RIESZ BASIS PROPERTY OF TIMOSHENKO BEAMS WITH BOUNDARY FEEDBACK CONTROL

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A Timoshenko beam equation with boundary feedback control is considered. By an abstract result on the Riesz basis generation for the discrete operators in the Hilbert spaces, we show that the closed-loop system is a Riesz system, that is, the sequence of generalized eigenvectors of the closed-loop system forms a Riesz basis in the state Hilbert space.

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1. Introduction. The boundary feedback stabilization problem of a hybrid system has been studied extensively in the last decade. Many important results have been obtained. Among them, most of studies in the literatures are concerned with Euler-Bernoulli and Rayleigh beams; there are a few results for Timoshenko beams (cf. [3, 5, 6, 7, 9]), which are mainly focused on the stability of the closed-loop system. Though it is important to obtain the exponential stability of the system, it is also very interesting to study the rate of the exponential decay of the system. It is well known that if the system satisfies the spectrum-determined growth assumption, then the rate of the exponential decay can be easily estimated via the spectra of the system operator, see [2]. Furthermore, if the system is a Riesz one, that is, the set of the generalized eigenvectors of the system operator forms a Riesz basis of the state Hilbert space, then the spectrum-determined growth assumption is satisfied. In [1], the Riesz basis property was used to give some quantitative information of the rate of the exponential decay for a simpler Euler-Bernoulli beam system with no tip mass. For Euler-Bernoulli and Rayleigh beam systems, some further results concerning the Riesz basis property of the systems can be found in [4, 8].

In the present note, we consider the following Timoshenko beam equation with a tip mass (see [7, 9]):

$$\rho \ddot{w}(x,t) - K(w''(x,t) - \varphi'(x,t)) = 0, \quad 0 < x < \ell,$$
$$I_\rho \ddot{\varphi}(x,t) - EI\varphi''(x,t) - K(w'(x,t) - \varphi(x,t)) = 0, \quad 0 < x < \ell,$$
$$w(0,t) = 0, \qquad \varphi(0,t) = 0, \tag{1.1}$$
$$M\ddot{w}(\ell,t) = -K(w'(\ell,t) - \varphi(\ell,t)) + u_1(t),$$
$$J\ddot{\varphi}(\ell,t) = -EI\varphi'(\ell,t) + u_2(t),$$

with boundary feedback control

$$
\begin{aligned}
u_1(t) &= -\alpha \dot{w}(\ell,t) - \gamma[\dot{w}'(\ell,t) - \dot{\varphi}(\ell,t)], \\
u_2(t) &= \beta \dot{\varphi}(\ell,t) - \nu \dot{\varphi}'(\ell,t).
\end{aligned}
\tag{1.2}
$$

Here, I_ρ, ρ, EI, K, and ℓ are mass moment of inertia, mass density, rigidity coefficient, shear modulus of elasticity, and length of the beam, respectively, and α, β, γ, and ν are positive feedback constants. Here, and henceforth, the dot and the prime denote derivatives with respect to time and space variables, respectively. In [9], the energy multiplier method is used to show the stability of the closed-loop system (1.1) with (1.2). In this note, the Riesz basis property of this system is proven, and hence the spectrum-determined growth assumption of the system is satisfied. Finally, based on this consideration, we show the exponential stability of the closed-loop system (1.1) with (1.2) via estimating the eigenvalues of $\mathscr{A}$.

2. The state space and eigenvalue problem. First, we recall the state space and the operator defined in [9]. Let

$$
\mathscr{H} = V_0^1 \times L_\rho^2(0,\ell) \times V_0^1 \times L_{I_\rho}^2(0,\ell) \times \mathbb{R} \times \mathbb{R},
\tag{2.1}
$$

where $V_0^k = \{\varphi \in H^k(0,\ell) \mid \varphi(0) = 0\}$, $k = 1,2$, with $H^k(0,\ell)$ the usual Sobolev space of order k. For $Y_1 = [w_1, z_1, \varphi_1, \psi_1, \xi_1, \eta_1]^T$, $Y_2 = [w_2, z_2, \varphi_2, \psi_2, \xi_2, \eta_2]^T \in \mathscr{H}$, where the superscript T denotes the transposition of a matrix, the inner product in $\mathscr{H}$ is defined by

$$
\begin{aligned}
\langle Y_1, Y_2 \rangle &= \int_0^\ell K w_1' \overline{w_2'} \, dx + \int_0^\ell \rho z_1 \overline{z_2} \, dx \\
&\quad + \int_0^\ell EI \varphi_1' \overline{\varphi_2'} \, dx + \int_0^\ell I_\rho \psi_1 \overline{\psi_2} \, dx + M^{-1} \xi_1 \overline{\xi_2} + J^{-1} \eta_1 \overline{\eta_2}.
\end{aligned}
\tag{2.2}
$$

Define the linear operator $\mathscr{A}$ in $\mathscr{H}$ by

$$
\mathscr{A}
\begin{pmatrix} w \\ z \\ \varphi \\ \psi \\ \xi \\ \eta \end{pmatrix}
=
\begin{pmatrix}
z \\
\dfrac{K}{\rho}(w'' - \varphi') \\
\psi \\
\dfrac{EI}{I_\rho}\varphi'' + \dfrac{K}{I_\rho}(w' - \varphi) \\
-\alpha z(\ell) - K(w'(\ell) - \varphi(\ell)) \\
-\beta \psi(\ell) - EI\varphi'(\ell)
\end{pmatrix}
\tag{2.3}
$$

with the domain

$$
\mathcal{D}(\mathcal{A}) = \Big\{ Y = [w,z,\varphi,\psi,\xi,\eta]^T \in \mathcal{H} \mid w, \varphi \in V_0^2, \ z,\psi \in V_0^1,
$$
$$
\xi = Mz(\ell) + \gamma(w'(\ell) - \varphi(\ell)), \ \eta = J\psi(\ell) + \nu\varphi'(\ell) \Big\}.
\tag{2.4}
$$

Then the closed-loop system (1.1) with (1.2) becomes the following evolution equation in $\mathcal{H}$:

$$
\frac{d}{dt}Y(t) = \mathcal{A}Y(t), \quad \forall t > 0,
\tag{2.5}
$$

where

$$
Y(t) = \big[w(\cdot,t), \dot{w}(\cdot,t), \varphi(\cdot,t), \dot{\varphi}(\cdot,t), M\dot{w}(\ell,t)
$$
$$
+ \gamma(w'(\ell,t) - \varphi(\ell,t)), J\dot{\varphi}(\ell,t) + \nu\varphi'(\ell,t) \big]^T.
\tag{2.6}
$$

The following lemma can be found in [9].

LEMMA 2.1. *The operator $\mathcal{A}$ is dissipative and generates a C_0-semigroup with exponential decay.*

In order to investigate the rate of the exponential decay of the closed-loop system (1.1) with (1.2), we study the Riesz basis property of the generalized eigenvector system of $\mathcal{A}$. For this purpose, we need the following lemma.

LEMMA 2.2. *The operator $\mathcal{A}$ has compact resolvent on $\mathcal{H}$.*

PROOF. It follows from Lemma 2.1 that $0 \in \rho(\mathcal{A})$. Then for any $F = [f_1, f_2, g_1, g_2, \zeta_1, \zeta_2]^T \in \mathcal{H}$, there is a unique element $Y = [w,z,\varphi,\psi,\xi,\eta]^T \in \mathcal{H}$ such that $\mathcal{A}Y = F$, that is,

$$
z = f_1, \quad \frac{K}{\rho}(w'' - \varphi') = f_2, \quad \psi = g_1, \quad \frac{EI}{I_\rho}\varphi'' + \frac{K}{I_\rho}(w' - \varphi) = g_2,
$$
$$
-\alpha z(\ell) - K(w'(\ell) - \varphi(\ell)) = \zeta_1, \quad -\beta\psi(\ell) - EI\varphi'(\ell) = \zeta_2,
\tag{2.7}
$$

from which we obtain

$$
Y = \begin{pmatrix} \int_0^x \varphi(s)\,ds - \dfrac{1}{K}(\alpha f_1(\ell) + \zeta_1)x + \dfrac{\rho}{K}\int_0^\ell k(x,s)f_2(s)\,ds \\ f_1(x) \\ \varphi(x) \\ g_1(x) \\ Mf_1(\ell) - \dfrac{\gamma}{K}[\alpha f_1(\ell) + \zeta_1] \\ J\psi(\ell) + \dfrac{\nu}{EI}[\beta\psi(\ell) + \zeta_2] \end{pmatrix},
\tag{2.8}
$$

where

$$\varphi(x) = -\frac{1}{EI}\Bigg[(\beta g_1(\ell) + \zeta_2)x + I_\rho \int_0^\ell k(x,s)g_2(s)ds$$

$$+ (\alpha f_1(\ell) + \zeta_1)\left(\ell x + \frac{x^2}{2}\right) + \rho\int_0^\ell k(x,s)ds\int_s^\ell f_2(r)dr\Bigg], \quad (2.9)$$

$$k(x,s) = \begin{cases} s, & 0 \le s \le x, \\ x, & x \le s \le \ell. \end{cases}$$

Then it is easy to see the compactness of $\mathscr{A}^{-1}$.   □

Now, we consider the eigenvalue problem of $\mathscr{A}$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathscr{A}$ and $Y = [w,z,\varphi,\psi,\xi,\eta]^T \in \mathscr{H}$ an eigenvector corresponding to λ, then the functions $w(x)$ and $\varphi(x)$ satisfy

$$\rho\lambda^2 w(x) - K(w''(x) - \varphi'(x)) = 0, \quad 0 < x < \ell,$$

$$I_\rho\lambda^2\varphi(x) - EI\varphi''(x) - K(w'(x) - \varphi(x)) = 0, \quad 0 < x < \ell,$$

$$w(0) = \varphi(0) = 0, \quad\quad\quad (2.10)$$

$$\lambda^2 Mw(\ell) + K(w'(\ell) - \varphi(\ell)) = -\alpha\lambda w(\ell) - \lambda\gamma(w'(\ell) - \varphi(\ell)),$$

$$\lambda^2 J\varphi(\ell) + EI\varphi(\ell) = \beta\lambda\varphi(\ell) - \lambda\nu\varphi'(\ell)).$$

Set

$$\rho_1^2 = \frac{\rho}{K}, \qquad \rho_2^2 = \frac{I_\rho}{EI}, \qquad a = \rho_1^2\lambda^2, \qquad b = \rho_2^2\lambda^2 + \frac{K}{EI}, \qquad c = -\frac{K}{EI}. \quad (2.11)$$

Denote by μ_1 and μ_2 the two roots of the quadratic equation

$$\mu^2 - (a+b+c)\mu + ab = 0, \quad\quad\quad (2.12)$$

that is,

$$\mu_1 = \frac{(a+b+c) + \sqrt{(a+b+c)^2 - 4ab}}{2},$$

$$\quad\quad\quad (2.13)$$

$$\mu_2 = \frac{(a+b+c) - \sqrt{(a+b+c)^2 - 4ab}}{2}.$$

In the case of $\mu_1 \neq \mu_2$, we define functions $w_j(\lambda, x)$ and $\varphi_j(\lambda, x)$ for $j = 3, 4$ by

$$w_3(\lambda, x) = \frac{1}{\mu_1 - \mu_2}((\mu_1 - b)\mu_1^{-1/2} \sinh \sqrt{\mu_1} x - (\mu_2 - b)\mu_2^{-1/2} \sinh \sqrt{\mu_2} x),$$

$$w_4(\lambda, x) = \frac{1}{\mu_1 - \mu_2}(\cosh \sqrt{\mu_1} x - \cosh \sqrt{\mu_2} x),$$

$$\varphi_3(\lambda, x) = \frac{1}{\mu_1 - \mu_2} c(\cosh \sqrt{\mu_1} x - \cosh \sqrt{\mu_2} x),$$

$$\varphi_4(\lambda, x) = \frac{1}{\mu_1 - \mu_2}((\mu_1 - a)\mu_1^{-1/2} \sinh \sqrt{\mu_1} x - (\mu_2 - a)\mu_2^{-1/2} \sinh \sqrt{\mu_2} x).$$

$$(2.14)$$

Set

$$w(\lambda, x) = Aw_3(\lambda, x) + Bw_4(\lambda, x),$$
$$\varphi(\lambda, x) = A\varphi_3(\lambda, x) + B\varphi_4(\lambda, x), \tag{2.15}$$

where A and B are two constants to be determined. Then $w(\lambda, x)$, $\varphi(\lambda, x)$ satisfy

$$\rho \lambda^2 w(x) - K(w''(x) - \varphi'(x)) = 0, \quad 0 < x < \ell,$$
$$I_\rho \lambda^2 \varphi(x) - EI\varphi''(x) - K(w''(x) - \varphi'(x)) = 0, \quad 0 < x < \ell, \tag{2.16}$$
$$w(0) = \varphi(0) = 0, \qquad w'(0) = A, \qquad \varphi'(0) = B.$$

From (2.10), we obtain

$$A[(\lambda^2 M + \alpha\lambda)w_3(\lambda, \ell) + (K + \gamma\lambda)(w_3'(\lambda, \ell) - \varphi_3(\lambda, \ell))]$$
$$+ B[(\lambda^2 M + \alpha\lambda)w_4(\lambda, \ell) + (K + \lambda\gamma)(w_4'(\lambda, \ell) - \varphi_4(\lambda, \ell))] = 0,$$
$$A[(\lambda^2 J + \beta\lambda)\varphi_3(\lambda, \ell) + (EI + \lambda\nu)\varphi_3'(\lambda, \ell)]$$
$$+ B[(\lambda^2 J + \beta\lambda)\varphi_4(\lambda, \ell) + (EI + \lambda\nu)\varphi_4'(\lambda, \ell)] = 0. \tag{2.17}$$

For an eigenpair (λ, Y) of $\mathcal{A}$, the determinant $\Gamma(\lambda)$ of the coefficient matrix of the above linear equation system must be vanishing. Here

$$\Gamma(\lambda) = \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}, \tag{2.18}$$

where

$$y_{11} = (\lambda^2 M + \alpha\lambda)w_3(\lambda, \ell) + (K + \lambda\gamma)(w_3'(\lambda, \ell) - \varphi_3(\lambda, \ell)),$$
$$y_{12} = (\lambda^2 M + \alpha\lambda)w_4(\lambda, \ell) + (K + \lambda\gamma)(w_4'(\lambda, \ell) - \varphi_4(\lambda, \ell)),$$
$$y_{21} = (\lambda^2 J + \beta\lambda)\varphi_3(\lambda, \ell) + (EI + \lambda\nu)\varphi_3'(\lambda, \ell),$$
$$y_{22} = (\lambda^2 J + \beta\lambda)\varphi_4(\lambda, \ell) + (EI + \lambda\nu)\varphi_4'(\lambda, \ell). \tag{2.19}$$

Therefore, we have the following result.

THEOREM 2.3. *Assume that $w(\lambda, x)$, $\varphi(\lambda, x)$, and $\Gamma(\lambda)$ are defined as before. Let $\lambda \in \mathbb{C}$ be such that $\mu_1 \neq \mu_2$. Then $\lambda \in \sigma(\mathcal{A})$ if and only if $\Gamma(\lambda) = 0$. In this case, an eigenvector of $\mathcal{A}$ corresponding to λ is*

$$Y = \begin{pmatrix} w(\lambda, x) \\ \lambda w(\lambda, x) \\ \varphi(\lambda, x) \\ \lambda \varphi(\lambda, x) \\ M\lambda w(\lambda, \ell) + \gamma(w'(\lambda, \ell) - \varphi(\lambda, \ell)) \\ J\lambda \varphi(\lambda, \ell) + \nu \varphi'(\lambda, \ell) \end{pmatrix}. \tag{2.20}$$

3. Riesz basis property of generalized eigenvector system of $\mathcal{A}$. In this section, we study the Riesz basis property of generalized eigenvector system of $\mathcal{A}$. We recall that the basis $\{\varphi_n \mid n \geq 1\}$ of a Hilbert space $\mathcal{H}$ is said to be a Riesz basis if it is equivalent to some orthonormal basis $\{e_n \mid n \geq 1\}$ of $\mathcal{H}$, that is, there is a bounded invertible linear operator T on $\mathcal{H}$ such that $T\varphi_n = e_n$ for all $n \geq 1$. For the linear system (2.5), if the set of the generalized eigenvectors of the operator $\mathcal{A}$ forms a Riesz basis of the state Hilbert space $\mathcal{H}$, then the linear system (2.5) is called a Riesz system.

In the sequel, we prove that (2.5) is indeed a Riesz system. The following lemma can be found in [4].

LEMMA 3.1 [4]. *Let $\mathcal{A}$ be a densely defined discrete operator in a Hilbert space $\mathcal{H}$ and $\{\phi_n \mid n \geq 1\}$ a Riesz basis of $\mathcal{H}$. Assume that there are an integer $N_1 \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n \mid n > N_1\}$ of $\mathcal{A}$ such that*

$$\sum_{n=N_1+1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty. \tag{3.1}$$

Then the following assertions hold.

(1) There are an integer $N_2 > N_1$ and generalized eigenvectors $\{\tilde{\psi}_n \mid 1 \leq n \leq N_2\}$ of $\mathcal{A}$ such that $\{\tilde{\psi}_n \mid 1 \leq n \leq N_2\} \cup \{\psi_n \mid n > N_2\}$ forms a Riesz basis of $\mathcal{H}$.

(2) Let $\{\tilde{\psi}_n \mid 1 \leq n \leq N_2\} \cup \{\psi_n \mid n > N_2\}$ correspond to eigenvalues $\{\sigma_n \mid n \geq 1\}$ of $\mathcal{A}$. Then $\sigma(\mathcal{A}) = \{\sigma_n \mid n \geq 1\}$, where each σ_n is counted according to its algebraic multiplicity.

(3) If there is an integer $N_3 > 0$ such that $\sigma_n \neq \sigma_m$ for all $n, m > N_3$, then there is an integer $N_4 > N_3$ such that any σ_n for $n > N_4$ is algebraically simple.

From Lemma 3.1, it follows that in order to obtain the Riesz basis property of generalized eigenvector system of $\mathcal{A}$, we need to know some eigenvalues with corresponding eigenvectors of $\mathcal{A}$ and their asymptotic behavior.

Now, we discuss the asymptotic behavior of eigenvalues of $\mathcal{A}$. A lengthy computation shows that when $J - \rho_2 \nu \neq 0$ and $M - \rho_1 \gamma \neq 0$,

$$\lim_{h \to -\infty} \inf_{\operatorname{Re}\lambda \leq h} |\Gamma(\lambda)| > 0. \tag{3.2}$$

Thus, for $\lambda \in \sigma(\mathscr{A})$, it is sufficient to consider λ lying in some vertical zone of complex plane, parallel to the imaginary axis. For λ in this zone with $|\lambda|$ large enough, we have

$$
\frac{\Gamma(\lambda)}{\lambda^2}
$$

$$
= \begin{vmatrix} \dfrac{M}{\rho_1} \sinh \ell \sqrt{\mu_1} + \gamma \cosh \ell \sqrt{\mu_1} + O(\lambda^{-1}) & O(\lambda^{-1}) \\[2mm] O(\lambda^{-1}) & \dfrac{J}{\rho_2} \sinh \ell \sqrt{\mu_2} + \nu \cosh \ell \sqrt{\mu_2} + O(\lambda^{-1}) \end{vmatrix}
$$

$$
= 0.
$$

(3.3)

In the case of $J - \rho_2 \nu \neq 0$ and $M - \rho_1 \gamma \neq 0$, for $n \in \mathbb{Z}$, the set of all integers denote

$$
\begin{aligned}
\omega_n^{(1)} &= \begin{cases} \dfrac{1}{2\ell} \ln \left| \dfrac{M - \rho_1 \gamma}{\rho_1 \gamma + M} \right| + \dfrac{n\pi}{\ell} i, & \text{if } M > \rho_1 \gamma, \\[3mm] \dfrac{1}{2\ell} \ln \left| \dfrac{M - \rho_1 \gamma}{\rho_1 \gamma + M} \right| + \dfrac{(2n+1)\pi}{2\ell} i, & \text{if } M < \rho_1 \gamma, \end{cases} \\[5mm]
\omega_n^{(2)} &= \begin{cases} \dfrac{1}{2\ell} \ln \left| \dfrac{J - \rho_2 \nu}{\rho_2 \nu + J} \right| + \dfrac{n\pi}{\ell} i, & \text{if } J > \rho_2 \nu, \\[3mm] \dfrac{1}{2\ell} \ln \left| \dfrac{J - \rho_2 \nu}{\rho_2 \nu + J} \right| + \dfrac{(2n+1)\pi}{2\ell} i, & \text{if } J < \rho_2 \nu. \end{cases}
\end{aligned}
$$

(3.4)

Then

$$
\begin{aligned}
M \sinh \ell \omega_n^{(1)} + \gamma \rho_1 \cosh \ell \omega_n^{(1)} &= 0, \\
J \sinh \ell \omega_n^{(2)} + \nu \rho_2 \cosh \ell \omega_n^{(2)} &= 0.
\end{aligned}
$$

(3.5)

Set

$$
\lambda_n^{(1)} = \rho_1^{-1} \omega_n^{(1)} + \alpha_n^{(1)}, \qquad \lambda_n^{(2)} = \rho_2^{-1} \omega_n^{(2)} + \alpha_n^{(2)},
$$

(3.6)

and let $\lambda_n^{(j)} \in \sigma(\mathscr{A})$ for $j = 1, 2$, then for $|n|$ large enough, we have $\alpha_n^{(j)} = O(n^{-1})$ for $j = 1, 2$.

Now, we consider the eigenvectors of $\mathscr{A}$. For $\lambda = \lambda_n^{(1)}$, take

$$
\begin{aligned}
w'(\lambda, 0) &= (\lambda^2 J + \lambda \beta) \varphi_4(\lambda, \ell) + (EI + \lambda \nu) \varphi_4'(\lambda, \ell), \\
\varphi'(\lambda, 0) &= -(\lambda^2 J + \lambda \beta) \varphi_3(\lambda, \ell) - (EI + \lambda \nu) \varphi_3'(\lambda, \ell),
\end{aligned}
$$

(3.7)

and for $\lambda = \lambda_n^{(2)}$, take

$$
\begin{aligned}
w'(\lambda, 0) &= (\lambda^2 M + \lambda \alpha) w_4(\lambda, \ell) + (K + \lambda \gamma) w_4'(\lambda, \ell), \\
\varphi'(\lambda, 0) &= -(\lambda^2 M + \lambda \alpha) w_3(\lambda, \ell) - (K + \lambda \gamma) w_3'(\lambda, \ell).
\end{aligned}
$$

(3.8)

Set

$$A_n = J\rho_2^{-1} \sinh\lambda_n^{(1)}\rho_2\ell + v\cosh\lambda_n^{(1)}\rho_2\ell,$$
$$B_n = M\rho_1^{-1} \sinh\lambda_n^{(2)}\rho_1\ell + v\cosh\lambda_n^{(2)}\rho_1\ell. \tag{3.9}$$

Then with $|n|$ large enough, we have

$$w'(\lambda_n^{(1)},0) = A_n\lambda_n^{(1)} + O(1), \qquad \varphi'(\lambda_n^{(1)},0) = O(1),$$
$$\varphi'(\lambda_n^{(2)},0) = B_n\lambda_n^{(2)} + O(1), \qquad w'(\lambda_n^{(2)},0) = O(1). \tag{3.10}$$

For $\lambda \in \sigma(\mathscr{A})$ and $j = 3,4$, denote

$$Y_j(\lambda) = \begin{pmatrix} w_j(\lambda,x) \\ \lambda w_j(\lambda,x) \\ \varphi_j(\lambda,x) \\ \lambda\varphi_j(\lambda,x) \\ M\lambda w_j(\lambda,\ell) + \gamma\left(w_j'(\lambda,\ell) - \varphi_j(\lambda,\ell)\right) \\ J\lambda\varphi_j(\lambda,\ell) + v\varphi_j'(\lambda,\ell) \end{pmatrix}, \tag{3.11}$$

then according to Theorem 2.3, as an eigenvector of $\mathscr{A}$, we can take

$$Y = \lambda^{-1}w'(\lambda,0)Y_3(\lambda) + \lambda^{-1}\varphi'(\lambda,0)Y_4(\lambda). \tag{3.12}$$

Based on the above discussion, now we are able to prove the main result of the present note.

THEOREM 3.2. *Let $\mathscr{H}$ and $\mathscr{A}$ be defined as before. If $J - \rho_2 v \neq 0$ and $M - \rho_1\gamma \neq 0$, then the generalized eigenvector system of $\mathscr{A}$ forms a Riesz basis of $\mathscr{H}$. Moreover, the eigenvalues of $\mathscr{A}$ with large module are algebraically simple.*

PROOF. Assume, without loss of the generality, that $\rho_1 \neq \rho_2$, then for $\lambda \in \mathbb{C}$ with $|\lambda|$ large enough, it follows that $\sqrt{\mu_1} = \rho_1\lambda + O(\lambda^{-1})$ and $\sqrt{\mu_2} = \rho_2\lambda + O(\lambda^{-1})$. So, for $\lambda = \lambda_n^{(1)}$ with $|n| \geq N$, where N is a sufficiently large positive integer, we have

$$\sqrt{\mu_1} = \rho_1\lambda_n^{(1)} + O(n^{-1}) = \omega_n^{(1)} + O(n^{-1}),$$
$$\sqrt{\mu_2} = \rho_2\lambda_n^{(1)} + O(n^{-1}) = \rho_2\rho_1^{-1}\omega_n^{(1)} + O(n^{-1}), \tag{3.13}$$

and for $\lambda = \lambda_n^{(2)}$ with $|n| \geq N$, we have

$$\sqrt{\mu_1} = \rho_1\lambda_n^{(2)} + O(n^{-1}) = \rho_1\rho_2^{-1}\omega_n^{(2)} + O(n^{-1}),$$
$$\sqrt{\mu_2} = \rho_2\lambda_n^{(2)} + O(n^{-1}) = \omega_n^{(2)} + O(n^{-1}). \tag{3.14}$$

Therefore, for $\lambda_n^{(1)} \in \sigma(\mathscr{A})$ with $|n|$ large enough, we have

$$Y_3(\lambda_n^{(1)}) = \left[\omega_n^{(1)\,-1}\sinh\omega_n^{(1)}x, \rho_1^{-1}\sinh\omega_n^{(1)}x, 0, 0, 0, 0\right]^T + G_1(\lambda_n^{(1)}), \quad (3.15)$$

where $\|G_1(\lambda_n^{(1)})\| = O(n^{-1})$ and $Y_4(\lambda_n^{(1)}) = O(1)$.

Similarly, for $\lambda_n^{(2)} \in \sigma(\mathscr{A})$ with $|n|$ large enough, we have

$$Y_4(\lambda_n^{(2)}) = \left[0, 0, \omega_n^{(2)\,-1}\sinh\omega_n^{(2)}x, \rho_2^{-1}\sinh\omega_n^{(2)}x, 0, 0\right]^T + G_2(\lambda_n^{(1)}), \quad (3.16)$$

where $\|G_2(\lambda_n^{(2)})\| = O(n^{-1})$ and $Y_3(\lambda_n^{(2)}) = O(1)$. Thus, it follows that

$$Y(\lambda_n^{(1)}) = A_n\left[\omega_n^{(1)\,-1}\sinh\omega_n^{(1)}x, \rho_1^{-1}\sinh\omega_n^{(1)}x, 0, 0, 0, 0\right]^T + F_1(\lambda_n^{(1)}) \quad (3.17)$$

with $\|F_1(\lambda_n^{(1)})\| = O(n^{-1})$, and that

$$Y(\lambda_n^{(2)}) = B_n\left[0, 0, \omega_n^{(2)\,-1}\sinh\omega_n^{(2)}x, \rho_2^{-1}\sinh\omega_n^{(2)}x, 0, 0\right]^T + F_2(\lambda_n^{(2)}) \quad (3.18)$$

with $\|F_2(\lambda_n^{(2)})\| = O(n^{-1})$.

Noticing that

$$0 < \inf_{n \in \mathbb{Z}}|A_n| < \sup_{n \in \mathbb{Z}}|A_n| < \infty, \qquad 0 < \inf_{n \in \mathbb{Z}}|B_n| < \sup_{n \in \mathbb{Z}}|B_n| < \infty, \quad (3.19)$$

it remains to prove that the sequences $\{\Phi_n \mid n \in \mathbb{Z}\} \cup \{\Psi_n \mid n \in \mathbb{Z}\}$, defined by

$$\begin{aligned}\Phi_n &= \left[\omega_n^{(1)\,-1}\sinh\omega_n^{(1)}x, \rho_1^{-1}\sinh\omega_n^{(1)}x, 0, 0, 0, 0\right]^T, \\ \Psi_n &= \left[0, 0, \omega_n^{(2)\,-1}\sinh\omega_n^{(2)}x, \rho_2^{-1}\sinh\omega_n^{(2)}x, 0, 0\right]^T,\end{aligned} \quad (3.20)$$

form a Riesz basis of the subspace $\mathscr{H}_1$ of $\mathscr{H}$, where

$$\mathscr{H}_1 = \{Y \mid Y = [w, z, \varphi, \psi, 0, 0]^T \in \mathscr{H}\}. \quad (3.21)$$

Obviously, it is equivalent to prove that the sequence

$$\left\{[\cosh\omega_n^{(1)}x, \rho_1^{-1}\sinh\omega_n^{(1)}x]^T \mid n \in \mathbb{Z}\right\} \quad (3.22)$$

forms a Riesz basis of $L_K^2(0,\ell) \times L_\rho^2(0,\ell)$ and that the sequence

$$\left\{ [\cosh\omega_n^{(2)}x, \rho_2^{-1}\sinh\omega_n^{(2)}x]^T \mid n \in \mathbb{Z} \right\} \tag{3.23}$$

forms a Riesz basis of $L_{EI}^2(0,\ell) \times L_{I_\rho}^2(0,\ell)$.

In the case of $M > \rho_1 \gamma$, we define operator $\mathcal{T}$ by

$$\mathcal{T} = \begin{pmatrix} \cosh\left(\dfrac{x}{2\ell}\ln\left|\dfrac{M-\rho_1\gamma}{M+\rho_1\gamma}\right|\right) & \rho_1\sinh\left(\dfrac{x}{2\ell}\ln\left|\dfrac{M-\rho_1\gamma}{M+\rho_1\gamma}\right|\right) \\ \rho_1^{-1}\sinh\left(\dfrac{x}{2\ell}\ln\left|\dfrac{M-\rho_1\gamma}{M+\rho_1\gamma}\right|\right) & \cosh\left(\dfrac{x}{2\ell}\ln\left|\dfrac{M-\rho_1\gamma}{M+\rho_1\gamma}\right|\right) \end{pmatrix}, \tag{3.24}$$

and in the case of $M < \rho_1 \gamma$, we define

$$\mathcal{T} = \begin{pmatrix} \cosh\left(\dfrac{x}{2\ell}\left[\ln\left|\dfrac{M-\rho_1\gamma}{M+\rho_1\gamma}\right|+i\pi\right]\right) & \rho_1\sinh\left(\dfrac{x}{2\ell}\left[\ln\left|\dfrac{M-\rho_1\gamma}{M+\rho_1\gamma}\right|+i\pi\right]\right) \\ \rho_1^{-1}\sinh\left(\dfrac{x}{2\ell}\left[\ln\left|\dfrac{M-\rho_1\gamma}{M+\rho_1\gamma}\right|+i\pi\right]\right) & \cosh\left(\dfrac{x}{2\ell}\left[\ln\left|\dfrac{M-\rho_1\gamma}{M+\rho_1\gamma}\right|+i\pi\right]\right) \end{pmatrix}. \tag{3.25}$$

Obviously, $\mathcal{T}$ is a bounded invertible operator on $L_K^2(0,\ell) \times L_\rho^2(0,\ell)$ and satisfies

$$\begin{pmatrix} \cosh\omega_n^{(1)}x \\ \rho_1^{-1}\sinh\omega_n^{(1)}x \end{pmatrix} = \mathcal{T}\begin{pmatrix} \cosh\dfrac{n\pi ix}{\ell} \\ \rho_1^{-1}\sinh\dfrac{n\pi ix}{\ell} \end{pmatrix}. \tag{3.26}$$

Therefore, the sequence $\{[\cosh\omega_n^{(1)}x, \rho_1^{-1}\sinh\omega_n^{(1)}x]^T \mid n \in \mathbb{Z}\}$ forms a Riesz basis of $L_K^2(0,\ell) \times L_\rho^2(0,\ell)$ because the sequence

$$\left\{ \left[\cosh\dfrac{n\pi ix}{\ell}, \rho_1^{-1}\sinh\dfrac{n\pi ix}{\ell}\right]^T \mid n \in \mathbb{Z} \right\} \tag{3.27}$$

is an orthonormal basis on $L_K^2(0,\ell) \times L_\rho^2(0,\ell)$.

The similar approach can be used to prove that the sequence

$$\left\{ [\cosh\omega_n^{(2)}x, \rho_2^{-1}\sinh\omega_n^{(2)}x]^T \mid n \in \mathbb{Z} \right\} \tag{3.28}$$

forms a Riesz basis of $L_{EI}^2(0,\ell) \times L_{I_\rho}^2(0,\ell)$. Thus, the sequence $\{\Phi_n \mid n \in \mathbb{Z}\} \cup \{\Psi_n \mid n \in \mathbb{Z}\}$ forms a Riesz basis of $\mathcal{H}_1$ and so does the sequence $\{A_n\Phi_n \mid n \in \mathbb{Z}\} \cup \{B_n\Psi_n \mid n \in \mathbb{Z}\}$. Therefore, there is a positive integer N such that

$$\sum_{|n|\geq N}^{\infty} \|Y(\lambda_n^{(1)}) - A_n\Phi_n\|^2 + \sum_{|n|\geq N}^{\infty} \|Y(\lambda_n^{(2)}) - B_n\Psi_n\|^2 < \infty. \tag{3.29}$$

Using Lemma 3.1, the required result follows.   □

According to the proof of Theorem 3.2, there exists an integer $N > 0$ such that $\lambda_n^{(1)}$, $\lambda_n^{(2)}$ for $|n| \geq N$ are simple eigenvalues of $\mathcal{A}$, and $Y(\lambda_n^{(1)})$ and $Y(\lambda_n^{(2)})$ are two eigenvectors of $\mathcal{A}$ associated with $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$, respectively. Denote

$$\sigma_0 = \sigma(\mathcal{A}) \setminus \{\lambda_n^{(1)}, \lambda_n^{(2)} \mid |n| \geq N\}, \tag{3.30}$$

then σ_0 is a finite set, that is,

$$\sigma_0 = \{\mu_1, \mu_2, \ldots, \mu_k\} \tag{3.31}$$

with $k \leq 4N - 2$. For each eigenvalue μ_j, let

$$\{Y_{j,1}, Y_{j,2}, \ldots, Y_{j,s_j}\} \tag{3.32}$$

be a basis of the corresponding root subspace. Therefore, by Theorem 3.2, the set

$$\{Y_{j,i} \mid 1 \leq j \leq k,\ 1 \leq i \leq s_j\} \cup \{Y(\lambda_n^{(1)}) \mid |n| \geq N\} \cup \{Y(\lambda_n^{(2)}) \mid |n| \geq N\} \tag{3.33}$$

forms a Riesz basis for $\mathcal{H}$. Let

$$\{Y_{j,i}^* \mid 1 \leq j \leq k,\ 1 \leq i \leq s_j\} \cup \{Y^*(\lambda_n^{(1)}) \mid |n| \geq N\} \cup \{Y^*(\lambda_n^{(2)}) \mid |n| \geq N\} \tag{3.34}$$

be the biorthogonal system associated with $\{Y_{j,i} \mid 1 \leq j \leq k,\ 1 \leq i \leq s_j\} \cup \{Y(\lambda_n^{(1)}) \mid |n| \geq N\} \cup \{Y(\lambda_n^{(2)}) \mid |n| \geq N\}$. For each $F \in \mathcal{H}$, we have

$$F = \sum_{j=1}^{k} \sum_{i=1}^{s_j} \langle F, Y_{j,i}^* \rangle Y_{j,i} + \sum_{|n| \geq N} \langle F, Y^*(\lambda_n^{(1)}) \rangle Y(\lambda_n^{(1)}) + \sum_{|n| \geq N} \langle F, Y^*(\lambda_n^{(2)}) \rangle Y(\lambda_n^{(2)}). \tag{3.35}$$

Let $T(t)$ be the C_0-semigroup generated by $\mathcal{A}$. Then, for each $Y \in \operatorname{span}\{Y_{j,i} \mid i = 1, 2, \ldots, s_j\}$, we have

$$T(t)Y = e^{\mu_j t} \sum_{i=1}^{s_j} P_{s_j, i}(Y, t) Y_{j,i}, \tag{3.36}$$

where $P_{s_j, i}(Y, t)$ is a polynomial of order less than s_j. Hence, we have the following corollary.

COROLLARY 3.3. *Let $\mathcal{H}$ and $\mathcal{A}$ be as before. Let $T(t)$ be the C_0-semigroup generated by $\mathcal{A}$. If $J - \rho_2 \nu \neq 0$ and $M - \rho_1 \gamma \neq 0$, then the solution of system (2.5) with initial data $F \in \mathcal{H}$ can be expressed as*

$$\begin{aligned}
T(t)F &= \sum_{j=1}^{k} \sum_{i=1}^{s_j} e^{\mu_j t} P_{s_j, i}(F, t) Y_{j,i} + \sum_{|n| \geq N} e^{\lambda_n^{(1)} t} \langle F, Y^*(\lambda_n^{(1)}) \rangle Y(\lambda_n^{(1)}) \\
&\quad + \sum_{|n| \geq N} e^{\lambda_n^{(2)} t} \langle F, Y^*(\lambda_n^{(2)}) \rangle Y(\lambda_n^{(2)}).
\end{aligned} \tag{3.37}$$

4. Exponential decay of the closed-loop system. In Section 3, we discussed the Riesz basis property of system (2.5) in $\mathcal{H}$ and gave the solution expression of the system. In this section, we discuss the exponential decay of the closed-loop system. Since (2.5) is a Riesz system, according to [2], we have the following theorem.

THEOREM 4.1. *Let $\mathcal{H}$ and $\mathcal{A}$ be as before. If $J - \rho_2 \nu \neq 0$ and $M - \rho_1 \gamma \neq 0$, then system (2.5) satisfies the spectrum-determined growth assumption in the state space $\mathcal{H}$.*

Let $T(t)$ be the C_0-semigroup generated by $\mathcal{A}$. The rate of exponential decay of $T(t)$ is defined by

$$\omega(T) = \lim_{t \to \infty} \frac{\ln \|T(t)\|}{t} \tag{4.1}$$

and the bound $s(\mathcal{A})$ of the spectrum of $\mathcal{A}$ is given by

$$s(\mathcal{A}) = \sup \{ \operatorname{Re} \lambda \mid \lambda \in \sigma(\mathcal{A}) \}. \tag{4.2}$$

Theorem 4.1 implies that $\omega(T) = s(\mathcal{A})$. In order to estimate $s(\mathcal{A})$, denote

$$\begin{aligned}
\omega_1 &= \max \left\{ \frac{1}{2\ell\rho_1} \ln \left| \frac{M - \rho_1 \gamma}{M + \rho_1 \gamma} \right|, \frac{1}{2\ell\rho_2} \ln \left| \frac{J - \rho_2 \nu}{J + \rho_2 \nu} \right| \right\}, \\
\omega_2 &= \min \left\{ \frac{1}{2\ell\rho_1} \ln \left| \frac{M - \rho_1 \gamma}{M + \rho_1 \gamma} \right|, \frac{1}{2\ell\rho_2} \ln \left| \frac{J - \rho_2 \nu}{J + \rho_2 \nu} \right| \right\}.
\end{aligned} \tag{4.3}$$

From the discussion in Section 3, we know that the lines $\operatorname{Re} \lambda = \omega_1$ and $\operatorname{Re} \lambda = \omega_2$ are two asymptotic lines of the spectrum of $\mathcal{A}$. Noticing that $\omega_2 < \omega_1 < 0$, obviously for any $\varepsilon > 0$ with $\omega_1 + \varepsilon < 0$, there are only finitely many eigenvalues of $\mathcal{A}$ outside the zone

$$\omega_2 - \varepsilon \leq \operatorname{Re} \lambda \leq \omega_1 + \varepsilon. \tag{4.4}$$

If $\operatorname{Re} \lambda \leq \omega_1$ for any $\lambda \in \sigma(\mathcal{A})$, then $s(\mathcal{A}) = \omega_1$. If there is an eigenvalue λ of $\mathcal{A}$ such that $\operatorname{Re} \lambda > \omega_1$, then there exists at least one eigenvalue λ_0 of $\mathcal{A}$ such that $\operatorname{Re} \lambda_0 = s(\mathcal{A})$. Since $\mathcal{A}$ is a dissipative operator and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \varnothing$, we have $s(\mathcal{A}) < 0$.

Summarizing the above discussion, we have the following result.

THEOREM 4.2. *Let $\mathcal{A}$ and $\mathcal{H}$ be as before and $T(t)$ the C_0-semigroup of contractions generated by $\mathcal{A}$. Then it holds that*

$$\omega(T) = s(\mathcal{A}) = \max_{\lambda \in \sigma(\mathcal{A})} \{ \omega_1, \operatorname{Re} \lambda \} < 0, \tag{4.5}$$

and hence the closed-loop system is exponentially stable.

We have proven that system (2.5) is a Riesz system. However, from the previous discussion we also can see that the asymptotic behavior of the eigenvalues of the closed-loop system operator $\mathscr{A}$ is dependent only upon the feedback parameters γ and ν. So, if we take $\alpha = 0$ and $\beta = 0$ in the feedback controls, then it is not difficult to prove that the corresponding closed-loop system is also a Riesz system. In this case, it is natural to ask whether the corresponding closed-loop system decays still exponentially. The answer is positive. In fact, if we denote by $\mathscr{A}_1$ the operator $\mathscr{A}$ with $\alpha = 0$ and $\beta = 0$, then $\mathscr{A}_1$ is also dissipative. In order to prove the exponential stability of the C_0-semigroup generated by $\mathscr{A}_1$, similar to above, it is enough to show that there is no eigenvalue of $\mathscr{A}_1$ on imaginary axis. It is easy to see that $0 \in \rho(\mathscr{A}_1)$. If $\lambda \in i\mathbb{R}$ is an eigenvalue of $\mathscr{A}_1$ and $Y = [w, z, \varphi, \psi, \xi, \eta]^T \in \mathscr{D}(\mathscr{A}_1)$ is an eigenvector of $\mathscr{A}_1$ corresponding to λ, then

$$\operatorname{Re}\langle \mathscr{A}_1 Y, Y \rangle = -\frac{\gamma K}{M}\left| w'(\ell) - \varphi(\ell) \right|^2 - \frac{\nu EI}{J}\left| \varphi'(\ell) \right|^2 = 0, \qquad (4.6)$$

from which it follows that $w'(\ell) - \varphi(\ell) = 0$ and $\varphi'(\ell) = 0$. Then $w(x)$ and $\varphi(x)$ satisfy

$$\begin{aligned}
\rho\lambda^2 w(x) - K(w''(x) - \varphi'(x)) &= 0, \quad 0 < x < \ell, \\
I_\rho \lambda^2 \varphi(x) - EI\varphi''(x) - K(w'(x) - \varphi(x)) &= 0, \quad 0 < x < \ell, \\
w(0) = \varphi(0) = 0, \qquad w'(\ell) - \varphi(\ell) &= 0, \qquad \varphi'(\ell) = 0, \\
w(\ell) = 0, \qquad \varphi(\ell) &= 0.
\end{aligned} \qquad (4.7)$$

Thus, $w(\ell) = w'(\ell) = 0$ and $\varphi(\ell) = \varphi'(\ell) = 0$, and hence according to the general theory of ordinary differential equations, it follows that $w(x) = \varphi(x) = 0$ for all $x \in [0, \ell]$, that is, $Y = 0$. This shows that there is no eigenvalue of $\mathscr{A}_1$ on the imaginary axis. Notice the fact that when $J - \rho_2\nu \neq 0$ and $M - \rho_1\gamma \neq 0$, the lines $\operatorname{Re}\lambda = \omega_1$ and $\operatorname{Re}\lambda = \omega_2$ are two asymptotic lines of $\sigma(\mathscr{A}_1)$ and the system associated with $\mathscr{A}_1$ is a dissipative Riesz system; we can assert that the closed-loop system decays still exponentially and the rate of the exponential decay of the system is just equal to the supremum of real parts of the spectra of $\mathscr{A}_1$. Let $T_1(t)$ be the contraction semigroup generated by $\mathscr{A}_1$. Similar to above, we have the following result.

THEOREM 4.3. *Let $\mathscr{A}_1$, $\mathscr{H}$, and ω_1 be defined as before. Let $T_1(t)$ be the contraction semigroup generated by $\mathscr{A}_1$. Then it holds that*

$$\omega(T_1) = s(\mathscr{A}_1) = \max_{\lambda \in \sigma(\mathscr{A}_1)} \{\omega_1, \operatorname{Re}\lambda\} < 0. \qquad (4.8)$$

Notice that the exponential decay rates of the systems associated with $\mathscr{A}$ and $\mathscr{A}_1$ are different. In fact, we can show that $\omega(T) \leq \omega(T_1)$. It is very interesting to clarify the dependence of the exponential decay rate on the parameters α and β.

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