It is a widely held view that people have a tendency to take excessive risks under limited liability. When an individual does not have to bear full responsibility for the downside risk, the payoff from a project with return X is $\max\{W_0 + X, 0\}$ if the person has initial wealth W_0. Since the payoff function is convex, a mean-preserving spread in the distribution of X will increase the expected payoff. In the economics literature, such a preference for risky prospects occupies a prominent place in discussions about adverse selection problems in the credit market (e.g., Stiglitz and Weiss 1981).

Excessive risk-taking is literally a last-period problem: the game is stopped at bankruptcy. Economists have treated bankruptcy as if individuals had no interests in its consequences. However, even in the absence of direct bankruptcy costs, people do have an incentive to avoid bankruptcy. If the expected value of an income flow is positive, ending the game means forgoing a valuable option. I argue that the one-shot gain from taking risky prospects is more than offset by the long-term loss resulting from a higher probability of bankruptcy. Thus the incentive to avoid bankruptcy will translate into a preference for safer prospects on the part of risk-neutral individuals.

Here it is assumed that bankruptcy is not subject to choice. If the capital market were perfect, investor with positive net present value projects may find it optimal not to declare bankruptcy even if their current wealth is negative. However investors will not be able to commit themselves into not declaring bankruptcy once their debt reaches a certain critical level (see also McDonald and Siegel 1986). Imperfect information about the true value of the investment project makes creditors even more skeptical. The analysis that follows will carry through as long as creditors force investors into bankruptcy once that debt reaches some fixed level.

In the options literature (e.g., Black and Scoles 1973; Merton 1973; McDonald and Siegel 1986), an increased in riskiness of returns will increase the value of the project. This is because, by their decision to exercise their options, investors can effectively remove one side of the risk. Here it is assumed that bankruptcy is not optional, and the conclusion about the effect of riskiness on investment value is also reversed.

1. The Probability of Bankruptcy

In a one-period model, the probability of bankruptcy need not increase with the riskiness of an investment project. Given an initial wealth of W_0, the probability of bankruptcy is $F(-W_0)$, where F is the distribution function of the investment returns X. A mean-preserving spread in the distribution may increase or decrease the bankruptcy probability. However in a multi-period framework the probability that an investment survives bankruptcy does change systematically with its degree of riskiness.

Consider a case when project returns are accruable for more than one period provided bankruptcy has not occurred. Then the accumulation of wealth with the project is represented by the following stochastic process:

$$W_n = \begin{cases} \max\{W_{n-1} + X_n, 0\} & \text{if } W_{n-1} > 0; \\ 0 & \text{otherwise;} \end{cases} \tag{1}$$

where the X_n's are assumed to be independent and identically distributed. The accumulation process represented by equation (1) is a general random walk with absorbing barrier at zero. The probability that the process will eventually be terminated at the absorbing barrier can be obtained by using Wald's identity for sequential analysis (Wald 1947).

Let the net flow return X be a random variable with positive mean, and let its p.d.f. be $f(x)$. Its moment generating function is $f^*(\theta) = \int_{-\infty}^{\infty} e^{-\theta x} f(x)dx$. For a process that starts at W_0 and hits the absorbing barrier at time N (the stopping time N is random and possibly infinite), Wald's identity gives

$$E[e^{-\theta(W_N - W_0)} f^*(\theta)^{-N} | W_N \le 0] \Pr[W_N \le 0] = 1. \tag{2}$$

Let θ^* be the non-zero root to the equation $f^*(\theta) = 1$. Then, for $\theta = \theta^*$, equation (2) can be simplified to give the probability, π, that wealth eventually reaches the absorbing barrier at zero (i.e., the probability of eventual bankruptcy):

$$\pi = \Pr[W_N \le 0] \approx e^{-\theta^* W_0}. \tag{3}$$

The probability of eventual bankruptcy depends on the value of θ^*, which in turn depends on the shape of the moment generating function $f^*(\theta)$. Figure 1 shows the graph of $f^*(\theta)$. It is drawn in such a way that it satisfies the general properties of a moment generating function (for details, see Cox and Miller 1965, pp. 48–50): (1) $f^*(\theta) > 0$; (2) $\lim_{\theta\to\pm\infty} f^*(\theta) = \infty$; (3) $f^*(0) = 1$; (4) $df^*(0)/d\theta = -E[X]$; and (5) $f^*(\theta)$ is convex.

It is evident from the figure that there are two roots to the equation $f^*(\theta) = 1$, and that the non-zero root (i.e., θ^*) must be strictly positive. Thus the probability of eventual bankruptcy is strictly less than one if the accumulation process has a positive drift. Moreover, for such positive net present value prospects, the probability of eventual bankruptcy increases with the riskiness of the prospect.

Proposition 1. If $E[X] > 0$, a mean-preserving spread in the distribution of X will increase π.

Proof. Equation (3) shows that the probability of eventual bankruptcy is decreasing in θ^*. Thus it is sufficient to demonstrate that a mean-preserving spread in the distribution of X will decrease θ^*. Since $f^*(\theta) = E[e^{-\theta X}]$ and $e^{-\theta X}$ is a convex function in X, a mean-preserving spread in the distribution of X will increase $f^*(\theta)$ for all values of θ. In Figure 2, $g^*(\theta)$ shows the moment generating function of a random variable that is more risky than X. The graph of $g^*(\theta)$ lies everywhere above $f^*(\theta)$ except at $\theta = 0$ where both functions are equal to one. θ_g^* is the non-zero root to the equation $g^*(\theta) = 1$ and θ_f^* is the non-zero root to the equation $f^*(\theta) = 1$. Figure 2 clearly shows that $\theta_g^* < \theta_f^*$. ♡

2. The Behavior of Expected Wealth

To study the behavior of expected wealth, first consider the survival function: $H_n(w) = \Pr[W_n > w]$. If $W_{n-1} = u$ where $u > 0$, the probability that $W_n > w$ is equal to the probability that $u + X_n > w$. Let $h_{n-1}(w)$ be the p.d.f. of the random variable W_{n-1}. Then the survival function can be determined recursively by the relationship:

$$H_n(w) = \int_0^\infty [1 - F(w-u)]h_{n-1}(u)du, \tag{4}$$

with the initial condition $H_1(w) = 1 - F(w - W_0)$. For n large, $1 - H_n(0)$ is approximately equal to the probability of eventually bankruptcy. Formally,

$$\lim_{n\to\infty} 1 - H_n(0) = \pi. \tag{5}$$

The expected value of W_n is just $\int_0^\infty H_n(w)dw$. Thus, upon integration by parts,

$$\begin{aligned}
E[W_n] &= H_{n-1}(0)\int_0^\infty 1 - F(w)dw + \int_0^\infty [1 - F(-u)]H_{n-1}(u)du \\
&= \int_0^\infty H_{n-1}(w)dw + H_{n-1}(0)\Big(\int_0^\infty 1 - F(w)dw - \int_0^\infty F(-w)dw\Big) \\
&\quad + \int_0^\infty F(-w)[H_{n-1}(0) - H_{n-1}(w)]dw \\
&= E[W_{n-1}] + \Pr[W_{n-1} > 0]E[X] + \int_0^\infty F(-w)[H_{n-1}(0) - H_{n-1}(w)]dw. \quad (6)
\end{aligned}$$

Equation (6) shows that the growth of expected wealth depends on the mean flow income times the survival probability, plus an adjustment term for the limited liability truncation at zero.

For n large, the adjustment term due to truncation will be insignificant. To show this, let $F(-x) = 0$ for $x \leq -B$ for some finite B. The last term in equation (6) is then equal to $\int_0^B F(-w)[H_{n-1}(0) - H_{n-1}(w)]dw$, which is in turn less than $[H_{n-1}(0) - H_{n-1}(B)]\int_0^B F(-w)dw$. The term in square brackets is the probability that W_{n-1} lies within $(0, B]$. This probability is less than the probability that an unrestricted random walk $W_0 + \sum_1^{n-1} X_i$ lies within $(0, B]$. If the random walk has a positive drift, by the central limit theorem, the latter probability approaches zero as n approaches infinity. Thus the assumption that the support of X is bounded below ensures that the gain from limited liability truncation will be finite. Equations (5) and (6) then together imply:

$$\lim_{n\to\infty} E[W_n] - E[W_{n-1}] = E[X](1 - \pi). \tag{7}$$

That is, the long run growth in expected wealth is equal to the mean flow income times the probability of long term survival.

Since a mean-preserving spread in the distribution of X will increase π, the following proposition is established:

Proposition 2. If the net income flow from two prospects have the same positive mean but different degree of riskiness, adopting the more risky prospect will result in a lower long run growth in expected wealth.

It follows from Proposition 2 that adopting the more risky prospect will result in a lower level of expected wealth $E[W_n]$ for n sufficiently large. When risk-neutral individuals are undertaking long term prospects, they will therefore tend to avoid risks.[1]

3. Additional Implications

A. Decreasing Risk Aversion

Suppose an investment project will last n periods provided it is not terminated by bankruptcy. When n is large, expected wealth at period n is approximately equal to initial wealth plus n times the long run growth. Thus $E[W_n] \approx W_0 + nE[X](1 - e^{-\theta^* W_0})$. Let the rate of return on wealth be ρ where ρ is defined by $W_0(1+\rho)^n = E[W_n]$. Taking logarithms on both sides and using the approximation $\log(1+x) \approx x$, then

$$\rho \approx E[X](1 - e^{-\theta^* W_0})/W_0. \tag{8}$$

The analysis in Section 2 shows that an increase in riskiness of an investment project will decrease long run growth in wealth. In terms of the rate of return on wealth, $\partial\rho/\partial\theta^* = E[X]e^{-\theta^* W_0} > 0$. However, as W_0 increases, the probability of bankruptcy falls for all θ^* and individuals will be less unwilling to undertake the risky projects. In particular,

$$\frac{\partial^2 \rho}{\partial\theta^* \partial W_0} = -E[X]\theta^* e^{-\theta^* W_0} < 0. \tag{9}$$

Thus one would expect that the more risky investment projects are undertaken primarily by people with large initial wealth.

B. Moral Hazard

Economists have argued that limited liability reduces the incentive for individuals to undertake costly actions that probabilistically increase output (see, for example, Sappington 1983). To illustrate this moral hazard effect, let $F(x;a)$ be the distribution function of output when input level is a and let $c(a)$ be the cost. Assume $c' > 0$, $c'' > 0$

[1] Our results need not be interpreted in terms of a long run model. One can think of a model where an investment project lasts for some fixed period. Investment returns are accruable at intervals of length Δ. Then, instead of taking the limit as n goes to infinity, one can take the limit as Δ approaches zero. The propositions in this paper still hold.

and $F_a \leq 0$ so that higher input increases output in the sense of first order stochastic dominance. With unlimited liability, a risk-neutral individual will choose a such that $c'(a) = \partial E[X]/\partial a = \int_{-\infty}^{\infty} -F_a(x;a)dx$. Under limited liability, on the other hand, the marginal private benefit from supplying the costly input is only $\int_0^{\infty} -F_a(x;a)dx$, which is less than $\partial E[X]/\partial a$. Limited liability therefore results in a sub-optimal level of input.

In the multi-period framework presented here, however, the moral hazard problem is less severe than what a one-period model would suggest. Proposition 3 below shows that shirking increases the probability of eventual bankruptcy. Thus the one-time gain from shirking may be offset by the long term loss resulting from the early termination of positive present value projects.

Proposition 3. A first order stochastic increase in the distribution of the flow returns X will reduce the probability of eventual bankruptcy.

Proof. From equation (3), it is sufficient to prove that θ^* increases when there is a first order stochastic increase in X. Since $f^*(\theta) = E[e^{-\theta X}]$ and $e^{-\theta X}$ is a decreasing function in X for $\theta > 0$ and it is an increasing function in X for $\theta < 0$, a first order stochastic increase in X will reduce $f^*(\theta)$ for $\theta > 0$ and will increase $f^*(\theta)$ for $\theta < 0$. In Figure 3, $g^*(\theta)$ is the moment generating function corresponding to a distribution that stochastically dominates X. It is evident from the figure that $\theta_g^* > \theta_f^*$. ♡

Consider now a project that could potentially last n periods. The marginal benefit from increasing the level of input is

$$\frac{\partial E[X]}{\partial a} n(1 - e^{-\theta^* W_0}) + \frac{\partial \theta^*}{\partial a} n e^{-\theta^* W_0} E[X] W_0. \tag{10}$$

The first term in (10) is less than $(\partial E[X]/\partial a)n$. It indicates the reduced incentive to supply inputs in a static model of limited liability. However, since $\partial\theta^*/\partial a > 0$ by Proposition 3, the second term in (10) is positive. Thus, when there is a cost to bankruptcy in terms of forgone future profit opportunities, the moral hazard problem resulting from limited liability will be mitigated.

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RISK AVOIDANCE UNDER LIMITED LIABILITY

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Abstract. When the expected flow returns from an investment is positive, bankruptcy carries a cost in terms of the future profit opportunities forgone. This paper demonstrates that, under limited liability, the one-shot gain from taking risky projects is offset by the long-term loss resulting from a higher probability of bankruptcy. In a multi-period model, the incentive problems associated with limited liability is less severe than what static models would suggest.

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