6) we obtain

$$|K_n(a, 2^m)| \leq 2^{\frac{m\alpha}{2}} 2^{\gamma - \frac{m}{2}} 2^{\frac{m-\gamma+2}{2}} = 2 \cdot 2^{\frac{\gamma}{2}} 2^{mn/2}.$$

This completes the proof of the theorem for the case of even m .

The case of odd m . Suppose that $m \geq 3$ is odd. If $\gamma = 1$ then we trivially have from Theorem 3.1 (iii) that $|K_n(a, 2^m)| \leq 2^{\frac{m\alpha+3}{2}}$, for there are at most two values of u satisfying the constraints on the sum. Indeed, if a is not an $(n+1)$ -th power $(\text{mod } 2^\beta)$ then the sum is zero, and if a is such a power then, provided $\beta \geq 3$, there are precisely four distinct $(n+1)$ -th roots of $a \pmod{2^\beta}$ of which exactly two are roots $(\text{mod } 2^{\beta+1})$. If $\beta = 1$ or 2 the assertion is also trivial.

Suppose now that $\gamma \geq 2$. By Theorem 3.1 (iv) we have

$$(3.14) \quad |K_n(a, 2^m)| = 2^{\frac{m\alpha-1}{2}} \left| \sum_{\substack{u=1 \\ 2^{\beta+1} | (u^{n+1} - a)}}^{2^\beta} e_{2^m}(nu + a\bar{u}^n) \right|.$$

If $\gamma \geq m - 2$ then by Lemma 3.1,

$$|K_n(a, 2^m)| \leq 2^{\beta-1} 2^{\frac{m\alpha+1}{2}} \leq 2^{\frac{m}{2}-1} 2^{mn/2} = 2^{\frac{1}{2} \min(\gamma, m-2)} 2^{mn/2}.$$

Suppose now that $2 \leq \gamma \leq m - 3$ and that a is an $(n+1)$ -th power $(\text{mod } 2^{\beta+1})$. If $2 \leq \gamma \leq \beta - 1$ then by Lemma 3.2, a is also an $(n+1)$ -th power $(\text{mod } 2^m)$ and the set of solutions of the congruence $u^{n+1} \equiv a \pmod{2^{\beta+1}}$ is given by

$$\{2^{\beta+1-\gamma}t \pm \alpha : 1 \leq t \leq 2^\gamma\},$$

where α satisfies $\alpha^{n+1} \equiv a \pmod{2^m}$. By Theorem 3.1 (iv) we can write

$$|K_n(a, 2^m)| = 2^{\frac{m\alpha+1}{2}} |S^+ + S^-|,$$

where S^\pm are as given in (3.10) with $j = \beta + 1 - \gamma$. The proof of the theorem follows as above. If $\gamma \geq \beta$ then the set of solutions is given similarly by

$$\{2t + \alpha : 1 \leq t \leq 2^{\beta-1}\},$$

where α is a fixed value satisfying $\alpha^{n+1} \equiv a \pmod{2^{\beta+1}}$. The theorem again follows as above.

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