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DEGENERATION OF MODULI SPACES AND GENERALIZED THETA FUNCTIONS

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Introduction

Let C be a smooth projective curve of genus g and $\mathcal{U}_C$ the moduli space of semistable vector bundles of rank r and degree d on C. There is a natural ample line bundle Θ on $\mathcal{U}_C$ that we call the theta line bundle of $\mathcal{U}_C$, which generalises the line bundle on the Jacobian of C defined by the Riemann theta divisor [DN]. A section of Θ^k over $\mathcal{U}_C$ is called a generalised theta function of order k. This definition of theta line bundle and generalised theta functions can be generalised to the moduli spaces of semistable torsion free sheaves of rank r and degree d on singular curves. A natural problem suggested by the conformal field theory is to study the space $H^0(\mathcal{U}_C, \Theta^k)$ by relating it to the space of generalised theta functions associated with a smooth curve of genus $g-1$.

We consider a family of curves $f : \mathcal{X} \to T$ of genus g, whose singular fibre $\mathcal{X}_0 = X$ is irreducible, smooth except for a single node, so that its normalisation $\widetilde{X}$ is a smooth curve of genus $g-1$. There exists a moduli scheme $\mathcal{M} \to T$ such that $\mathcal{M}_t$ for any $t \in T$ is the moduli space $\mathcal{U}_{\mathcal{X}_t}$ of semistable torsion free sheaves of rank r and degree d. One can define a line bundle on $\mathcal{M}$ such that its restriction on $\mathcal{M}_t = \mathcal{U}_{\mathcal{X}_t}$ is the theta line bundle Θ_t on $\mathcal{U}_{\mathcal{X}_t}$. Moreover, if we have a vanishing theorem $H^1(\Theta_t^k) = 0$ for any $t \in T$, one would have that $dim(H^0(\Theta_t^k))$ is constant. Thus we need to relate the space $H^0(\mathcal{U}_X, \Theta^k)$ with the spaces of generalised theta functions associated with $\widetilde{X}$. Let x_0 be the node of X and $\pi : \widetilde{X} \to X$ the normalisation of X with $\pi^{-1}(x_0) = \{x_1, x_2\}$. The expected factorisation rule is

$$H^0(\mathcal{U}_X, \Theta^k) = \bigoplus_{\mu} H^0(\mathcal{U}^{\mu}_{\widetilde{X}}, \Theta_{\mu}), \tag{A}$$

where μ runs through a certain indexing set depending on k, $\mathcal{U}^{\mu}_{\widetilde{X}}$ is the moduli space of parabolic vector bundles of rank r and degree d on $\widetilde{X}$ with parabolic

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structures at x_1 and x_2 (with weights depending on μ), and Θ_μ is the generalised theta line bundle. It is clear that to carry through the induction on genus, one has to start with moduli spaces of parabolic torsion free sheaves of rank r on a nodal curve X with parabolic structure at a finite number of smooth points and prove a factorisation rule for generalised theta functions on them, as well as a vanishing theorem for H^1. This was done in the case of rank two by [NR]. We will treat the general case of any rank in this paper.

Now we are going to state the main result. First, some preliminaries:

(1) Let X be an irreducible projective curve of genus g, smooth but for one node x_0. Let $\pi : \widetilde{X} \to X$ be the normalization of X, and $\pi^{-1}(x_0) = \{x_1, x_2\}$.

(2) Let I be a finite set of smooth points on X. Fix integers d, k, r and

$$\vec{a}(x) = (a_1(x), a_2(x), \cdots, a_{l_x+1}(x)),$$
$$\vec{n}(x) = (n_1(x), n_2(x), \cdots, n_{l_x+1}(x))$$

with $0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) \leq k$ for each $x \in I$. Take $(\alpha_x)_{x\in I} \in \mathbb{Z}^I_{\geq 0}$ and $\ell > 0$ satisfying

$$(*) \qquad \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\sum_{x\in I}\alpha_x + r\ell = k(d + r(1-g)),$$

where $d_i(x) = a_{i+1}(x) - a_i(x)$ and $r_i(x) = n_1(x) + \cdots + n_i(x)$.

(3) Let $\mathcal{U}_X$ be the moduli space of (s-equivalence classes of) parabolic torsion free sheaves of rank r and degree d on X, with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$, semistable with respect to the weights $\{\vec{a}(x)\}_{x\in I}$. The definitions can be extended to cover the case that $l_x = 0$ for $x \in Q \subset I$ (Remark 1.1).

(4) For $\mu = (\mu_1, \cdots, \mu_r)$ with $0 \leq \mu_r \leq \cdots \leq \mu_1 \leq k-1$, let

$$\{d_i = \mu_{r_i} - \mu_{r_i+1}\}_{1\leq i\leq l}$$

be the subset of nonzero integers in $\{\mu_i - \mu_{i+1}\}_{i=1,\cdots,r-1}$, and for $j = 1, 2$ set

$$\vec{a}(x_j) = (\mu_r, \mu_r + d_1(x_j), \cdots, \mu_r + \sum_{i=1}^{l-1} d_i(x_j), \mu_r + \sum_{i=1}^{l} d_i(x_j)),$$
$$\vec{n}(x_j) = (r_1(x_j), r_2(x_j) - r_1(x_j), \cdots, r_l(x_j) - r_{l-1}(x_j)),$$

where $r_i(x_1) = r_i$, $d_i(x_1) = d_i$, and $r_i(x_2) = r - r_{l-i+1}$, $d_i(x_2) = d_{l-i+1}$.

Let $\mathcal{U}^\mu_{\widetilde{X}}$ be the moduli space of semistable parabolic bundles on $\widetilde{X}$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I\cup\{x_1,x_2\}}$ at points $\{x\}_{x\in I\cup\{x_1,x_2\}}$ and weights $\{\vec{a}(x)\}_{x\in I\cup\{x_1,x_2\}}$. We can extend the definition to cover the case that $l=0$, namely, $\mu_1=\cdots=\mu_r$.

(5) For any data $\omega=(k,r,d,\ell,I,\{\vec{a}(x),\vec{n}(x),\alpha_x\}_{x\in I})$ satisfying the condition $(*)$, we will define a natural ample line bundle

$$\Theta_{\mathcal{U}_X} \doteq \Theta(k,r,d,\ell,I,\{\vec{a}(x),\vec{n}(x),\alpha_x\}_{x\in I})$$

on $\mathcal{U}_X$, and $\Theta_{\mathcal{U}^{\mu}_{\widetilde{X}}}$ is defined similarly with $\alpha_{x_1}=\mu_r$ and $\alpha_{x_2}=k-\mu_1$.

Factorization Theorem. *There exists a (noncanonical) isomorphism*

$$H^0(\mathcal{U}_X,\Theta_{\mathcal{U}_X})\cong\bigoplus_{\mu}H^0(\mathcal{U}^{\mu}_{\widetilde{X}},\Theta_{\mathcal{U}^{\mu}_{\widetilde{X}}})$$

where $\mu=(\mu_1,\cdots,\mu_r)$ runs through the integers $0\le\mu_r\le\cdots\le\mu_1\le k-1$.

Vanishing Theorem. (1) *Suppose that C is a smooth projective curve of genus $g\ge 2$. Then $H^1(\mathcal{U}_C,\Theta_{\mathcal{U}_C})=0$.* (2) *Assume that $g\ge 3$. Then $H^1(\mathcal{U}_X,\Theta_{\mathcal{U}_X})=0$.*

The Factorization theorem is proved in §4 (Theorem 4.1) and the Vanishing theorems are proved in §5 (Theorem 5.1 and Theorem 5.3). Next we describe briefly the main steps in the proof of the main theorems.

We adopt a variant of a concept in [B1], GPS, to relate $\mathcal{U}_X$ with a suitable moduli space $\mathcal{P}$ of GPS on $\widetilde{X}$. Such a GPS of rank r is given by a pair (E,Q) where E is a sheaf, torsion free outside $\{x_1,x_2\}$, of rank r on $\widetilde{X}$ and Q an r-dimensional quotient of $E_{x_1}\oplus E_{x_2}$ such that the torsion of E injects to Q. Given such a GPS, one defines a torsion free sheaf F on X by the exact sequence

$$0\to F\to\pi_*E\to {}_{x_0}Q\to 0$$

where ${}_{x_0}Q$ is the skyscraper sheaf on X with support $\{x_0\}$ and fibre Q. One can define the notion of a semistable GPS, and prove that F is semistable iff (E,Q) is semistable. All this goes through if there are additional parabolic structures at $\{x\}_{x\in I}$. There is therefore a morphism $\phi:\mathcal{P}\to\mathcal{U}_X$, which is actually the normalization of $\mathcal{U}_X$ (§2).

Setting $\Theta_{\mathcal{P}}=\phi^*\Theta_{\mathcal{U}_X}$, we will characterize the image of $H^0(\mathcal{U}_X,\Theta_{\mathcal{U}_X})$ in $H^0(\mathcal{P},\Theta_{\mathcal{P}})$. Our strategy is to consider the filtrations ($j=1,2$)

$$\mathcal{P}\supset\mathcal{D}_j:=\mathcal{D}_j(r-1)\supset\cdots\supset\mathcal{D}_j(a)\supset\mathcal{D}_j(a-1)\supset\cdots\supset\mathcal{D}_j(0)$$

of subvarieties of $\mathcal{P}$ (Notation 2.3 in §2), and the filtration

$$\mathcal{U}_X\supset\mathcal{W}_{r-1}\supset\cdots\supset\mathcal{W}_a\supset\mathcal{W}_{a-1}\supset\cdots\supset\mathcal{W}_0$$

of subvarieties of $\mathcal{U}_X$ (Notation 2.4 in §2). We will prove in §3 and §4 that $\mathcal{P}$, $\mathcal{D}_j(a)$ are reduced, irreducible, normal with only rational singularities, and $\mathcal{U}_X$, $\mathcal{W}_a$ are seminormal (Proposition 3.2 and Theorem 4.2). Moreover, we will prove that the restriction ϕ_a of ϕ gives the normalization $\phi_a : \mathcal{D}_1(a) \to \mathcal{W}_a$ of $\mathcal{W}_a$, and $\phi_a^{-1}(\mathcal{W}_{a-1}) = \mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1)$ (Proposition 2.1). All of these properties are essentially used to prove that there exists a (noncanonical) isomorphism $H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \cong H^0(\mathcal{P}, \Theta_{\mathcal{P}}(-\mathcal{D}_2))$ in §4 (Proposition 4.3). Note that Proposition 2.1 is essential for the story. To prove it for the general rank case, we have to clarify a fact: if $\mathcal{R}_a$ (Notation 2.4 in §2) are saturated sets for the quotient map? We prove that $\mathcal{R}_a$ are indeed saturated sets for the quotient map (Lemma 2.6), which is not known in [NR] and [S2] (see Notation 3.1 of [NR] and the "Remarque" on page 172 of [S2]); thus we can even simplify the arguments of [NR] for the case of rank two by using our lemma.

Let $\widetilde{\mathcal{R}}_F$ be the variety parametrizing a certain locally universal family of rank r vector bundles $\mathcal{E}$ on $\widetilde{X}$ with degree d and parabolic structures at $\{x\}_{x\in I}$. $\mathcal{U}_{\widetilde{X}}$ is a geometric invariant theory (GIT) quotient of the semistable points of $\widetilde{\mathcal{R}}_F$ with respect to the action of a suitable reductive group and certain linearization by a line bundle $\hat{\Theta}$. Let $\rho : \widetilde{\mathcal{R}}'_F \to \widetilde{\mathcal{R}}_F$ denote the grassmannian bundle of r-dimensional quotients of $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}$. One will see that

$$H^0(\mathcal{P}, \Theta_{\mathcal{P}}(-\mathcal{D}_2)) = H^0(\widetilde{\mathcal{R}}'_F, \rho^*\hat{\Theta} \otimes \mathbf{L})^{inv} = H^0(\widetilde{\mathcal{R}}_F, \hat{\Theta} \otimes \rho_*\mathbf{L})^{inv}$$

where $\mathbf{L}$ is essentially the line bundle $\mathcal{O}(k-1)$ along the fibres of the grassmannian bundle, and $\{\ \ \}^{inv}$ denotes a space of invariants for the group action. The computation of $\rho_*\mathbf{L}$ amounts to the following classical problem in representation theory. Let Gr be the grassmannian of r-dimensional subspaces of $\mathbb{C}^{2r}$ and m a positive integer. The question is how to decompose the irreducible representation $H^0(Gr, \mathcal{O}(m))$ of $GL(2r)$ into irreducible representations of $GL(r) \times GL(r) \subset GL(2r)$ (Lemma 4.5). The factorization theorem follows from this.

We turn next to the vanishing theorem (1) for a smooth curve C. For the given data ω satisfying the condition $(*)$, one has a line bundle $\hat{\Theta}_\omega$ on $\widetilde{\mathcal{R}}_F$, where $\mathcal{U}_C$ is the GIT quotient of semistable points $\widetilde{\mathcal{R}}_\omega^{ss} \subset \widetilde{\mathcal{R}}_F$ with respect to the action of $SL(n)$ ($n = d + r(1-g)$) and the linearization by the line bundle $\hat{\Theta}_\omega$, which descends to the ample line bundle $\Theta_{\mathcal{U}_C}$ on $\mathcal{U}_C$. We can write

$$\hat{\Theta}_\omega = \omega_{\widetilde{\mathcal{R}}_F} \otimes \hat{\Theta}_{\bar{\omega}} \otimes Det^*\Theta_y^{-2}$$

on $\widetilde{\mathcal{R}}_F$ (Proposition 2.2) for a new data $\bar{\omega}$ satisfying the condition $(*)$. Let $\mathcal{U}_{C,\bar{\omega}}$ be the GIT quotient of semistable points $\widetilde{\mathcal{R}}_{\bar{\omega}}^{ss} \subset \widetilde{\mathcal{R}}_F$ for the $SL(n)$ action under the new linearization by $\hat{\Theta}_{\bar{\omega}}$, which descends to an ample line bundle $\Theta_{\bar{\omega}}$ on $\mathcal{U}_{C,\bar{\omega}}$. Using the fact that the complements of $\widetilde{\mathcal{R}}^{ss}_{\omega}$, $\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}$ and $\widetilde{\mathcal{R}}^{s}_{\bar{\omega}}$ in $\widetilde{\mathcal{R}}_F$ and $\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}$ are of high codimensions (one needs here the restriction on genus, see Proposition 5.1), we have

$$H^1(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = H^1(\widetilde{\mathcal{R}}_F, \hat{\Theta}_{\omega})^{inv} = H^1(\mathcal{U}_{C,\bar{\omega}}, \Theta_{\bar{\omega}} \otimes Det^*\Theta_y^{-2} \otimes \omega_{\mathcal{U}_{C,\bar{\omega}}})$$

where Det denotes the determinant map and Θ_y the theta bundle on the Jacobian J^d_C of C. Then we prove that $\Theta_{\bar{\omega}} \otimes Det^*\Theta_y^{-2}$ is ample (Lemma 5.3) and thus prove the vanishing of $H^1(\mathcal{U}_{C,\bar{\omega}}, \Theta_{\bar{\omega}} \otimes Det^*\Theta_y^{-2} \otimes \omega_{\mathcal{U}_{C,\bar{\omega}}})$ by applying a Kodaira-type vanishing theorem (Theorem 7.80(f) of [SS]).

The vanishing theorem (2) for the singular curve X is reduced to proving the vanishing of $H^1(\mathcal{P}, \Theta_{\mathcal{P}})$ (Lemma 5.5). There exists a flat morphism $Det : \mathcal{P} \to J^d_{\widetilde{X}}$ extending the determinant morphism on the open set of stable torsion free GPS (Lemma 5.7), and a decomposition

$$(Det)_*\Theta_{\mathcal{P}} = \bigoplus_{\mu} (Det_{\mu})_*\Theta_{\mathcal{U}^{\mu}_{\widetilde{X}}}$$

where $Det_{\mu} : \mathcal{U}^{\mu}_{\widetilde{X}} \to J^d_{\widetilde{X}}$ is the determinant morphism. Thus $H^1(J^d_{\widetilde{X}}, (Det)_*\Theta_{\mathcal{P}}) = 0$ by using the vanishing theorem (1) for smooth curves, and we are left with the task of proving $R^1 Det_*\Theta_{\mathcal{P}} = 0$. To prove that $H^1(\mathcal{P}^L, \Theta_{\mathcal{P}}) = 0$, where $\mathcal{P}^L$ denotes the fibre of Det at any $L \in J^d_{\widetilde{X}}$, we follow the same line as in the proof of the vanishing theorem (1) except that $Det^*\Theta_y^{-2}$ disappears. We do need here the properties that $\mathcal{P}$ is Gorenstein with only rational singularities. It also takes more work to prove a formula for the dualizing sheaf of $\mathcal{P}$ (Proposition 3.4 and Lemma 5.6).

We introduce the moduli spaces and theta line bundles in §1. A detailed study of the morphism $\phi : \mathcal{P} \to \mathcal{U}_X$ is given in §2. We prove in §3 that $\mathcal{P}$ and its subvarieties $\mathcal{D}_j(a)$ ($j = 1, 2$, $0 \leq a \leq r-1$) are normal with only rational singularities, and we also prove a formula expressing the canonical (dualizing) sheaf of $\mathcal{H}$ (see §2 for the definition) where we need to prove $\mathcal{H}$ is Gorenstein (it is actually a complete intersection by using a dimension formula for double determinant varieties). The factorization theorem and the seminormality of $\mathcal{U}_X$ and its subvarieties $\mathcal{W}_a$ ($0 \leq a \leq r-1$) are proved in §4. §5 is devoted to the estimation of codimensions and the proof of vanishing theorems.

§1. Moduli spaces and theta bundles

We introduce the notation in this section by recalling the construction of moduli spaces and theta bundles, whose proofs are contained in [NR], where they deal with rank two, but the proof there goes through for any rank. We also refer to [BR] and [Pa] for theta bundles on moduli spaces of parabolic bundles of any rank.

Let X be an irreducible projective curve of genus g over the complex number field $\mathbb{C}$, which has at most one node x_0. Let I be a finite set of smooth points of X, and E be a torsion free sheaf of rank r and degree d on X.

Definition 1.1. *By a quasi-parabolic structure on E at a smooth point $x \in X$, we mean a choice of flag*

$$E_x = F_0(E)_x \supset F_1(E)_x \supset \cdots \supset F_{l_x}(E)_x \supset F_{l_x+1}(E)_x = 0$$

of the fibre E_x of E at x. If, in addition, a sequence of integers called the parabolic weights

$$0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) \leq k$$

is given, we say that E has a parabolic structure at x.

Let $n_i(x) = dim(F_{i-1}(E)_x/F_i(E)_x)$ and $r_i(x) = dim(E_x/F_i(E)_x)$. Write

$$\vec{a}(x) := (a_1(x), a_2(x), \cdots, a_{l_x+1}(x)),$$
$$\vec{n}(x) := (n_1(x), n_2(x), \cdots, n_{l_x+1}(x)).$$

We use $\vec{a}$ (resp., $\vec{n}$) to denote the map $x \mapsto \vec{a}(x)$ (resp., $x \mapsto \vec{n}(x)$) from I to a suitable set. Let E' be a subsheaf of E such that E/E' is torsion free. Then the induced parabolic structure on E' is defined as follows: the quasi-parabolic structure is defined by $F_i(E')_x := F_i(E)_x \cap E'_x$, and the weights by $a'_j(x) = a_i(x)$ where i is the biggest integer satisfying that $F_j(E')_x \subset F_i(E)_x$.

Definition 1.2. The parabolic degree of a parabolic sheaf E is

$$pardeg(E) := deg(E) + \frac{1}{k}\sum_{x\in I}\sum_{i=1}^{l_x+1} n_i(x)a_i(x).$$

E is called semistable (resp., stable) for $(k, \vec{a})$ if for any subsheaf $E' \subset E$ such that E/E' is torsion free with the induced parabolic structure, one has

$$pardeg(E') \leq \frac{pardeg(E)}{rk(E)} \cdot rk(E') \text{ (resp., } <).$$

By a family of rank r parabolic sheaves parametrised by a variety T, we mean a sheaf $\mathcal{F}_T$ on $X \times T$, flat over T, and torsion free (with rank r and degree d) on $X \times \{t\}$ for every point $t \in T$, together with, for each $x \in I$, a flag of subbundles of $\mathcal{F}_T|_{\{x\}\times T}$. The following theorem was proved in the Appendix of [NR].

Theorem 1.1. *There exists a (coarse) moduli space* $\mathcal{U}_X^s(d,r,I,k,\vec{a},\vec{n})$ *of stable parabolic sheaves* F. *We have an open immersion*

$$\mathcal{U}_X^s(d,r,I,k,\vec{a},\vec{n}) \hookrightarrow \mathcal{U}_X(d,r,I,k,\vec{a},\vec{n})$$

where $\mathcal{U}_X(d,r,I,k,\vec{a},\vec{n})$ *denotes the space of s-equivalent classes of semistable parabolic sheaves. The latter is a seminormal projective variety. If* X *is smooth, then it is normal, with only rational singularities.*

Fixing I, k, $\vec{a}$, and $\vec{n}$, we set $\mathcal{U}_X := \mathcal{U}_X(d,r,I,k,\vec{a},\vec{n})$ and $\mathcal{U}_X^s := \mathcal{U}_X^s(d,r,I,k,\vec{a},\vec{n})$. Let us recall the construction of $\mathcal{U}_X$.

Let $\mathbf{Q}$ be the Quot scheme of coherent sheaves (of rank r and degree d) over X that are quotients of $\mathcal{O}^n$, where $n = d + r(1-g)$. Thus there is on $X \times \mathbf{Q}$ a sheaf $\mathcal{F}_{\mathbf{Q}}$, flat over $\mathbf{Q}$, and $\mathcal{O}^n_{X\times\mathbf{Q}} \to \mathcal{F}_{\mathbf{Q}} \to 0$. Let $\mathcal{F}_x$ be the sheaf given by restricting $\mathcal{F}_{\mathbf{Q}}$ to $\{x\} \times \mathbf{Q}$. Let $Flag_{\vec{n}(x)}(\mathcal{F}_x)$ be the relative flag scheme of type $\vec{n}(x)$, and $\mathcal{R}$ be the fibre product over $\mathbf{Q}$:

$$\mathcal{R} = \underset{x\in I}{\times_{\mathbf{Q}}} Flag_{\vec{n}(x)}(\mathcal{F}_x).$$

Let $\mathcal{R}^s$ (resp., $\mathcal{R}^{ss}$) be the open subscheme of $\mathcal{R}$ corresponding to stable (resp., semistable) parabolic sheaves, which is generated by global sections and whose first cohomology vanishes when d is large enough. The variety $\mathcal{U}_X$ is the good quotient of R^{ss} by $SL(n)$ acting through $PGL(n)$. We denote the projection by

$$\psi : \mathcal{R}^{ss} \to \mathcal{U}_X.$$

Choose an ample line bundle of degree 1 on X, denoted by $\mathcal{O}_X(1)$ from now on. For large enough m, we have an $SL(n)$-equivariant embedding $\mathcal{R} \hookrightarrow \mathbf{G}$, where $\mathbf{G}$ is defined to be

$$Grass_{P(m)}(\mathbb{C}^n \otimes W) \times \prod_{x\in I}\{Grass_r(\mathbb{C}^n) \times Grass_{r_1(x)}(\mathbb{C}^n) \times \cdots \times Grass_{r_{l_x}(x)}(\mathbb{C}^n)\}$$

where $P(m) = n + rm$, and $W = H^0(\mathcal{O}_X(m))$. For any $(\alpha_x)_{x\in I} \in \mathbb{Z}^I_{\geq 0}$ and $\ell > 0$ satisfying

$$(*) \qquad \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) r_i(x) + r\sum_{x\in I}\alpha_x + r\ell = kn,$$

where $d_i(x) = a_{i+1}(x) - a_i(x)$, we give $\mathbf{G}$ the polarisation (using the obvious notation):

$$\frac{\ell}{m} \times \prod_{x\in I}\{\alpha_x, d_1(x), \cdots, d_{l_x}(x)\}$$

and take the induced polarisation on $\mathcal{R}$. It was proved in [NR] that the set of semistable points for the $SL(n)$ action on $\mathcal{R}$ is precisely $\mathcal{R}^{ss}$. One remarks that this fact is independent of the choice of $\vec{\alpha} := (\alpha_x)_{x\in I}$ satisfying the condition $(*)$. $\mathcal{R}^{ss}$ is reduced and irreducible and $\mathcal{U}_X$ is its GIT quotient.

For any family of parabolic sheaves $\mathcal{F}$ of type $\vec{n}(x)$ at $x \in I$ parametrised by T, we denote the quotients $\mathcal{F}_{\{x\}\times T}/F_i(\mathcal{F}_{\{x\}\times T})$ by $\mathcal{Q}_{\{x\}\times T,i}$, and we define

$$\Theta_T := (det\, R\pi_T\mathcal{F})^k \otimes \bigotimes_{x\in I}\{(det\,\mathcal{F}_{\{x\}\times T})^{\alpha_x} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{\{x\}\times T,i})^{d_i(x)}\} \otimes (det\,\mathcal{F}_{\{y\}\times T})^{\ell}$$

where π_T is the projection $X \times T \to T$, and $det\, R\pi_T\mathcal{F}$ is the determinant bundle defined as

$$\{det\, R\pi_T\mathcal{F}\}_t := \{det\, H^0(X,\mathcal{F}_t)\}^{-1} \otimes \{det\, H^1(X,\mathcal{F}_t)\}.$$

If we take $T = \mathcal{R}^{ss}$ and $\vec{a}$, $\vec{\alpha}$, k, ℓ satisfying the condition $(*)$, it is easy to check that $\Theta_{\mathcal{R}^{ss}}$ is a $PGL(n)$-bundle, which descends to $\mathcal{U}_X$. Moreover, we have the following theorem for whose proof we refer to [NR], [Pa] and [BR].

Theorem 1.2. *There is a unique ample line bundle $\Theta_{\mathcal{U}_X} = \Theta(k,\ell,\vec{a},\vec{\alpha},I)$ on $\mathcal{U}_X$ such that for any given family of semistable parabolic sheaves $\mathcal{F}$ parametrised by T, we have $\phi_T^*\Theta_{\mathcal{U}_X} = \Theta_T$, where ϕ_T is the induced map $T \to \mathcal{U}_X$.*

Remark 1.1. (1) It is known that the analytic local ring of $\mathcal{R}^{ss}$ is determined (up to smooth morphisms) by $\mathbb{C}[X,Y]/(XY,YX)$, where X, Y are $r\times r$ matrices (see [Fa] and [S2]). Thus, by Lemma 3.8 and Lemma 3.13 of [NR], the seminormality of $\mathcal{U}_X$ is equivalent to that of $\mathbb{C}[X,Y]/(XY,YX)$, which is known to be seminormal (see [Tr]).

(2) If we replace, in the construction of Theorem 1.2, $(det\,\mathcal{F}_{\{y\}\times\mathcal{R}^{ss}})^{\ell}$ by

$$\bigotimes_{q\in Q}(det\,\mathcal{F}_{\{q\}\times\mathcal{R}^{ss}})^{\beta_q} \otimes (det\,\mathcal{F}_{\{y\}\times\mathcal{R}^{ss}})^{\ell+\ell_0},$$

where Q is a set of smooth points of X, and $\sum_{q\in Q}\beta_q = -\ell_0$, we get ample line bundles on $\mathcal{U}_X$, which are all algebraically equivalent to $\Theta_{\mathcal{U}_X}$.

(3) We can extend the above definitions to cover the case that $l_q = 0$ for $q \in Q \subset I$. In this case, $\mathcal{U}_X$ denotes the moduli space of semistable parabolic sheaves with parabolic structures at $\{x\}_{x\in I\setminus Q}$ and parabolic weights $\{\vec{a}(x)\}_{x\in I\setminus Q}$. When $Q = I$, $\mathcal{U}_X$ is the ordinary moduli space of semistable torsion free sheaves (i.e., no quasi-parabolic structure is considered), and the definition of $\Theta_{\mathcal{U}_X}$ in Theorem 1.2 gives ample line bundles $\Theta(\vec{\alpha},I)$ on $\mathcal{U}_X$, all of them algebraically equivalent to the descendant of

$$(det\, R\pi_{\mathcal{R}^{ss}}\mathcal{F})^k \otimes (det\,\mathcal{F}_{\{y\}\times\mathcal{R}^{ss}})^{\frac{kn}{r}}.$$

These $\Theta(\vec{\alpha}, I)$ will appear naturally in the decomposition theorems, induced by the 1-dimensional representations of $GL(r)$.

Now we are going to recall the notion of "generalised parabolic sheaf" (GPS) and the construction of its moduli space ([B1], [B2] and [NR]). We do not define the general notation (as in [B1] and [B2]), but we have to consider the sheaves with torsion as in [NR]. Let $\pi : \widetilde{X} \to X$ be the normalisation of X and $\pi^{-1}(x_0) = \{x_1, x_2\}$. Then we have

Definition 1.3. *Let E be a sheaf on $\widetilde{X}$, torsion free of rank r outside $\{x_1, x_2\}$. A generalised parabolic structure on E over the divisor $x_1 + x_2$ is an r-dimensional quotient Q*

$$E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0.$$

(E, Q) is said to be a generalised parabolic sheaf, namely GPS.

We will consider generalised parabolic sheaves E with, in addition, parabolic structures at the points of $\pi^{-1}(I)$ (we will identify I with $\pi^{-1}(I)$). Furthermore, by a family of GPS over T, we mean the following:

(1) a rank r sheaf $\mathcal{E}$ on $\widetilde{X} \times T$ flat over T and locally free outside $\{x_1, x_2\} \times T$;
(2) a locally free rank r quotient $\mathcal{Q}$ of $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}$ on T;
(3) a flag bundle $Flag(\mathcal{E}_x)$ on T with given weights for each $x \in I$.

Definition 1.4. A GPS (E, Q) is called semistable (resp., stable), if for every nontrivial subsheaf $E' \subset E$ such that E/E' is torsion free outside $\{x_1, x_2\}$, we have

$$pardeg(E') - dim(Q^{E'}) \leq rk(E') \cdot \frac{pardeg(E) - dim(Q)}{rk(E)} \quad \text{(resp., <)},$$

where $Q^{E'} = q(E'_{x_1} \oplus E'_{x_2}) \subset Q$.

Set $\tilde{n} = d + r(1 - \tilde{g})$, where $\tilde{g}$ $(= g - 1)$ is the genus of $\widetilde{X}$, and let $\widetilde{\mathbf{Q}}$ be the Quot scheme of coherent sheaves (of degree d and rank r) over $\widetilde{X}$ that are quotients of $\mathcal{O}^{\tilde{n}}_{\widetilde{X}}$. Taking d to be large enough, we can assume that for any semistable generalised parabolic sheaf E of rank r and degree d we have $H^1(E(-x_1 - x_2 - x)) = 0$, $x \in \widetilde{X}$, which means that $\mathbb{C}^{\tilde{n}} \to H^0(E)$ is an isomorphism, where E is generated by global sections and $H^0(E) \to E_{x_1} \oplus E_{x_2}$ is onto, where $E(-x_1 - x_2)$ is generated by global sections. Let $\mathcal{F}$ be the universal quotient $\widetilde{\mathcal{O}}^{\tilde{n}} := \mathcal{O}^{\tilde{n}}_{\widetilde{X} \times \widetilde{\mathbf{Q}}} \to \mathcal{F} \to 0$ on $\widetilde{X} \times \widetilde{\mathbf{Q}}$ and

$$\widetilde{\mathcal{R}}' := Grass_r(\mathcal{F}_{x_1} \oplus \mathcal{F}_{x_2}) \times_{\widetilde{\mathbf{Q}}} \left\{ \underset{x \in I}{\times_{\widetilde{\mathbf{Q}}}} Flag_{\vec{n}(x)}(\mathcal{F}_x) \right\}.$$

There is a locally universal family of GPS parametrised by $\widetilde{\mathcal{R}}'$ that we denote by $\mathcal{E}$, which is actually the pull-back of $\mathcal{F}$ by the natural projection. Let

$\tilde{P}(m) = \tilde{n} + rm$ and

$$\tilde{\mathbf{G}}' := Grass_{\tilde{P}(m)}(\mathbb{C}^{\tilde{n}} \otimes W) \times Grass_r(\mathbb{C}^{\tilde{n}} \otimes \mathbb{C}^2) \times \mathbf{Flag},$$

where **Flag** denotes the variety

$$\prod_{x\in I}\{Grass_r(\mathbb{C}^{\tilde{n}}) \times Grass_{r_1(x)}(\mathbb{C}^{\tilde{n}}) \times \cdots \times Grass_{r_{l_x}(x)}(\mathbb{C}^{\tilde{n}})\}.$$

Then we have an $SL(\tilde{n})$-equivariant embedding $\widetilde{\mathcal{R}}' \hookrightarrow \tilde{\mathbf{G}}'$. Take the polarisation

$$\frac{(\tilde{\ell}-k)}{m} \times k \times \prod_{x\in I}\{\alpha_x, d_1(x), \cdots, d_{l_x}(x)\}$$

such that

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\sum_{x\in I}\alpha_x + r\tilde{\ell} = k\tilde{n},$$

which is nothing but $(*)$ with $\tilde{n} = n + r$ and $\tilde{\ell} = \ell + k$. Then one proves that the GIT-semistable (stable) points of $\widetilde{\mathcal{R}}'$ are precisely the semistable (stable) generalised parabolic sheaves, namely $\widetilde{\mathcal{R}}'^{ss}$. Let $\mathcal{P} := \mathcal{P}_{\widetilde{X}}$ be the GIT quotient of $\widetilde{\mathcal{R}}'^{ss}$ by $SL(\tilde{n})$ with the projection

$$\tilde{\psi}' : \widetilde{\mathcal{R}}'^{ss} \to \mathcal{P}.$$

One defines an s-equivalence of GPS such that

(1.1)
$$E \sim E' \iff \text{there exist } E_1 = E, \cdots, E_{s+1} = E' \text{ with } \overline{o(E_i)} \cap \overline{o(E_{i+1})} \neq \emptyset,$$

where $\overline{o(E_i)}$ denotes the schematic closure of the orbit of E_i under $SL(\tilde{n})$. It is clear that if E_1 and E_2 are stable, then $E_1 \sim E_2$ iff $E_1 \cong E_2$. Then we have

Theorem 1.3. *There exists a (coarse) moduli space $\mathcal{P}^s$ of stable GPS on $\widetilde{X}$, which is a smooth variety. We have an open immersion $\mathcal{P}^s \hookrightarrow \mathcal{P}$, where $\mathcal{P}$ is the space of s-equivalence classes of semistable GPS on $\widetilde{X}$, which is a reduced, irreducible and normal projective variety with rational singularities.*

The existence of $\mathcal{P}$ is known as we have shown above. We will prove in §3 that it is reduced, irreducible and normal with rational singularities. In fact $\mathcal{P}$ is the normalisation of $\mathcal{U}_X$ as we will see in next section. We complete this section by introducing a sheaf-theoretic description of s-equivalence of GPS, which was given in Appendix B of [NR] in the case of rank 2. We enlarge the category of GPS by adopting the following more general definition, and assume that $|I| = 0$ for simplicity.

Definition 1.5. A generalised m-parabolic structure on a sheaf E over the divisor $x_1 + x_2$ is a choice of an m-dimensional quotient Q of $E_{x_1} \oplus E_{x_2}$. A sheaf with a generalised m-parabolic structure will be called an m-GPS, or GPS for short. A GPS E is said to be semistable (resp., stable) if E is torsion free outside $\{x_1, x_2\}$ and

(1) if $rank(E) > 0$, then for every proper subsheaf E' such that E/E' is torsion free outside $\{x_1, x_2\}$, we have

$$rank(E)(deg(E') - dim(Q^{E'})) \leq rank(E')(deg(E) - m) \quad \text{(resp., <);}$$

(2) if $rank(E) = 0$, then $E_{x_1} \oplus E_{x_2} = Q$ (resp., $E_{x_1} \oplus E_{x_2} = Q$ and $dim(Q) = 1$).

Definition 1.6. If (E, Q) is a GPS and $rank(E) > 0$, we set

$$\mu_G[(E, Q)] = \frac{deg(E) - dim(Q)}{rank(E)}.$$

It is useful to think of an m-GPS as a sheaf E on $\widetilde{X}$ together with a map $\pi_* E \to {}_{x_0}Q \to 0$ and $h^0({}_{x_0}Q) = m$. Let K_E denote the kernel of $\pi_* E \to Q$.

Definition 1.7. A morphism of GPS $(E, Q) \to (E', Q')$ is a sheaf map $E \to E'$ that maps K_E to $K_{E'}$ (and therefore induces a map $Q \to Q'$).

Definition 1.8. Given an exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of sheaves on $\widetilde{X}$, and $\pi_* E \to Q \to 0$ a generalised parabolic structure on E, we define the generalised parabolic structures on E' and E'' via the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_* E' & \longrightarrow & \pi_* E & \longrightarrow & \pi_* E'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & Q'' & \longrightarrow & 0 \end{array}$$

The first horizontal sequence is exact because π is finite, Q' is defined as the image in Q of $\pi_* E'$ so that the first vertical arrow is onto, Q'' is defined by demanding that the second horizontal sequence is exact, and finally the third vertical arrow is onto by the snake lemma. We will write

$$0 \to (E', Q') \to (E, Q) \to (E'', Q'') \to 0$$

whose meaning is clear.

Proposition 1.1. *Fix a rational number μ. Then the category $\mathcal{C}_\mu$ of semistable GPS (E,Q) such that $rank(E)=0$ or, $rank(E)>0$ with $\mu_G[(E,Q)]=\mu$, is an abelian, Artinian, Noetherian category whose simple objects are the stable GPS in the category.*

One can conclude, as usual, that given a semistable GPS (E,Q) it has a Jordan-Hölder filtration, and the associated graded GPS $gr(E,Q)$ is uniquely determined by (E,Q). Thus we have

Definition 1.9. Two semistable GPS (E_1,Q_1) and (E_2,Q_2) are said to be s-equivalent if they have the same associated graded GPS, namely,

$$(E_1,Q_1)\sim(E_2,Q_2)\quad\Longleftrightarrow\quad gr(E_1,Q_1)\cong gr(E_2,Q_2).$$

Remark 1.2. Any stable GPS (E,Q) with $rank(E)>0$ must be a GPB (i.e., E is a vector bundle) such that $E_{x_j}\to Q$ $(j=1,2)$ are isomorphisms, and two stable GPS are s-equivalent iff they are isomorphic. In fact, let Q_j be the image of $E_{x_j}\to Q$ and $\bar{q}: E_{x_1}\oplus E_{x_2}\xrightarrow{q} Q\to Q/Q_1=\bar{Q}$. Then we define E' by the exact sequence

$$0\to E'\to E\to_{x_2}\bar{Q}_2\to 0,$$

where $\bar{Q}_2=Q_2/Q_1\cap Q_2$ is the image of $E_{x_2}\to Q\to\bar{Q}$. Thus $Q^{E'}=Q_1$ and

$$\mu_G[E',Q_1)]=\mu_G[(E,Q)]+\frac{dim(Q)-dim(Q_1)-dim(Q_2)+dim(Q_1\cap Q_2)}{rank(E)}.$$

If (E,Q) is stable, we must have $Q_1=Q_2=Q$. One can imitate the proof of Lemma 4.7 and Theorem 4.8 in [Gi] to show that this s-equivalence satisfies the requirement (1.1).

§2. The normalization of parabolic moduli spaces on a nodal curve

Let X be an irreducible projective curve of genus g and smooth except for one node x_0, and $\pi:\widetilde{X}\to X$ the normalisation, $\pi^{-1}(x_0)=\{x_1,x_2\}$. It is clear that we have the canonical exact sequence

$$0\to\mathcal{O}_X\to\pi_*\mathcal{O}_{\widetilde{X}}\to{}_{x_0}k(x_0)\to 0,$$

where $k(x_0)$ denotes the residue field of x_0, and we will use ${}_xW$ to denote the "skyscraper sheaf" supported at $\{x\}$, with fibre W.

Given a GPS (E,Q) on $\widetilde{X}$, we have the exact sequence

$$0 \to F \to \pi_* E \to {}_{x_0}Q \to 0.$$

It is clear that $\phi(E,Q) := F$ (which has the natural parabolic structures at points of I) is a torsion free sheaf on X of rank r if and only if

$$\text{(T)} \qquad (\mathrm{Tor}E)_{x_1} \oplus (\mathrm{Tor}E)_{x_2} \overset{q}{\hookrightarrow} Q.$$

Note that, for any sheaf E on $\widetilde{X}$, we have $deg(\pi_* E) = deg(E) + rank(E)$; thus $deg(F) = deg(E)$.

Lemma 2.1. *Let (E,Q) satisfy condition (T), and let $F = \phi(E,Q)$ be the associated torsion free sheaf on X. We have*

1. *If E is a vector bundle and the maps $E_{x_i} \to Q$ are isomorphisms, then F is a vector bundle.*
2. *If F is a vector bundle on X, then there is a unique (E,Q) such that $\phi(E,Q) = F$. In fact, $E = \pi^* F$.*
3. *If F is a torsion free sheaf, then there is an (E,Q), with E a vector bundle on $\widetilde{X}$, such that $\phi(E,Q) = F$ and $E_{x_2} \to Q$ is an isomorphism. The rank of the map $E_{x_1} \to Q$ is a iff $F \otimes \hat{\mathcal{O}}_{x_0} \cong \hat{\mathcal{O}}_{x_0}^{\oplus a} \oplus m_{x_0}^{\oplus(r-a)}$. The roles of x_1 and x_2 can be reversed.*
4. *Every torsion free rank r sheaf F on X comes from an (E,Q) such that E is a vector bundle.*

Proof. Here we only check (3) since we will need the construction later. The proof of Lemma 4.6 in [NR] easily extends to the other statements for any rank. Let $F \otimes \hat{\mathcal{O}}_{x_0} \cong \hat{\mathcal{O}}_{x_0}^{\oplus a} \oplus m_{x_0}^{\oplus(r-a)}$, and define a vector bundle $\widetilde{E}$ on $\widetilde{X}$ by

$$0 \to \mathrm{Tor}(\pi^* F) \to \pi^* F \to \widetilde{E} \to 0.$$

By the canonical exact sequence

$$0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_{\widetilde{X}} \to {}_{x_0}k(x_0) \to 0,$$

we get (note that $\pi_*\pi^* F = F \otimes \pi_* \mathcal{O}_{\widetilde{X}}$ and F is torsion free)

$$0 \to F \to \pi_*\pi^* F \to {}_{x_0}Q_F \to 0,$$

where $Q_F := k(x_0) \otimes_{\mathcal{O}_X} F$ is a vector space of dimension $2r - a$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & \pi_*\pi^* F & \longrightarrow & {}_{x_0}Q_F & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \xrightarrow{d} & \pi_*\widetilde{E} & \longrightarrow & {}_{x_0}\widetilde{Q} & \longrightarrow & 0
\end{array}$$

where the vertical arrows are surjections and $\widetilde{Q} = Q_F/\pi_*\mathrm{Tor}(\pi^*F)$ is of dimension a. Note that $\pi_*\pi^*F \to {}_{x_0}Q_F$ induces two surjective maps $(\pi^*F)_{x_i} \to Q_F$, and so is $\pi_*\widetilde{E} \to \widetilde{Q}$. Denoting their kernel by K_i, we have exact sequences

$$0 \to K_i \to \widetilde{E}_{x_i} \to \widetilde{Q} \to 0. \tag{2.1}$$

Let $h : \widetilde{E} \to E$ be the Hecke modification at x_2 (see Remark 1.4 of [NS]) such that $ker(h_{x_2}) = K_2$, where $h_{x_2} : \widetilde{E}_{x_2} \to E_{x_2}$ is the induced map of h between the fibres at x_2. Then one has the exact sequence for some Q_2 of dimension $r - a$

$$0 \to \widetilde{E} \xrightarrow{h} E \to {}_{x_2}Q_2 \to 0,$$

namely, $pardeg(E) = pardeg(F)$ and $E_{x_1} \cong \widetilde{E}_{x_1}$. We define Q by the exact sequence

$$0 \to F \xrightarrow{(\pi_*h)\cdot d} \pi_*E \to {}_{x_0}Q \to 0 \tag{Q}$$

which is clearly of dimension r and $\phi(E, Q) = F$. To check that the induced $E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0$ (by (Q)) satisfies the requirement, we consider the restriction of (Q) at x_0

$$F_{x_0} \xrightarrow{d_0} \widetilde{E}_{x_1} \oplus \widetilde{E}_{x_2} \xrightarrow{h_{x_1}\oplus h_{x_2}} E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0$$

which implies that $Im(d_0) \cap \widetilde{E}_{x_i} = K_i$ and $ker(q_i) = h_{x_i}(K_i)$, where $q_i : E_{x_i} \to Q$. Thus q_1 has rank a (since h_{x_1} is an isomorphism), and q_2 is an isomorphism (since $ker(h_{x_2}) = K_2$)).

Lemma 2.2. *Let $F = \phi(E, Q)$. Then F is semistable if and only if (E, Q) is semistable. Moreover, one has*

(1) *if (E, Q) is stable, then F is stable;*
(2) *if F is a stable vector bundle, then (E, Q) is stable.*

Proof. See the proof of Proposition 4.7 of [NR], or [B2].

Given a family $\mathcal{E}_T$ of GPS parametrised by T, we can define a family $\mathcal{F}_T$ of sheaves on X by the exact sequence

$$0 \to \mathcal{F}_T \to (\pi \times I_T)_*\mathcal{E}_T \to {}_{x_0}\mathcal{Q}_T \to 0. \tag{2.2}$$

Since $\mathcal{E}_T$ is flat on T and $\mathcal{Q}_T$ locally free on T, $\mathcal{F}_T$ is flat on T, namely a flat family. Thus we have

$$\hat{\phi} : T \to \mathcal{R}. \tag{2.3}$$

If $\mathcal{E}_T$ is a semistable family, we get a morphism $\phi_T : T \xrightarrow{\hat{\phi}} \mathcal{R}^{ss} \xrightarrow{\psi} \mathcal{U}_X$ by Lemma 2.1. Thus, taking $T = \widetilde{\mathcal{R}}'^{ss}$, the morphism $\phi_{\widetilde{\mathcal{R}}'^{ss}}$ induces a morphism

$$\phi : \mathcal{P} \to \mathcal{U}_X.$$

We use the notation in §1, and let

$$\widetilde{\mathcal{R}} = \times_{\widetilde{\mathbf{Q}}} {}_{x\in I} Flag_{\vec{n}(x)}(\mathcal{F}_x).$$

From now on, we will understand that sheaves in $\widetilde{\mathbf{Q}}$ have torsions only at $\{x_1, x_2\}$; so $\widetilde{\mathcal{R}} \to \widetilde{\mathbf{Q}}$ is a flag bundle. Let

$$\rho : \widetilde{\mathcal{R}}' \to \widetilde{\mathcal{R}}$$

be the natural projection. Then we defined in §1 that

$$\Theta_{\widetilde{\mathcal{R}}} = (det\, R\pi_{\widetilde{\mathcal{R}}}\mathcal{F})^k \otimes \bigotimes_{x\in I}\{(det\,\mathcal{F}_x)^{\alpha_x} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{d_i(x)}\} \otimes (det\,\mathcal{F}_y)^{\tilde{\ell}},$$

where $\tilde{\ell} = \ell + k$. Let $(\mathcal{E}, \mathcal{Q})$ be the universal family of GPS on $\widetilde{\mathcal{R}}'$. We define on $\widetilde{\mathcal{R}}'^{ss}$ that

$$\hat{\Theta}' = \rho^*\Theta_{\widetilde{\mathcal{R}}} \otimes (det\,\mathcal{Q})^k \otimes (det\,\mathcal{E}_y)^{-k}.$$

It is easy to check that $\hat{\Theta}'$ is the (restriction of) the ample line bundle on $\widetilde{\mathcal{R}}'^{ss}$ used to linearise the action of $SL(\tilde{n})$ (note that $\mathcal{E}$ is the pull-back of $\mathcal{F}$ by ρ), and descends to an ample line bundle $\Theta_{\mathcal{P}}$ on $\mathcal{P}$.

Lemma 2.3. *Let $\eta_x := (det\,\mathcal{Q})(det\,\mathcal{E}_x)^{-1}$ for a point $x \in \widetilde{X}$ and denote $\Theta_{\widetilde{\mathcal{R}}}$ by $\hat{\Theta}$. Then*

(1) $\hat{\Theta}' = \rho^*\hat{\Theta} \otimes \eta_y^k$;

(2) $\Theta_{\mathcal{P}} = \phi^*\Theta_{\mathcal{U}_X}$.

Proof. (1) is the definition of $\hat{\Theta}'$. To check (2), consider the morphism $\phi_{\widetilde{\mathcal{R}}'^{ss}} : \widetilde{\mathcal{R}}'^{ss} \to \mathcal{U}_X$. It is enough to prove that

$$\phi^*_{\widetilde{\mathcal{R}}'^{ss}}(\Theta_{\mathcal{U}_X}) = \hat{\Theta}'.$$

From the exact sequence (2.2), we have

$$detR\pi_T\mathcal{F}_T = (detR\pi_T(\pi_*\mathcal{E}_T)) \otimes (det\mathcal{Q}_T) = (detR\pi_T\mathcal{E}_T) \otimes (det\mathcal{Q}_T),$$

which result and Theorem 1.2 in §1 imply (2).

Notation 2.1. Define $\mathcal{H}$ to be the subscheme of $\widetilde{\mathcal{R}}'$ parametrising the generalised parabolic sheaves $(\mathcal{O}^{\tilde{n}} \to E \to 0, Q)$ satisfying

(1) $\mathbb{C}^{\tilde{n}} \cong H^0(E)$, and $H^1(E(-x_1 - x_2 - x)) = 0$ for any $x \in \widetilde{X}$;

(2) Tor$E$ is supported on $\{x_1, x_2\}$ and $(\mathrm{Tor}E)_{x_1} \oplus (\mathrm{Tor}E)_{x_2} \hookrightarrow Q$.

Notation 2.2. Define $\widetilde{\mathbf{Q}}_F$ to be the open subscheme of $\widetilde{\mathbf{Q}}$ consisting of locally free quotients $(\mathcal{O}^{\tilde{n}} \to E \to 0)$ such that

(1) $\mathbb{C}^{\tilde{n}} \to H^0(E)$ is an isomorphism, and

(2) $H^1(E(-x_1 - x_2 - x)) = 0$ for any $x \in \widetilde{X}$.

Remark 2.1. The assumption $H^1(E(-x_1 - x_2 - x)) = 0$ implies that $H^1(E) = 0$, E is generated by global sections, $H^0(E) \to E_{x_1} \oplus E_{x_2}$ is onto, and $E(-x_1 - x_2)$ is generated by global sections. It is not difficult to see ([NR]) that $\mathcal{H}$ is an irreducible open subscheme of $\widetilde{\mathcal{R}}'$ and

$$\widetilde{\mathcal{R}}'^{ss} \overset{\text{open}}{\hookrightarrow} \mathcal{H} \overset{\text{open}}{\hookrightarrow} \widetilde{\mathcal{R}}'.$$

Notation 2.3. Let $\widetilde{\mathcal{R}}_F = \times_{\substack{\widetilde{\mathbf{Q}}_F \\ x \in I}} Flag_{\vec{n}(x)}(\mathcal{F}_x)$ and $\widetilde{\mathcal{R}}'_F = \rho^{-1}(\widetilde{\mathcal{R}}_F)$. Then

$$\rho : \widetilde{\mathcal{R}}'_F \to \widetilde{\mathcal{R}}_F$$

is a grassmannian bundle over $\widetilde{\mathcal{R}}_F$, and $\widetilde{\mathcal{R}}'_F \subset \mathcal{H}$. We define

$$R^1_{F,a} := \{(E, Q) \in \widetilde{\mathcal{R}}'_F \mid E_{x_1} \to Q \text{ has rank } a\},$$

and $\hat{\mathcal{D}}_{F,1}(i) := R^1_{F,0} \cup \cdots \cup R^1_{F,i}$, which have the natural scheme structures. The subschemes $R^2_{F,a}$ and $\hat{\mathcal{D}}_{F,2}(i)$ are defined similarly. Let $\hat{\mathcal{D}}_1(i)$ and $\hat{\mathcal{D}}_2(i)$ be the Zariski closure of $\hat{\mathcal{D}}_{F,1}(i)$ and $\hat{\mathcal{D}}_{F,2}(i)$ in $\widetilde{\mathcal{R}}'$. Then they are reduced, irreducible and $SL(\tilde{n})$-invariant closed subschemes of $\widetilde{\mathcal{R}}'$, thus inducing closed subschemes $\mathcal{D}_1(i)$, $\mathcal{D}_2(i)$ of $\mathcal{P}$. Clearly, we have (for $j = 1, 2$) that

$$\widetilde{\mathcal{R}}' \supset \hat{\mathcal{D}}_j(r-1) \supset \hat{\mathcal{D}}_j(r-2) \supset \cdots \supset \mathcal{D}_j(1) \supset \hat{\mathcal{D}}_j(0),$$

$$\mathcal{P} \supset \mathcal{D}_j(r-1) \supset \mathcal{D}_j(r-2) \supset \cdots \supset \mathcal{D}_j(1) \supset \mathcal{D}_j(0).$$

Notation 2.4. Let $\mathcal{R}_a = \{F \in \mathcal{R} \mid F \otimes \hat{\mathcal{O}}_{x_0} = \hat{\mathcal{O}}_{x_0}^{\oplus a} \oplus m_{x_0}^{\oplus(r-a)}\}$, and let

$$\hat{\mathcal{W}}_i = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_i$$

(which are closed in $\mathcal{R}$) be endowed with their reduced scheme structures. The subschemes $\hat{\mathcal{W}}_i$ are $SL(n)$-invariant and yield closed reduced subschemes of $\mathcal{U}_X$. It is clear that

$$\mathcal{R} \supset \hat{\mathcal{W}}_{r-1} \supset \hat{\mathcal{W}}_{r-2} \supset \cdots \supset \hat{\mathcal{W}}_1 \supset \hat{\mathcal{W}}_0 = \mathcal{R}_0,$$

$$\mathcal{U}_X \supset \mathcal{W}_{r-1} \supset \mathcal{W}_{r-2} \supset \cdots \supset \mathcal{W}_1 \supset \mathcal{W}_0.$$

Lemma 2.4. *With the above notation and* $\hat{\phi}: \widetilde{\mathcal{R}}'_F \to \widetilde{\mathcal{R}}$ *defined as in* (2.3), *we have*

(1) $\hat{\phi}(R^1_{F,a} \cap R^2_{F,b}) = \mathcal{R}_{a+b-r}$,

(2) $\hat{\phi}(R^1_{F,a}) = \hat{\phi}(R^2_{F,a}) = \hat{\mathcal{W}}_a$,

(3) $\hat{\phi}(\hat{\mathcal{D}}_{F,1}(i)) = \hat{\phi}(\hat{\mathcal{D}}_{F,2}(i)) = \hat{\mathcal{W}}_i$.

Proof. (1) follows from Proposition 4.2 and Proposition 4.7 (1) of [B2]. To check (2), we note that (1) implies that $R^1_{F,a} \cap R^2_{F,j} = \emptyset$ if $j < r - a$. Thus

$$\hat{\phi}(R^1_{F,a}) = \bigcup_{j=r-a}^{r} \mathcal{R}_{a-(r-j)} = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_a = \hat{\mathcal{W}}_a.$$

(3) follows (2) immediately.

Proposition 2.1. *With the above notation and denoting* $\mathcal{D}_1(r-1)$, $\mathcal{D}_2(r-1)$, $\mathcal{W}_{r-1}$ *by* $\mathcal{D}_1$, $\mathcal{D}_2$ *and* $\mathcal{W}$, *we have*

(1) $\phi: \mathcal{P} \to \mathcal{U}_X$ *is finite and surjective, and* $\phi(\mathcal{D}_1(a)) = \phi(\mathcal{D}_2(a)) = \mathcal{W}_a$,

(2) $\phi(\mathcal{P} \setminus \{\mathcal{D}_1 \cup \mathcal{D}_2\}) = \mathcal{U}_X \setminus \mathcal{W}$ *and induces an isomorphism on* $\mathcal{P} \setminus \{\mathcal{D}_1 \cup \mathcal{D}_2\}$,

(3) $\phi|_{\mathcal{D}_1(a)}: \mathcal{D}_1(a) \to \mathcal{W}_a$ *is finite and surjective,*

(4) $\phi(\mathcal{D}_1(a) \setminus \{\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1)\}) = \mathcal{W}_a \setminus \mathcal{W}_{a-1}$ *and induces an isomorphism on* $\mathcal{D}_1(a) \setminus \{\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1)\}$,

(5) $\phi: \mathcal{P} \to \mathcal{U}_X$ *is the normalisation of* $\mathcal{U}_X$,

(6) $\phi|_{\mathcal{D}_1(a)}: \mathcal{D}_1(a) \to \mathcal{W}_a$ *is the normalisation of* $\mathcal{W}_a$,

(7) $\phi(\mathcal{D}_1(a) \cap \mathcal{D}_2) = \mathcal{W}_{a-1}$, *and* $\mathcal{W}_{a-1}$ *is the non-normal locus of* $\mathcal{W}_a$.

Proof. (5) and (6) are corollaries of the above (1)-(4), and Proposition 3.2 of §3.

(1) and (3) follow Lemmas (2.1)-(2.4). In fact, the surjectivity follows from Lemma 2.1 and Lemma 2.4, and the finiteness follows from Lemma 2.3 and the ampleness of $\Theta_{\mathcal{U}_X}$ and $\Theta_{\mathcal{P}}$.

To prove (2) and (4), we need the following Lemmas (2.5)-(2.7). We will check (4) here; (2) follows similarly. For any $\tilde{\psi}'(E,Q) \in \mathcal{D}_1(a) \setminus \{\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1)\}$, we can assume that E is a vector bundle by Lemma 2.5, and $E_{x_2} \to Q$ is an isomorphism since $\tilde{\psi}'(E,Q) \notin \mathcal{D}_2$. Thus $\phi(E,Q) \in \mathcal{W}_a \setminus \mathcal{W}_{a-1}$ by Lemma 2.6 and Lemma 2.4; so ϕ induces a morphism

$$\phi: \mathcal{D}_1(a) \setminus \{\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1)\} \to \mathcal{W}_a \setminus \mathcal{W}_{a-1}$$

whose surjectivity follows from Lemma 2.1(3) and $\phi(\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1)) = \mathcal{W}_{a-1}$ by Lemma 2.4(1). Now, taking $T = \mathcal{R}_a$ in Lemma 2.7 and using the universal property of $\widetilde{\mathcal{R}}'$, we get a section of ϕ on $\mathcal{W}_a \setminus \mathcal{W}_{a-1}$, which proves (4).

(7) is easy to see. In fact, $\phi(\mathcal{D}_1(a) \cap \mathcal{D}_2) = \mathcal{W}_{a-1}$ is clear by Lemma 2.4(1), and the non-normal locus of $\mathcal{W}_a$ is contained in $\mathcal{W}_{a-1}$ by the above (4). On the other hand, $\mathcal{W}_{a-1}$ is irreducible since $\mathcal{D}_j(a-1)$ is so (Proposition 3.2) and

$$\phi(\mathcal{D}_j(a-1)) = \mathcal{W}_{a-1}.$$

Thus it suffices to prove that $\phi_a := \phi|_{\mathcal{D}_1(a)} : \mathcal{D}_1(a) \to \mathcal{W}_a$ is not an isomorphism unless $\mathcal{W}_{a-1}$ is empty. If $\mathcal{W}_{a-1}$ is not empty, neither are $\mathcal{D}_1(a) \cap \mathcal{D}_2$ and $\mathcal{D}_1(a-1)$. One sees easily that the fibre of ϕ_a at the generic point of $\mathcal{W}_{a-1}$ contains at least two points since $\mathcal{D}_1(a-1) \not\subseteq \mathcal{D}_1(a) \cap \mathcal{D}_2$ clearly. Therefore ϕ_a is not an isomorphism at the generic point of $\mathcal{W}_{a-1}$.

Lemma 2.5. *Every semistable GPS (E', Q') with $rank(E') > 0$ is s-equivalent to a semistable GPS (E, Q) with E locally free.*

Proof. For given $(E', Q') \in \mathcal{C}_\mu$ with $rank(E') > 0$, if $\mathrm{Tor}(E') = 0$, then we are done. Thus we assume that one of $\mathrm{Tor}(E')_{x_i}$, say $\mathrm{Tor}(E')_{x_1}$, is non-trivial and the lemma is true for all $(\widetilde{E}', \widetilde{Q}') \in \mathcal{C}_\mu$ with $dim(\mathrm{Tor}(\widetilde{E}')) < dim(\mathrm{Tor}(E'))$.

Let $q_i' : E'_{x_i} \to Q'$ be the maps induced by the generalised parabolic structure of (E', Q'), and choose a projection $p : Q' \to \mathbb{C}$ such that $p(q_1'(\mathrm{Tor}(E')_{x_1})) \neq 0$. Let $E' \to {}_{x_1}\mathbb{C} \to 0$ be the morphism

$$E' \to E'_{x_1} \xrightarrow{q_1'} Q' \xrightarrow{p} \mathbb{C}$$

and $\widetilde{E}'$ its kernel, which has a smaller torsion than E' by the choice of p. Then we have an exact sequence

$$0 \to \widetilde{E}' \to E' \to {}_{x_1}\mathbb{C} \to 0$$

which induces an exact sequence of GPS (see Definition 1.6) if we set $\tau = {}_{x_1}\mathbb{C}$:

$$0 \to (\widetilde{E}', \widetilde{Q}') \to (E', Q') \to (\tau, \mathbb{C}) \to 0.$$

One can check that $(\widetilde{E}', \widetilde{Q}') \in \mathcal{C}_\mu$; thus, there exists a $(\widetilde{E}, \widetilde{Q}) \in \mathcal{C}_\mu$ with $\widetilde{E}$ locally free such that $gr(\widetilde{E}, \widetilde{Q}) = gr(\widetilde{E}', \widetilde{Q}')$. Since $(\tau, \mathbb{C})$ is stable, we have

$$gr(E', Q') = gr(\widetilde{E}, \widetilde{Q}) \oplus (\tau, \mathbb{C}).$$

Let $\tilde{q} : \widetilde{E}_{x_1} \oplus \widetilde{E}_{x_2} \to \widetilde{Q} \to 0$ and $K_1 = ker(\tilde{q}_1 : \widetilde{E}_{x_1} \to \widetilde{Q})$. Choosing a Hecke modification $h : \widetilde{E} \to E$ at x_1 (see Remark 1.4 of [NS]) such that $\widetilde{K}_1 := ker(h_{x_1}) \subset K_1$ and $dim(\widetilde{K}_1) = 1$, we get the extension

$$0 \to \widetilde{E} \xrightarrow{h} E \xrightarrow{\gamma} \tau \to 0.$$

Let $Q = \widetilde{Q} \oplus \mathbb{C}$ and $E_{x_1} = h_{x_1}(\widetilde{E}_{x_1}) \oplus V_1$ for a subspace V_1. We define a morphism $f : E_{x_1} \oplus E_{x_2} \to Q$ such that $E_{x_2} \to Q$ to be

$$E_{x_2} \xrightarrow{h_{x_2}^{-1}} \widetilde{E}_{x_2} \xrightarrow{\tilde{q}_2} \widetilde{Q} \hookrightarrow Q$$

and $E_{x_1} \to Q$ to be

$$h_{x_1}(\widetilde{E}_{x_1}) \oplus V_1 \xrightarrow{(\bar{h}_{x_1}^{-1}, \gamma_{x_1})} \frac{\widetilde{E}_{x_1}}{\widetilde{K}_1} \oplus \mathbb{C} \xrightarrow{(\tilde{q}_1, id)} \widetilde{Q} \oplus \mathbb{C} = Q$$

where $\bar{h}_{x_1} : \widetilde{E}_{x_1}/\widetilde{K}_1 \cong h_{x_1}(\widetilde{E}_{x_1})$ and $\tilde{q}_1 : \widetilde{E}_{x_1}/\widetilde{K}_1 \to \widetilde{Q}$ (note that $\widetilde{K}_1 \subset K_1$). Thus the following diagram is commutative:

$$\begin{tikzcd}
 & \widetilde{E}_{x_1} \oplus \widetilde{E}_{x_2} \arrow[r, "(h_{x_1}, h_{x_2})"] \arrow[d, "q"] & E_{x_1} \oplus E_{x_2} \arrow[r, "(\gamma_{x_1}, 0)"] \arrow[d, "f"] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widetilde{Q} \arrow[r] & \widetilde{Q} \oplus \mathbb{C} \arrow[r] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

One checks that f is surjective by this diagram, and thus

$$0 \to (\widetilde{E}, \widetilde{Q}) \to (E, Q) \to (\tau, \mathbb{C}) \to 0.$$

It is easy to see that $(E, Q) \in \mathcal{C}_\mu$ and is s-equivalent to (E', Q').

It is well known that for any $F \in \mathcal{R}^{ss}$ there is an integer a_F such that $F \otimes \hat{\mathcal{O}}_{x_0} \cong \hat{\mathcal{O}}_{x_0}^{\oplus a_F} \oplus m_{x_0}^{\oplus (rk(F) - a_F)}$, thus defining a function

$$\mathbf{a} : \mathcal{R}^{ss} \to \mathbb{Z}_{\geq 0}$$

with $\mathbf{a}(F) = a_F$. It is not clear if this function induces a function

$$\mathbf{a} : \mathcal{U}_X \to \mathbb{Z}_{\geq 0}$$

(see the "Remarque" on page 172 of [S2]). However, the following lemma implies that $\mathbf{a}$ is invariant under s-equivalence, in particular, descends to $\mathcal{U}_X$.

Lemma 2.6. *Let $0 \to F_1 \to F \to F_2 \to 0$ be an exact sequence of torsion free sheaves. Then*

$$\mathbf{a}(F) = \mathbf{a}(F_1) + \mathbf{a}(F_2).$$

In particular, if F is s-equivalent to F', then $\mathbf{a}(F) = \mathbf{a}(F')$.

Proof. For any torsion free sheaf F_i, we define a vector bundle on $\widetilde{X}$ to be $E_i = \pi^* F_i / \mathrm{Tor}(\pi^* F_i)$. Note that we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & 0 \\
 & \pi_* \pi^* F_1 \arrow[r] & \pi_* \pi^* F \arrow[r] & \pi_* \pi^* F_2 \arrow[r] & 0
\end{tikzcd}$$

which induces

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F & \longrightarrow & F_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_*E_1 & \longrightarrow & \pi_*E & \longrightarrow & \pi_*E_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_{x_0}Q_1 & \longrightarrow & {}_{x_0}Q & \longrightarrow & {}_{x_0}Q_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where Q_1, Q, Q_2 are defined such that each vertical sequence is exact, and the third horizontal sequence is defined such that the diagram is commutative, which must be exact.

For a torsion free sheaf F, if we define Q_F by

$$0 \to F \to \pi_*\pi^*F \to {}_{x_0}Q_F \to 0,$$

then one can see that $Q_i = Q_{F_i}/\pi_*\mathrm{Tor}(\pi^*F_i)$ and $Q = Q_F/\pi_*\mathrm{Tor}(\pi^*F)$ in the above diagram (see the proof of Lemma 2.1). Thus $dim(Q_i) = \mathbf{a}(F_i)$ and $dim(Q) = \mathbf{a}(F)$, which proves the lemma.

Lemma 2.7. *Let T be a reduced scheme, $\mathcal{F}$ a sheaf on $X \times T$, flat over T, such that for $t \in T$ the sheaf $\mathcal{F}_t$ on X is torsion free of rank r and $\mathbf{a}(\mathcal{F}_t) = a$ is constant. Then there exists a vector bundle $\mathcal{E}$ of rank r on $\widetilde{X} \times T$ and a locally free rank r quotient $q : \mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2} \to \mathcal{Q} \to 0$ on T such that $q_2 : \mathcal{E}_{x_2} \to \mathcal{Q}$ is an isomorphism, $q_1 : \mathcal{E}_{x_1} \to \mathcal{Q}$ has rank a at each fibre, and*

$$0 \to \mathcal{F} \to (\pi \times I)_*\mathcal{E} \to {}_{x_0 \times T}\mathcal{Q} \to 0.$$

Proof. We can assume that $\mathcal{F}$ is torsion free (see Lemma 4.13 of [NR]), and define $\mathcal{Q}_{\mathcal{F}}$ by

$$0 \to \mathcal{F} \to \pi_*\pi^*\mathcal{F} \to {}_{x_0 \times T}\mathcal{Q}_{\mathcal{F}} \to 0 \tag{2.4}$$

(we will write π for $\pi \times I$), which gives for any $t \in T$ an exact sequence

$$0 \to \mathcal{F}_t \to \pi_*\pi^*\mathcal{F}_t \to {}_{x_0}(\mathcal{Q}_{\mathcal{F}})_t \to 0$$

since $\mathcal{F}_t$ is torsion free. This shows that $(\mathcal{Q}_{\mathcal{F}})_t \cong \mathcal{Q}_{\mathcal{F}_t}$ has constant dimension $2r - a$; hence, $\mathcal{Q}_{\mathcal{F}}$ is flat (in fact, a vector bundle) on T, which result and (2.4) imply that $\pi^*\mathcal{F}$ is flat on T.

Taking a resolution $0 \to \mathcal{K} \to \mathcal{L}^{-m} \to \pi^*\mathcal{F} \to 0$ of $\pi^*\mathcal{F}$, where $\mathcal{L}$ is a line bundle, and dualising it, we get

$$0 \to (\pi^*\mathcal{F})^\vee \to \mathcal{L}^m \to \mathcal{K}^\vee \to \mathcal{E}xt^1(\pi^*\mathcal{F}, \mathcal{O}_{\widetilde{X}\times T}) \to 0.$$

Noting that $\mathcal{K}$ is a vector bundle on $\widetilde{X} \times T$ (since $\pi^*\mathcal{F}$ is flat over T and $\widetilde{X}$ is smooth) and $(\mathcal{K}^\vee)_t \cong (\mathcal{K}_t)^\vee$, we have the following diagram:

$$\begin{tikzcd}
((\pi^*\mathcal{F})^\vee)_t \arrow[r] \arrow[d] & (\mathcal{L}^m)_t \arrow[r] \arrow[d, equal] & (\mathcal{K}^\vee)_t \arrow[r] \arrow[d, equal] & \mathcal{E}xt^1(\pi^*\mathcal{F}, \mathcal{O}_{\widetilde{X}\times T})_t \arrow[r] \arrow[d] & 0 \\
(\pi^*\mathcal{F}_t)^\vee \arrow[r] & (\mathcal{L}_t)^m \arrow[r] & (\mathcal{K}_t)^\vee \arrow[r] & \mathcal{E}xt^1(\pi^*\mathcal{F}_t, \mathcal{O}_{\widetilde{X}}) \arrow[r] & 0
\end{tikzcd}$$

which implies that $\mathcal{E}xt^1(\pi^*\mathcal{F}, \mathcal{O}_{\widetilde{X}\times T})_t \cong \mathcal{E}xt^1(\pi^*\mathcal{F}_t, \mathcal{O}_{\widetilde{X}})$. Thus, $\mathcal{E}xt^1(\pi^*\mathcal{F}, \mathcal{O}_{\widetilde{X}\times T})$ is flat over T. This shows that $(\pi^*\mathcal{F})^\vee$ is locally free and $((\pi^*\mathcal{F})^\vee)_t \cong (\pi^*\mathcal{F}_t)^\vee$. Let $\widetilde{\mathcal{E}} := (\pi^*\mathcal{F})^{\vee\vee}$ be the double dual of $\pi^*\mathcal{F}$. Then we have

$$0 \to \mathcal{T} \to \pi^*\mathcal{F} \to \widetilde{\mathcal{E}} \to 0 \tag{2.5}$$

which specialises for any $t \in T$ to

$$0 \to \operatorname{Tor}(\pi^*\mathcal{F}_t) \to \pi^*\mathcal{F}_t \to \widetilde{\mathcal{E}}_t \to 0.$$

By (2.4) and (2.5), we get

$$0 \to \mathcal{F} \to \pi_*\widetilde{\mathcal{E}} \to {}_{x_0\times T}\widetilde{\mathcal{Q}} \to 0,$$

where $\mathcal{Q}$ is a vector bundle of rank a on T. Now the same construction in the proof of Lemma 2.1 proves our lemma.

Lemma 2.8. *Let E' be a rank r (stable) semistable parabolic bundle of degree $d - r$ on $\widetilde{X}$. Then its direct image $F = \pi_*E'$ is a (stable) semistable parabolic sheaf of degree d on X such that $F \otimes \hat{\mathcal{O}}_{x_0} \cong m_{x_0}^{\oplus r}$. This construction gives a morphism*

$$\mathcal{U}_{\widetilde{X}}(d-r) \to \mathcal{W}_0.$$

Proof. The proof of Lemma 2.1 clearly shows that $E' \to F = \pi_*E'$ and $F \to E' = \pi^*F/\operatorname{Tor}(\pi^*F)$ gives a bijection between the set of isomorphism classes of rank r bundles E' of degree $d - r$ on $\widetilde{X}$ and the set of torsion free sheaves F of degree d on X with $F \otimes \hat{\mathcal{O}}_{x_0} \cong m_{x_0}^{\oplus r}$.

We check now that the (stability) semistability of E' implies that of F. For any subsheaf $F_1 \subset F$ of rank r_1 such that $F/F_1 = F_2$ is a torsion free sheaf of rank r_2, then $\mathbf{a}(F_1) = \mathbf{a}(F_2) = 0$ since $\mathbf{a}(F) = 0$ and $\mathbf{a}(F) = \mathbf{a}(F_1) + \mathbf{a}(F_2)$ by Lemma 2.6. Thus we have $\pi_* E_i' = F_i$, where $E_i' = \pi^* F_i / \mathrm{Tor}(\pi^* F_i)$, and an exact sequence $0 \to E_1' \to E' \to E_2' \to 0$. One computes that

$$\frac{pardeg(E_i')}{rank(E_i')} - \frac{pardeg(E')}{rank(E')} = \frac{pardeg(F_i)}{rank(F_i)} - \frac{pardeg(F)}{rank(F)},$$

which proves that E' is (stable) semistable if and only if $F = \pi_* E'$ is so. Since $\mathcal{W}_0$ has reduced scheme structure, the above construction gives a morphism

$$\mathcal{U}_{\widetilde{X}}(d-r) \to \mathcal{W}_0.$$

Corollary 2.1. *Suppose that $g > 1$. Then $\mathcal{W}_0$ is nonempty and contains stable parabolic sheaves if $|I| > 0$. In particular, for any $0 \le a < r$, $\mathcal{W}_a \neq \mathcal{W}_{a+1}$.*

Proof. We will prove in §5 that $codim(\widetilde{\mathcal{R}}_F \setminus \widetilde{\mathcal{R}}^{ss}) \ge (r-1)(\tilde{g}-1)+1$ and $codim(\widetilde{\mathcal{R}}^{ss} \setminus \widetilde{\mathcal{R}}^{s}) \ge (r-1)(\tilde{g}-1)+1$ if $|I| > 0$ (see Proposition 5.1). Thus $\mathcal{U}_{\widetilde{X}}(d-r)$ is nonempty if $\tilde{g} = g-1 > 0$, and there exist stable parabolic bundles of degree $d-r$ on $\widetilde{X}$ if moreover $|I| > 0$. Now using Lemma 2.8, we conclude that $\mathcal{W}_0$ is nonempty and contains stable parabolic sheaves if $|I| > 0$.

Since semistability is an open condition and $\hat{\mathcal{W}}_{a+1} \setminus \hat{\mathcal{W}}_a$ is a nonempty open set of $\hat{\mathcal{W}}_{a+1}$, there is a semistable sheaf $F \in \hat{\mathcal{W}}_{a+1} \setminus \hat{\mathcal{W}}_a$ (because we have shown that $\hat{\mathcal{W}}_{a+1}^{ss}$ is nonempty), and we can see that $\psi(F) \in \mathcal{W}_{a+1} \setminus \mathcal{W}_a$ by Lemma 2.6.

Remark 2.2. When X is a nodal curve of $g = 1$ and $|I| = 0$, it is possible that $\mathcal{W}_0$ is empty (I am not saying that every $\mathcal{W}_a$ is empty). In fact, if $\mathcal{W}_0$ is nonempty in this case, then there exists a semistable bundle of degree $d-r$ on $\widetilde{X} = \mathbb{P}^1$ by Lemma 2.8, which implies that $r|d$.

We will finish this section by computing the canonical sheaf of $\widetilde{\mathcal{R}}_F$. Recall that $\pi : \widetilde{\mathcal{R}}_F = \times_{\widetilde{\mathbf{Q}}_F, x\in I} Flag_{\vec{n}(x)}(\mathcal{F}_x) \to \widetilde{\mathbf{Q}}_F$, let $\mathcal{E} = \pi^*\mathcal{F}$, and let

$$\mathcal{E}_x = F_0(\mathcal{E})_x \supset F_1(\mathcal{E})_x \supset \cdots \supset F_{l_x}(\mathcal{E})_x \supset F_{l_x+1}(\mathcal{E})_x = 0$$

be the universal flag on $\widetilde{\mathcal{R}}_F$. Write $\mathcal{Q}_{x,i} = \mathcal{E}_x / F_i(\mathcal{E})_x$ and let $\pi_{\widetilde{\mathcal{R}}_F} : \widetilde{X} \times \widetilde{\mathcal{R}}_F \to \widetilde{\mathcal{R}}_F$ be the projection. Then we have

Proposition 2.2. *Let $\omega_{\widetilde{\mathcal{R}}_F}$ be the canonical sheaf of $\widetilde{\mathcal{R}}_F$ and $\omega_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(\sum_q q)$ the canonical sheaf of $\widetilde{X}$. Then*

$$\omega_{\widetilde{\mathcal{R}}_F}^{-1} = (det\, R\pi_{\widetilde{\mathcal{R}}_F}\mathcal{E})^{2r} \otimes \bigotimes_{x\in I} \left\{ (det\, \mathcal{E}_x)^{n_{l_x+1}-r} \otimes \bigotimes_{i=1}^{l_x} (det\, \mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)} \right\}$$
$$\otimes \bigotimes_q (det\, \mathcal{E}_q)^{1-r} \otimes (det\, R\pi_{\widetilde{\mathcal{R}}_F} det\mathcal{E})^{-2}.$$

Proof. Noting that $\omega_{\widetilde{\mathcal{R}}_F} = \omega_{\widetilde{\mathcal{R}}_F/\widetilde{\mathbf{Q}}_F} \otimes \pi^*\omega_{\widetilde{\mathbf{Q}}_\mathbf{F}}$, the proposition is clearly a corollary of the following two lemmas.

Lemma 2.9. *Let E be a vector bundle of rank r on M, and let $F(l,E) = Flag_{\vec{n}}(E)$ be of type $\vec{n} = (n_1, \cdots, n_{l+1})$, with the universal flag*

$$E = F_0(E) \supset F_1(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = 0$$

on $F(l,E)$ and $\mathcal{Q}_i = E/F_i(E)$. Then

$$\omega_{F(l,E)/M} = (det\, E)^{r-n_{l+1}} \otimes \bigotimes_{i=1}^{l} (det \mathcal{Q}_i)^{-(n_i+n_{i+1})}.$$

Proof. One considers $F(l,E)$ as the grassmannian bundle

$$p : Grass_{rk(F_l(E))}(F_{l-1}(E)) \to F(l-1,E)$$

over $F(l-1,E)$. Then $\omega_{F(l,E)/F(l-1,E)} = det(F_l(E) \otimes (F_{l-1}(E)/F_l(E))^\vee)$, and one has

$$\omega_{F(l,E)/M} = \bigotimes_{i=1}^{l} det(F_i(E) \otimes (F_{i-1}(E)/F_i(E))^\vee).$$

Thus one can compute that

$$\omega_{F(l,E)/M} = (det\, E)^{r-n_{l+1}} \otimes \bigotimes_{i=1}^{l} (det \mathcal{Q}_i)^{-(n_i+n_{i+1})}.$$

Lemma 2.10. *Let $\mathcal{O}^{\tilde{n}}_{\widetilde{X}\times\widetilde{\mathbf{Q}}_\mathbf{F}} \to \mathcal{F} \to 0$ be the universal quotient on $\widetilde{X} \times \widetilde{\mathbf{Q}}_\mathbf{F}$. Then*

$$\omega^{-1}_{\widetilde{\mathbf{Q}}_\mathbf{F}} = (det\, R\pi_{\widetilde{\mathbf{Q}}_\mathbf{F}} \mathcal{F})^{2r} \otimes \bigotimes_q (det\, \mathcal{F}_q)^{1-r} \otimes (det\, R\pi_{\widetilde{\mathbf{Q}}_\mathbf{F}} det\, \mathcal{F})^{-2}.$$

Proof. We have, on $\widetilde{X} \times \widetilde{\mathbf{Q}}_\mathbf{F}$, the exact sequence $0 \to \mathcal{K} \to \mathcal{O}^{\tilde{n}} \to \mathcal{F} \to 0$. The tangent space of $\widetilde{\mathbf{Q}}_\mathbf{F}$ at a point $(0 \to K \to \mathcal{O}^{\tilde{n}} \to E \to 0)$ is $H^0(\widetilde{X}, K^\vee \otimes E)$. From the properties of $\widetilde{\mathbf{Q}}_\mathbf{F}$ (the Notation 2.2), it follows that

$$\omega^{-1}_{\widetilde{\mathbf{Q}}_\mathbf{F}} = det\, R\pi_{\widetilde{\mathbf{Q}}_\mathbf{F}}(\mathcal{F} \otimes \mathcal{F}^\vee).$$

We will now use a variant of the method of [DN] to evaluate $det\, R\pi_{\widetilde{\mathbf{Q}}_\mathbf{F}}(\mathcal{F}\otimes\mathcal{F}^\vee)$.

Let $\mathcal{M} = \pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}*}(\mathcal{F})$ be the direct image sheaf of $\mathcal{F}$, which is local free of rank $\tilde{n} = d + r(1-\tilde{g})$. Let Gr be the grassmannian of rank $r-1$ subbundles of $\mathcal{M}$, and let

$$p : Gr \to \widetilde{\mathbf{Q}}_{\mathbf{F}}$$

be the projection. We consider the canonical exact sequence on Gr

$$0 \to U_{Gr} \to p^*\mathcal{M} \to Q_{Gr} \to 0,$$

where U_{Gr} is the relative universal subbundle of $p^*\mathcal{M}$ on Gr, and Q_{Gr} is the relative universal quotient. Let $\mathcal{O}_{Gr}(-1) = det(U_{Gr})$ and consider

$$\begin{array}{ccc} \widetilde{X} \times Gr & \xrightarrow{\pi_{Gr}} & Gr \\ {\scriptstyle 1\times p}\downarrow & & \downarrow{\scriptstyle p} \\ \widetilde{X} \times \widetilde{\mathbf{Q}}_{\mathbf{F}} & \xrightarrow{\pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}}} & \widetilde{\mathbf{Q}}_{\mathbf{F}}. \end{array}$$

We have the induced morphism

$$\pi_{Gr}^* U_{Gr} \hookrightarrow \pi_{Gr}^* p^* \mathcal{M} = (1\times p)^* \pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}}^* \pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}*}(\mathcal{F}) \to (1\times p)^*\mathcal{F}. \tag{2.6}$$

Let Gr_0 be the open set of Gr such that, on $\widetilde{X} \times Gr_0$, the above induced morphism

$$\pi_{Gr}^* U_{Gr} \to (1\times p)^*\mathcal{F}$$

in (2.6) is injective. Then, if we write $D = Gr \setminus Gr_0$, there is on $\widetilde{X} \times (Gr \setminus D)$ an exact sequence

$$0 \to \pi_{Gr}^* U_{Gr} \to (1\times p)^*\mathcal{F} \to (1\times p)^* det(\mathcal{F}) \otimes \pi_{Gr}^* \mathcal{O}_{Gr}(1) \to 0. \tag{2.7}$$

We will denote the morphisms $Gr \setminus D \to \widetilde{\mathbf{Q}}_{\mathbf{F}}$ and $\widetilde{X} \times (Gr \setminus D) \to Gr \setminus D$ by the same p and π_{Gr}. By using (2.7), we compute that

$$\begin{aligned} det\, R\pi_{Gr}(1\times p)^*(\mathcal{F}\otimes\mathcal{F}^\vee) = {} & (det\, R\pi_{Gr}(1\times p)^*\mathcal{F}^\vee)^{r-1} \otimes \mathcal{O}_{Gr}(-rd) \\ & \otimes det\, R\pi_{Gr}(1\times p)^*(\mathcal{F}^\vee \otimes det\,\mathcal{F}), \end{aligned} \tag{2.8}$$

$$det\, R\pi_{Gr}(1\times p)^*\mathcal{F} = det\, R\pi_{Gr}(1\times p)^* det(\mathcal{F}) \otimes \mathcal{O}_{Gr}(-d). \tag{2.9}$$

Using

$$0 \to \pi_{Gr}^*\mathcal{O}_{Gr}(-1) \to (1\times p)^*(\mathcal{F}^\vee \otimes det\,\mathcal{F}) \to (1\times p)^* det\mathcal{F} \otimes \pi_{Gr}^* U_{Gr}^\vee \to 0,$$

we get
(2.10)
$$det\, R\pi_{Gr}(1 \times p)^*(\mathcal{F}^\vee \otimes det\,\mathcal{F}) = (det\, R\pi_{Gr}(1 \times p)^* det\,\mathcal{F})^{r-1} \otimes \mathcal{O}_{Gr}(-d).$$

Thus, by (2.8)–(2.10) and the base change theorem, we have

$$\begin{aligned} p^* det\, R\pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}}(\mathcal{F} \otimes \mathcal{F}^\vee) = p^*\{&(det\, R\pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}}\mathcal{F})^{r+1} \otimes (det\, R\pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}}\mathcal{F}^\vee)^{r-1} \\ &\otimes (det\, R\pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}} det\,\mathcal{F})^{-2}\}. \end{aligned}$$

By duality and the exact sequence

$$0 \to \mathcal{F} \to \mathcal{F} \otimes \omega_{\widetilde{X} \times \widetilde{\mathbf{Q}}_{\mathbf{F}}/\widetilde{\mathbf{Q}}_{\mathbf{F}}} \to \bigoplus_q \mathcal{F}_q \to 0,$$

one has that

$$det\, R\pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}}\mathcal{F}^\vee = det\, R\pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}}(\mathcal{F} \otimes \omega_{\widetilde{X} \times \widetilde{\mathbf{Q}}_{\mathbf{F}}/\widetilde{\mathbf{Q}}_{\mathbf{F}}}) = (det\, R\pi_{\widetilde{\mathbf{Q}}_{\mathbf{F}}}\mathcal{F}) \otimes \bigotimes_q (det \mathcal{F}_q)^{-1}.$$

Thus the lemma follows if $p^* : Pic(\widetilde{\mathbf{Q}}_{\mathbf{F}}) \to Pic(Gr \setminus D)$ is injective, which will be proved in the next lemma.

Lemma 2.11. *$p^* : Pic(\widetilde{\mathbf{Q}}_{\mathbf{F}}) \to Pic(Gr \setminus D)$ is injective.*

Proof. It is well known that $Pic(Gr) = Pic(\widetilde{\mathbf{Q}}_{\mathbf{F}}) \oplus \mathbb{Z}\mathcal{O}_{Gr}(1)$. For each fibre $Gr(E) = p^{-1}(E)$ of $p : Gr \to \widetilde{\mathbf{Q}}_{\mathbf{F}}$, $D \cap Gr(E)$ is an irreducible hypersurface of $Gr(E)$ (see Lemma 7.3 of [DN]). Thus the ideal sheaf

$$\mathcal{O}_{Gr}(-D) = p^*(\Lambda) \otimes \mathcal{O}_{Gr}(a_0), \quad \text{for some } \Lambda \in Pic(\widetilde{\mathbf{Q}}_{\mathbf{F}}),$$

with $a_0 \neq 0$. One has the exact sequence (see Chapter II, Proposition 6.5, of [Ha])

$$\mathbb{Z} \xrightarrow{i} Pic(Gr) \to Pic(Gr \setminus D) \to 0,$$

where $i(1) = \mathcal{O}_{Gr}(-D)$. For any $\mathcal{L} \in Pic(\widetilde{\mathbf{Q}}_{\mathbf{F}})$, if $p^*\mathcal{L}|_{Gr \setminus D}$ is trivial, then there exists $m \in \mathbb{Z}$ such that

$$p^*\mathcal{L} = i(m) = p^*(\Lambda^m) \otimes \mathcal{O}_{Gr}(ma_0).$$

The m has to be zero, namely, $p^*\mathcal{L} = \mathcal{O}_{Gr}$, which implies clearly that $\mathcal{L}$ is trivial.

§3. Geometry of moduli spaces of generalized parabolic sheaves

We will prove in this section that $\mathcal{P}$ and its subvarieties $\mathcal{D}_j(a)$ and $\mathcal{D}_1(a)\cap\mathcal{D}_2(b)$ are reduced, normal with rational singularities. In particular, we will prove that $\mathcal{H}$, $\hat{\mathcal{D}}_j(a)$, and $\hat{\mathcal{D}}_1(a)\cap\hat{\mathcal{D}}_2(b)$ are reduced, normal with rational singularities, and prove a formula to express the canonical (dualizing) sheaf of $\mathcal{H}$. We will use the following device to analyse singularities of a variety V: find varieties W and V' and smooth morphisms $f: V\to W$ and $f': V'\to W$, such that the singularities of V' are easy to analyse. We will call V' (or its complete local ring at a point) the smooth model of V (or local smooth model at a point of V). For simplicity, we assume that $|I|=0$, which will not affect the generality. Let Y be a scheme of finite type, $\mathcal{F}$ a locally free $\mathcal{O}_Y$-module of rank r and

$$\mathbb{H}om(\mathcal{O}_Y^r,\mathcal{F}) := Spec\, S(\mathcal{H}om(\mathcal{O}_Y^r,\mathcal{F})^\vee)\to Y,$$

which parametrizes homomorphisms from $\mathcal{O}_Y^r$ to $\mathcal{F}$. Let

$$F_Y := Isom(\mathcal{O}_Y^r,\mathcal{F})\subset \mathbb{H}om(\mathcal{O}_Y^r,\mathcal{F})$$

be the open subscheme corresponding to isomorphisms. Then we call $F_Y\to Y$ the frame bundle associated to $\mathcal{F}$. Moreover, if $\mathcal{E}$ is an $\mathcal{O}_Y$-module, then the functor

$$\mathbb{H}om(\mathcal{E},\mathcal{F}): T\longmapsto Hom_{\mathcal{O}_T}(\mathcal{E}_T,\mathcal{F}_T)$$

from the category of Y-schemes to the category of sets is representable by

$$V(\mathcal{F}^\vee\otimes_{\mathcal{O}_Y}\mathcal{E}) := Spec\, S(\mathcal{F}^\vee\otimes_{\mathcal{O}_Y}\mathcal{E})$$

(Proposition 9.6.1 of [EGA-I]), where $\mathcal{E}_T$ and $\mathcal{F}_T$ denote the pulling backs of $\mathcal{E}$ and $\mathcal{F}$ to T. We will also need that

Lemma 3.1. *Let $f: W\to V$ be a smooth morphism. Then W is reduced (respectively, normal, Cohen-Macaulay, Gorenstein) with only rational singularities if and only if V is.*

Proof. See Proposition 4.19 of [NR] for the statement about the rational singularity. The other statements are well-known commutative algebraic facts.

By a point $(E,h)\in\mathcal{H}$, we mean that $(\mathcal{O}^{\tilde{n}}\xrightarrow{e}E\to 0, E_{x_1}\oplus E_{x_2}\xrightarrow{h}\mathbb{C}^r\to 0)$ such that E is locally free outside $\{x_1,x_2\}$, $(\mathrm{Tor}E)_{x_1}\oplus(\mathrm{Tor}E)_{x_2}\overset{h}{\hookrightarrow}\mathbb{C}^r$, and $H^1(E(-x_1-x_2-x))=0$ for $x\in\widetilde{X}$, e induces isomorphism $\mathbb{C}^{\tilde{n}}\cong H^0(E)$. Let Λ be the category of Artinian local $\mathbb{C}$-algebras, and consider the functor

$$\Phi_1:\Lambda\to\mathbf{Set}$$

defined by (write $S = Spec(A)$ for any $A \in \Lambda$):

$$\Phi_1(A) := \left\{ \begin{array}{l} \text{Equivalence classes of flat families } (\mathcal{E}^S, h^S) \\ \text{such that } (\mathcal{E}^S, h^S)|_{Spec(A/m)\times \widetilde{X}} = (E, h) \end{array} \right\},$$

where $(\mathcal{E}^S, h^S) = (\mathcal{O}^{\tilde{n}}_{S\times\widetilde{X}} \xrightarrow{e^S} \mathcal{E}^S \to 0, \mathcal{E}^S_{x_1} \oplus \mathcal{E}^S_{x_2} \xrightarrow{h^S} \mathcal{O}^r_S \to 0)$.

Given a point $z = (E, h) \in \mathcal{H}$, let $dim(h_1(E_{x_1})) = r_1$ and $dim(h_2(E_{x_2})) = r_2$. The map e induces two maps $\mathbb{C}^{\tilde{n}} \xrightarrow{e_i} E_{x_i} \to 0$. We denote images of the canonical base of $\mathbb{C}^{\tilde{n}}$ (under h_i) by

$$(\alpha^i_{j1}, \cdots, \alpha^i_{jr}) \in \mathbb{C}^r$$

where $j = 1, ..., \tilde{n}$. Without loss of generality, we assume that

$$rk \begin{pmatrix} \alpha^i_{11} & \dots & \alpha^i_{1r_i} \\ \vdots & \ddots & \vdots \\ \alpha^i_{\tilde{n}1} & \dots & \alpha^i_{\tilde{n}r_i} \end{pmatrix} = r_i.$$

For any algebra A, we use $A^r \xrightarrow{r_i} A^{r_i}$ and $A^r \xrightarrow{[r-r_i]} A^{r-r_1}$ to denote the projections $(y_1, ..., y_r) \mapsto (y_1, ..., y_{r_i})$ and $(y_1, ..., y_r) \mapsto (y_{r_i+1}, ..., y_r)$ respectively. Thus for any $(\mathcal{E}^S, h^S) \in \Phi_1(A)$,

$$\mathcal{E}^S_{x_i} \xrightarrow{h^S_i} \mathcal{O}^r_S \xrightarrow{r_i} \mathcal{O}^{r_i}_S$$

are surjective since it is so at the fibre then by Nakayama's lemma, from which we get a surjection

$$\mathcal{E}^S \xrightarrow{q^S} {}_{x_1}\mathcal{O}^{r_1}_S \oplus {}_{x_2}\mathcal{O}^{r_2}_S \to 0.$$

Letting $\widetilde{\mathcal{E}}^S = ker(q^S)$, we get

$$0 \to \widetilde{\mathcal{E}}^S \to \mathcal{E}^S \xrightarrow{q^S} {}_{x_1}\mathcal{O}^{r_1}_S \oplus {}_{x_2}\mathcal{O}^{r_2}_S \to 0,$$

$$\mathcal{E}^S_{x_1} \xrightarrow{h^S_1} \mathcal{O}^r_S \xrightarrow{[r-r_1]} \mathcal{O}^{r-r_1}_S, \quad \mathcal{E}^S_{x_2} \xrightarrow{h^S_2} \mathcal{O}^r_S \xrightarrow{[r-r_2]} \mathcal{O}^{r-r_2}_S$$

which we denote by $(\widetilde{\mathcal{E}}^S, q^S, [r - r_1] \cdot h^S_1, [r - r_2] \cdot h^S_2) := \varphi_A(\mathcal{E}^S, h^S)$. It is clear that the restriction of $\varphi_A(\mathcal{E}^S, h^S)$ at the fibre $Spec(A/m) \times \widetilde{X}$ is

$$0 \to \widetilde{E} \to E \xrightarrow{q^0} {}_{x_1}\mathbb{C}^{r_1} \oplus {}_{x_2}\mathbb{C}^{r_2} \to 0,$$

$$E_{x_1} \xrightarrow{h_1} \mathbb{C}^r \xrightarrow{[r-r_1]} \mathbb{C}^{r-r_1}, \quad E_{x_2} \xrightarrow{h_2} \mathbb{C}^r \xrightarrow{[r-r_2]} \mathbb{C}^{r-r_2}$$

which is $\varphi_{\mathbb{C}}(E,h)$. Thus the above construction gives a morphism

$$\varphi : \Phi_1 \to \Phi_2$$

of functors, where Φ_2 will be defined later.

Let $\widetilde{\mathbf{Q}}^1$ be the Quot scheme of rank r, degree $d - r_1 - r_2$ quotients

$$\mathcal{O}_{\widetilde{X}}^{\tilde{n}-r_1-r_2} \to \widetilde{E} \to 0$$

and $\widetilde{\mathbf{Q}}_F^1$ the open subset of locally free quotients with vanishing $H^1(\widetilde{E})$ such that $\mathbb{C}^{\tilde{n}-r_1-r_2} \to H^0(\widetilde{E})$ is an isomorphism. It is known that $\widetilde{\mathbf{Q}}_F^1$ is smooth. Let $f : \widetilde{\mathbf{Q}}_F^1 \times \widetilde{X} \to \widetilde{\mathbf{Q}}_F^1$ be the projection, $\widetilde{\mathcal{E}}$ the universal quotient on $\widetilde{\mathbf{Q}}_F^1 \times \widetilde{X}$. Then the sheaf (see [La] for the definition)

$$\mathcal{G} := Ext_f^1({}_{x_1}\mathcal{O}^{r_1} \oplus {}_{x_2}\mathcal{O}^{r_2}, \widetilde{\mathcal{E}})$$

where $\mathcal{O} = \mathcal{O}_{\widetilde{\mathbf{Q}}_F^1}$, is locally free. Write $V := V(\mathcal{G}^\vee) \to \widetilde{\mathbf{Q}}_F^1$ and

$$p_V : V \times \widetilde{X} \to \widetilde{\mathbf{Q}}_F^1 \times \widetilde{X}.$$

Then there exists a universal extension on $V \times \widetilde{X}$

$$0 \to p_V^*\widetilde{\mathcal{E}} \to \mathcal{E} \to {}_{x_1}\mathcal{O}_V^{r_1} \oplus {}_{x_2}\mathcal{O}_V^{r_2} \to 0.$$

Let W_i be the total space of $\mathcal{H}om_{\mathcal{O}_V}(\mathcal{E}_{x_i}, \mathcal{O}_V^{r-r_i})$, namely, the V-scheme

$$W_i = V(\mathcal{E}_{x_i}^{\oplus(r-r_i)}) \to V$$

that represents the functor $\mathbb{H}om(\mathcal{E}_{x_i}, \mathcal{O}_V^{r-r_i})$ (see Proposition (9.6.1) of [EGA-I]) and let $Y := W_1 \times_V W_i$. Then the S-points of Y can be expressed as

$$\left\{ (\widetilde{\mathcal{E}}^S, q, e^S, \xi_1, \xi_2) := \begin{pmatrix} 0 \to \widetilde{\mathcal{E}}^S \to \mathcal{E}^S \xrightarrow{q} {}_{x_1}\mathcal{O}_S^{r_1} \oplus {}_{x_2}\mathcal{O}_S^{r_2} \to 0, \\ \mathcal{O}^{\tilde{n}-r_1-r_2} \xrightarrow{e^S} \widetilde{\mathcal{E}}^S \to 0,\ \mathcal{E}_{x_1}^S \xrightarrow{\xi_1} \mathcal{O}_S^{r-r_1},\ \mathcal{E}_{x_2}^S \xrightarrow{\xi_2} \mathcal{O}_S^{r-r_2} \end{pmatrix} \right\}$$

where $\mathcal{O}^{\tilde{n}-r_1-r_2} \xrightarrow{e^S} \widetilde{\mathcal{E}}^S \to 0$ is induced from $\widetilde{\mathcal{E}}$ by an S-point of $\widetilde{\mathbf{Q}}_F^1$. Thus we can define the functor $\Phi_2 : \Lambda \to \mathbf{Set}$ as

$$\Phi_2(A) = \left\{ \begin{array}{l} (\widetilde{\mathcal{E}}^S, q, \xi_1, \xi_2) \quad \text{such that its restriction at } Spec(A/m) \times \widetilde{X} \\ \text{is } (\widetilde{E}, q^0, [r-r_1]\cdot h_1, [r-r_2]\cdot h_2), \text{ which is } y := \varphi_{\mathbb{C}}(E,h) \end{array} \right\}$$

where $\widetilde{\mathcal{E}}^S$ is a flat family of bundles parametrized by S, which are generated by global sections and with vanishing H^1. We have an obvious smooth morphism $Y(-) \to \Phi_2$ by forgetting the frames of $\pi_{S*}\widetilde{\mathcal{E}}^S$, where $Y(-)$ denotes the functor defined by the variety Y.

Lemma 3.2. *The morphism $\varphi : \Phi_1 \to \Phi_2$ is formally smooth.*

Proof. For any $A \to A_0 \to 0$, where $A_0 = A/I$, $I^2 = 0$, we consider

$$\begin{array}{ccc} \Phi_1(A) & \longrightarrow & \Phi_1(A_0) \\ {\scriptstyle\varphi_A}\downarrow & & {\scriptstyle\varphi_{A_0}}\downarrow \\ \Phi_2(A) & \longrightarrow & \Phi_2(A_0) \end{array}$$

For any given points $(\mathcal{E}^{S_0}, h^{S_0}) \in \Phi_1(A_0)$ and $(\widetilde{\mathcal{E}}^S, q, \xi, \xi_2) \in \Phi_2(A)$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathcal{E}}^S|_{S_0 \times \widetilde{X}} & \longrightarrow & \mathcal{E}^S|_{S_0 \times \widetilde{X}} & \xrightarrow{q|_{S_0 \times \widetilde{X}}} & {}_{x_1}\mathcal{O}_{S_0}^{r_1} \oplus {}_{x_2}\mathcal{O}_{S_0}^{r_2} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \widetilde{\mathcal{E}}^{S_0} & \longrightarrow & \mathcal{E}^{S_0} & \xrightarrow{q^{S_0}} & {}_{x_1}\mathcal{O}_{S_0}^{r_1} \oplus {}_{x_2}\mathcal{O}_{S_0}^{r_2} & \longrightarrow & 0 \end{array}$$

and $\xi_1|_{S_0} = [r - r_1] \cdot h_1^{S_0}$, $\xi_2|_{S_0} = [r - r_2] \cdot h_2^{S_0}$, we need to show that there exists a point

$$(\mathcal{E}', h') = (\mathcal{O}^{\tilde{n}} \xrightarrow{e'} \mathcal{E}' \to 0, \mathcal{E}'_{x_1} \oplus \mathcal{E}'_{x_2} \xrightarrow{h'} \mathcal{O}_S^r \to 0) \in \Phi_1(A)$$

such that $\varphi_A(\mathcal{E}', h') = (\widetilde{\mathcal{E}}^S, q, \xi, \xi_2)$ and $(\mathcal{E}', h')|_{S_0} = (\mathcal{E}^{S_0}, h^{S_0})$.

We take $\mathcal{E}' = \mathcal{E}^S$ and $e' = e^S$ to be a lifting of e^{S_0}, which always exists and is surjective since $I^2 = 0$. To define h', let $q_i : \mathcal{E}_{x_i}^S \to \mathcal{O}_S^{r_i} \to 0$ be the two induced surjections by q, and define $h'_i : \mathcal{E}_{x_i}^S \xrightarrow{(q_i, \xi_i)} \mathcal{O}_S^r$, which gives a surjective morphism $h' : \mathcal{E}_{x_1}^S \oplus \mathcal{E}_{x_2}^S \to \mathcal{O}_S^r \to 0$ since its restriction on S_0 is h^{S_0}. One can check that it is what we want.

By the above lemma, we are reduced to considering the singularities of Y. To analyse the singularities of Y, we can fix a $\widetilde{E} \in \widetilde{\mathbf{Q}}_F^1$ since

$$Y = W_1 \times_V W_2 \to \widetilde{\mathbf{Q}}_F^1$$

is locally trivial, namely, the singularities of Y are the same with that of any fibre (note that $\widetilde{\mathbf{Q}}_F^1$ is smooth).

Proposition 3.1. *Let $\widetilde{E}$ be a vector bundle of rank r on $\widetilde{X}$, $x_1, x_2 \in \widetilde{X}$ and $V = V(Ext^1({}_{x_1}\mathbb{C}^{r_1} \oplus {}_{x_2}\mathbb{C}^{r_2}, \widetilde{E})^\vee)$, $p : V \times \widetilde{X} \to \widetilde{X}$. Consider the universal extension*

$$0 \to p^*\widetilde{E} \to E \to {}_{x_1}\mathcal{O}_V^{r_1} \oplus {}_{x_2}\mathcal{O}_V^{r_2} \to 0$$

on $\widetilde{X} \times V$. Then the space $\mathbf{E} = V(E_{x_1}^{\oplus(r-r_1)}) \times_V V(E_{x_2}^{\oplus(r-r_2)})$ is reduced, irreducible and normal with rational singularities.

Proof. Replace $\widetilde{X}$ by an affine neighbourhood of $\{x_1, x_2\}$ where $\widetilde{E}$ is trivial. Furthermore, we can assume that $\widetilde{X} = \mathbb{A}^1$ such that $x_1 = \{t = 0\}$ and $x_2 = \{t = 1\}$, namely,

$$\widetilde{X} = Spec\, \mathbb{C}[t] \supset \{(t), (1-t)\}, \qquad \widetilde{E} = \mathbb{C}[t]^r.$$

Let $F = Ext^1({}_{x_1}\mathbb{C}^{r_1} \oplus {}_{x_2}\mathbb{C}^{r_2}, \widetilde{E})$, let $\{e_{ij}\}$ be a $\mathbb{C}$-basis of F, and let $\{x_{ij}\} = \{e_{ij}^*\}$ be the dual basis of $\{e_{ij}\}$. Then $V = Spec(S(F^\vee)) = Spec(\mathbb{C}[\{x_{ij}\}])$ and the element

$$\begin{aligned}\gamma = \sum_{i,j} x_{ij}e_{ij} &\in F^\vee \otimes F \hookrightarrow S(F^\vee) \otimes F \\ &= Ext^1({}_{x_1}\mathbb{C}[\{x_{ij}\}]^{r_1} \oplus {}_{x_2}\mathbb{C}[\{x_{ij}\}]^{r_2}), p^*\widetilde{E})\end{aligned}$$

determines the universal extension

$$0 \to p^*\widetilde{E} \to E \to {}_{x_1}\mathbb{C}[\{x_{ij}\}]^{r_1} \oplus {}_{x_2}\mathbb{C}[\{x_{ij}\}]^{r_2} \to 0.$$

To construct the universal extension E, we need a resolution of ${}_{x_1}\mathbb{C}^{r_1} \oplus {}_{x_2}\mathbb{C}^{r_2}$,

$$0 \to \mathbb{C}[t]^{r_1} \oplus \mathbb{C}[t]^{r_2} \xrightarrow{\alpha} \mathbb{C}[t]^{r_1} \oplus \mathbb{C}[t]^{r_2} \to {}_{x_1}\mathbb{C}^{r_1} \oplus {}_{x_2}\mathbb{C}^{r_2} \to 0 \tag{3.1}$$

where α is defined by

$$\mathbb{C}[t]^{r_1} \oplus \mathbb{C}[t]^{r_2} = \bigoplus_{i=1}^{r_1+r_2} \mathbb{C}[t]e_i, \qquad \alpha(e_i) = \begin{cases} te_i, & \text{for } i \le r_1, \\ (1-t)e_i, & \text{for } i > r_1. \end{cases}$$

We have

$$F = \frac{Hom_{\mathbb{C}[t]}(\mathbb{C}[t]^{r_1} \oplus \mathbb{C}[t]^{r_2}, \mathbb{C}[t]^r)}{\alpha^* Hom_{\mathbb{C}[t]}(\mathbb{C}[t]^{r_1} \oplus \mathbb{C}[t]^{r_2}, \mathbb{C}[t]^r)}.$$

Define $e_{ij} : \mathbb{C}[t]^{r_1} \oplus \mathbb{C}[t]^{r_2} \to \mathbb{C}[t]^r$ $(1 \le i \le r_1 + r_2, 1 \le j \le r)$ to be

$$e_{ij}(e_i) = (\overbrace{0, \cdots, 0, 1}^{j}, 0, \cdots, 0), \qquad e_{ij}(e_k) = (0, \cdots, 0) \quad \text{if } k \ne i.$$

Then $\{e_{ij}\}$ is a basis of F. Let $p^*\widetilde{E} = \mathbb{C}[t]^r \otimes_{\mathbb{C}} \mathbb{C}[x_{ij}] = \mathbb{C}[t, x_{ij}]^r = \mathcal{O}^r_{V \times \widetilde{X}}$, and let

$$\begin{aligned}0 \to \mathbb{C}[t, x_{ij}]^{r_1} \oplus \mathbb{C}[t, x_{ij}]^{r_2} &\xrightarrow{\alpha} \mathbb{C}[t, x_{ij}]^{r_1} \oplus \mathbb{C}[t, x_{ij}]^{r_2} \\ &\to {}_{x_1}\mathbb{C}[x_{ij}]^{r_1} \oplus {}_{x_2}\mathbb{C}[x_{ij}]^{r_2} \to 0\end{aligned}$$

be the pull-back of (3.1). The extension E determined by

$$\gamma : \mathbb{C}[t, x_{ij}]^{r_1} \oplus \mathbb{C}[t, x_{ij}]^{r_2} \to p^*\widetilde{E} = \mathbb{C}[x_{ij}, t]^r$$

is

$$E = \frac{\mathbb{C}[x_{ij}, t]^r \oplus \mathbb{C}[x_{ij}, t]^{r_1+r_2}}{W} = \frac{\sum_{k=1}^{r+r_1+r_2} \mathbb{C}[x_{ij}, t]y_k}{W},$$

where $W = \{(\gamma(a), -\alpha(a)) \mid a \in \mathbb{C}[x_{ij}, t]^{r_1+r_2}\}$. We can describe E by the following exact sequence:

$$0 \to \bigoplus_{k=1}^{r_1+r_2} \mathbb{C}[x_{ij}, t]e_k \xrightarrow{\beta} \bigoplus_{k=1}^{r+r_1+r_2} \mathbb{C}[x_{ij}, t]y_k \to E \to 0$$

where $\beta(e_k) = \gamma(e_k) \oplus (-\alpha(e_k))$ equals

$$\sum_{j=1}^{r} x_{kj}y_j - \begin{cases} ty_{r+k}, & k \leq r_1, \\ (1-t)y_{r+k}, & k > r_1. \end{cases}$$

Thus we get

$$0 \to \bigoplus_{k=1}^{r_1+r_2} \mathbb{C}[x_{ij}]e_k \xrightarrow{\beta_{x_i}} \bigoplus_{k=1}^{r+r_1+r_2} \mathbb{C}[x_{ij}]y_k \to E_{x_i} \to 0. \tag{3.2}$$

where

$$\beta_{x_1}(e_k) = \begin{cases} \sum_{j=1}^{r} x_{kj}y_j, & \text{if } k \leq r_1, \\ \sum_{j=1}^{r} x_{kj}y_j - y_{r+k} & \text{if } k > r_1, \end{cases}$$

$$\beta_{x_2}(e_k) = \begin{cases} \sum_{j=1}^{r} x_{kj}y_j - y_{r+k}, & \text{if } k \leq r_1, \\ \sum_{j=1}^{r} x_{kj}y_j & \text{if } k > r_1. \end{cases}$$

Let

$$\mathbf{X}_1 = \begin{pmatrix} x_{11} & \dots & x_{1r} \\ \vdots & \ddots & \vdots \\ x_{r_1 1} & \dots & x_{r_1 r} \end{pmatrix}, \qquad \mathbf{X}_2 = \begin{pmatrix} x_{r_1+1,1} & \dots & x_{r_1+1,r} \\ \vdots & \ddots & \vdots \\ x_{r_1+r_2,1} & \dots & x_{r_1+r_2,r} \end{pmatrix}.$$

We have

$$S(E_{x_1}) = \frac{\mathbb{C}[\mathbf{X}_1, \mathbf{X}_2, \vec{\mathbf{y}}, y_{r+1}, \cdots, y_{r+r_1}]}{(\mathbf{X}_1 \cdot \vec{\mathbf{y}})} \qquad \text{where } \vec{\mathbf{y}} = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}$$

and

$$S(E_{x_2}) = \frac{\mathbb{C}[\mathbf{X}_1, \mathbf{X}_2, \vec{\mathbf{y}}, y_{r+r_1+1}, \cdots, y_{r+r_1+r_2}]}{(\mathbf{X}_2 \cdot \vec{\mathbf{y}})}.$$

Note that

$$\mathcal{O}_{\mathbf{E}} = \overbrace{S(E_{x_1}) \otimes_{\mathbb{C}[x_{ij}]} \cdots \otimes_{\mathbb{C}[x_{ij}]} S(E_{x_1})}^{r-r_1} \otimes_{\mathbb{C}[x_{ij}]} \overbrace{S(E_{x_2}) \otimes_{\mathbb{C}[x_{ij}]} \cdots \otimes_{\mathbb{C}[x_{ij}]} S(E_{x_2})}^{r-r_2}$$

and set

$$y_{jl} = \overbrace{1 \otimes \cdots \otimes y_j}^{l} \otimes \cdots \otimes 1 \quad (1 \leq j \leq r + r_1 + r_2,\ 1 \leq l \leq 2r - r_1 - r_2).$$

We have

$$\mathcal{O}_{\mathbf{E}} = \frac{\mathbb{C}[\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2]}{(\mathbf{X}_1 \cdot \mathbf{Y}_1,\ \mathbf{X}_2 \cdot \mathbf{Y}_2)} \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{Y}_3, \mathbf{Y}_4],$$

where

$$\mathbf{Y}_1 = \begin{pmatrix} y_{11} & \cdots & y_{1,r-r_1} \\ \vdots & \ddots & \vdots \\ y_{r1} & \cdots & y_{r,r-r_1} \end{pmatrix}, \quad \mathbf{Y}_4 = \begin{pmatrix} y_{r+r_1+1,1} & \cdots & y_{r+r_1+1,2r-r_1-r_2} \\ \vdots & \ddots & \vdots \\ y_{r+r_1+r_2,1} & \cdots & y_{r+r_1+r_2,2r-r_1-r_2} \end{pmatrix}$$

and

$$\mathbf{Y}_2 = \begin{pmatrix} y_{1,r-r_1+1} & \cdots & y_{1,2r-r_1-r_2} \\ \vdots & \ddots & \vdots \\ y_{r,r-r_1+1} & \cdots & y_{r,2r-r_1-r_2} \end{pmatrix},$$

$$\mathbf{Y}_3 = \begin{pmatrix} y_{r+1,1} & \cdots & y_{r+1,2r-r_1-r_2} \\ \vdots & \ddots & \vdots \\ y_{r+r_1,1} & \cdots & y_{r+r_1,2r-r_1-r_2} \end{pmatrix}.$$

By taking $a = r_i$ and $b = r - r_i$ in the following Lemma 3.3, and noting that $I_{r_i}(\mathbf{X}_i) = \{0\}$ and $I_{r-r_i}(\mathbf{Y}_i) = \{0\}$, the proposition is proved.

Lemma 3.3. *Let $\mathbf{X} = (x_{ij})_{p\times r}$, $\mathbf{Y} = (y_{ij})_{r\times q}$ be two matrices and let $I_a(\mathbf{X})$ (resp. $I_b(\mathbf{Y})$) denote the set of rank $a+1$ (resp. $b+1$) subdeterminants of $\mathbf{X}$ (resp. $\mathbf{Y}$). Then*

$$D_{a,b} = Spec \frac{\mathbb{C}[\mathbf{X}, \mathbf{Y}]}{(\mathbf{X} \cdot \mathbf{Y}, I_a(\mathbf{X}), I_b(\mathbf{Y}))}$$

is reduced, irreducible and normal with rational singularities if $a + b \leq r$.

Proof. The fact that the variety is reduced, normal and Cohen-Macaulay is a special case of theorems in [CS]. Theorems in [He] imply that it has only rational singularities (see Example 6.5 of [He]).

Remark 3.1. The above varieties $D_{a,b}$ were called double determinantal varieties in [He], whose dimension formula is

$$dim(D_{a,b}) = a(r + p) + b(r + q) - a^2 - b^2 - ab.$$

Taking $a = p = r_i$, $b = q = r - r_i$ $(i = 1, 2)$, we have $D_{r_i,r-r_i} = \operatorname{Spec}\mathbb{C}[\mathbf{X}_i, \mathbf{Y}_i]/(\mathbf{X}_i \cdot \mathbf{Y}_i)$ and $dim(D_{r_i,r-r_i}) = r^2 + r_i^2 - rr_i$. It is easy to see that the ideal $(\mathbf{X}_i \cdot \mathbf{Y}_i)$ has $r_i(r - r_i)$ generators at most and

$$dim(\operatorname{Spec}\mathbb{C}[\mathbf{X}_i, \mathbf{Y}_i]) - dim(D_{r_i,r-r_i}) = r_i(r - r_i) = height(I).$$

Thus $D_{r_i,r-r_i}$ are complete intersections. In particular, $\mathcal{H}$ and $\mathcal{P}$ are Gorenstein.

Proposition 3.2. *$\mathcal{H}$, $\hat{\mathcal{D}}_j(a)$ and $\hat{\mathcal{D}}_1(a)\cap\hat{\mathcal{D}}_2(b)$ are reduced, normal with rational singularities. In particular, $\mathcal{P}$, $\mathcal{D}_j(a)$ and $\mathcal{D}_1(a)\cap\mathcal{D}_2(b)$ are reduced, normal with rational singularities.*

Proof. By Lemma 3.2 and Proposition 3.1, it is true for $\mathcal{H}$. We only need to show the proposition for $\hat{\mathcal{D}}_j(a)$ and $\hat{\mathcal{D}}_1(a)\cap\hat{\mathcal{D}}_2(b)$. Let us rewrite (3.2) into

$$0\to\bigoplus_{k=1}^{r_1}\mathbb{C}[x_{ij}]e_k\xrightarrow{\beta_{x_1}}\bigoplus_{k=1}^{r+r_1}\mathbb{C}[x_{ij}]y_k\to E_{x_1}\to 0,$$

$$0\to\bigoplus_{k=r_1+1}^{r_1+r_2}\mathbb{C}[x_{ij}]e_k\xrightarrow{\beta_{x_2}}\bigoplus_{k=1}^{r+r_2}\mathbb{C}[x_{ij}]y_k\to E_{x_2}\to 0$$

where $\beta_{x_i}(e_k)=\sum_{j=1}^{r}x_{kj}y_j$. The universal maps $E_{x_j}\to\mathcal{O}^{r-r_j}$ are induced by

$$(f_1,\cdots,f_{r+r_1})\xmapsto{u_1}(\sum_{i=1}^{r+r_1}f_iy_{i1},\cdots,\sum_{i=1}^{r+r_1}f_iy_{i,r-r_1}),$$

$$(f_1,\cdots,f_{r+r_2})\xmapsto{u_2}(\sum_{i=1}^{r+r_2}f_iy_{i,r-r_1+1},\cdots,\sum_{i=1}^{r+r_2}f_iy_{i,2r-r_1-r_2}).$$

Let $E_{x_j}\to\mathcal{O}^{r_j}$ be the induced projections by the projection in the universal extension $((f_1,\cdots,f_{r+r_j})\xmapsto{p_j}(f_{r+1},\cdots,f_{r+r_j}))$. Then the maps $E_{x_j}\to\mathcal{O}^r$ are induced by $\mathcal{O}^{r+r_j}\xrightarrow{(p_j,u_j)}\mathcal{O}^r$, matrices of which are

$$\begin{pmatrix}\mathbf{0}&\mathbf{Y}_1\\ \mathbf{I}_{r_1}&\mathbf{Y}_3'\end{pmatrix},\quad\begin{pmatrix}\mathbf{0}&\mathbf{Y}_2\\ \mathbf{I}_{r_2}&\mathbf{Y}_3''\end{pmatrix}$$

where $\mathbf{I}_{r_i}$ denote $r_i\times r_i$ unit matrices and $(\mathbf{Y}_3',\mathbf{Y}_3'')=\mathbf{Y}_3$ (we use the notions in Proposition 3.1). It is not difficult to see that the local smooth models for $\hat{\mathcal{D}}_j(a)$ and $\hat{\mathcal{D}}_1(a)\cap\hat{\mathcal{D}}_2(b)$ at $z=(E,h)$ are

$$\operatorname{Spec}\frac{\mathbb{C}[\mathbf{X}_j,\mathbf{Y}_j]}{(\mathbf{X}_j\cdot\mathbf{Y}_j,I_{a-r_j}(\mathbf{Y}_j))}\times\operatorname{Spec}\frac{\mathbb{C}[\mathbf{X}_i,\mathbf{Y}_i]}{(\mathbf{X}_i\cdot\mathbf{Y}_i)}\quad(j=1,2,\ i\neq j)$$

and

$$\operatorname{Spec}\frac{\mathbb{C}[\mathbf{X}_1,\mathbf{Y}_1]}{(\mathbf{X}_1\cdot\mathbf{Y}_1,I_{a-r_1}(\mathbf{Y}_1))}\times\operatorname{Spec}\frac{\mathbb{C}[\mathbf{X}_2,\mathbf{Y}_2]}{(\mathbf{X}_2\cdot\mathbf{Y}_2,I_{b-r_2}(\mathbf{Y}_2))}.$$

The proposition follows Lemma 3.3 (note that $a\le r$).

Remark 3.2. (1) It is easy to see from the proof that a point $(E,h)\in\mathcal{H}$ is a smooth point in the following cases: (i) E is torsion free at x_1 and $h_2:E_{x_2}\to\mathbb{C}^r$ is surjective, i.e., $r_2=r$ (the roles of x_1 and x_2 are reversed).

(ii) Both of $h_j : E_{x_j} \to \mathbb{C}^r$ are surjective (i.e., $r_1 = r_2 = r$). In particular, one can see that $\hat{\mathcal{D}}_j(0)$ are smooth.

(2) The locus of non-locally-free extensions is

$$Spec\frac{\mathbb{C}[\mathbf{X}_1,\mathbf{Y}_1,\mathbf{X}_2,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4]}{(\mathbf{X}_1\cdot\mathbf{Y}_1,\mathbf{X}_2\cdot\mathbf{Y}_2,I_{r_1-1}(\mathbf{X}_1))}\cup Spec\frac{\mathbb{C}[\mathbf{X}_1,\mathbf{Y}_1,\mathbf{X}_2,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4]}{(\mathbf{X}_1\cdot\mathbf{Y}_1,\mathbf{X}_2\cdot\mathbf{Y}_2,I_{r_2-1}(\mathbf{X}_2))}.$$

More precisely, the non-locally-free locus $\mathcal{H}\setminus\mathcal{H}_F$ of $\mathcal{H}$ has two components $\hat{\mathcal{D}}^t_j$ $(j=1,2)$: $\hat{\mathcal{D}}^t_1$ is the component of $\mathcal{H}\setminus\mathcal{H}_F$ parametrising sheaves with nonzero torsion at x_2 ($\hat{\mathcal{D}}^t_2$ is defined similarly), whose local smooth models are

$$Spec\frac{\mathbb{C}[\mathbf{X}_1,\mathbf{Y}_1,\mathbf{X}_2,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4]}{(\mathbf{X}_1\cdot\mathbf{Y}_1,\mathbf{X}_2\cdot\mathbf{Y}_2,I_{r_1-1}(\mathbf{X}_1))}\text{ and }Spec\frac{\mathbb{C}[\mathbf{X}_1,\mathbf{Y}_1,\mathbf{X}_2,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4]}{(\mathbf{X}_1\cdot\mathbf{Y}_1,\mathbf{X}_2\cdot\mathbf{Y}_2,I_{r_2-1}(\mathbf{X}_2))}.$$

We will give more information about the subschemes $\hat{\mathcal{D}}^t_j$ of $\mathcal{H}$ in the following Proposition 3.3.

Proposition 3.3. *Let $\tilde{\psi}' : \widetilde{\mathcal{R}}'^{ss} \to \mathcal{P}$ be the projection. Then we have*

(1) $\tilde{\psi}'(\hat{\mathcal{D}}^t_1\cap\widetilde{\mathcal{R}}'^{ss}) = \mathcal{D}_1$, $\tilde{\psi}'(\hat{\mathcal{D}}^t_2\cap\widetilde{\mathcal{R}}'^{ss}) = \mathcal{D}_2$ *and*

(2) *the codimension-one subschemes $\hat{\mathcal{D}}^t_j$ in $\mathcal{H}$ are reduced, irreducible and normal.*

Proof. (1) We will prove that $\tilde{\psi}'(\hat{\mathcal{D}}^t_2\cap\widetilde{\mathcal{R}}'^{ss}) = \mathcal{D}_2$; the other one is similar. For any $(E',Q')\in\hat{\mathcal{D}}^t_2\cap\widetilde{\mathcal{R}}'^{ss}$, $\mathrm{Tor}(E')_{x_1}\neq 0$ by the definition. Thus it is s-equivalent to a semistable GPS (E,Q) with E locally free by Lemma 2.5. Moreover, by checking the proof of Lemma 2.5, one find that $E_{x_2}\to Q$ has rank $r-1$; so $\tilde{\psi}'((E,Q))\in\mathcal{D}_2$. Each point of $\mathcal{D}_2$ is the image of a GPS (E,Q) with E locally free, and $E_{x_2}\to Q$ is not surjective. Thus $E_{x_1}\to Q$ is nonzero, and we can choose a projection $Q\to\mathbb{C}$ such that $E_{x_1}\to Q\to\mathbb{C}$ is nonzero. Taking $\widetilde{E}$ to be the kernel of $E\to{}_{x_1}\mathbb{C}\to 0$ and $\widetilde{Q}$ to be the image of $\widetilde{E}_{x_1}\oplus\widetilde{E}_{x_2}$ under $E_{x_1}\oplus E_{x_2}\to Q$, we get an extension

$$0\to(\widetilde{E},\widetilde{Q})\to(E,Q)\to({}_{x_1}\mathbb{C},\,\mathbb{C})\to 0,$$

and one checks that (E,Q) is s-equivalent to $(\widetilde{E}\oplus{}_{x_1}\mathbb{C},\widetilde{Q}\oplus\mathbb{C})$. Hence we have proved that $\tilde{\psi}'(\hat{\mathcal{D}}^t_2\cap\widetilde{\mathcal{R}}'^{ss}) = \mathcal{D}_2$.

To prove (2), we only need to check the irreducibility, and the other facts follow Remark 3.2 (2) and Lemma 3.3. On $\widetilde{X}\times\mathcal{H}$, there is an exact sequence

$$0\to\mathcal{K}\to\mathcal{O}^{\tilde{n}}\to\mathcal{E}\to 0$$

where $\mathcal{K}$ is a vector bundle. It is easy to see that $\hat{\mathcal{D}}^t_1$ is the subscheme of $\mathcal{H}$ defined by

$$\hat{\mathcal{D}}^t_1 = \{h\in\mathcal{H}\,|\,\mathrm{rank}(\mathcal{K}_{(x_2,h)}\to\mathcal{O}^{\tilde{n}}_{(x_2,h)})\le\tilde{n}-r-1\}.$$

Thus we only need to prove that the open subset

$$\hat{\mathcal{D}}_1^{t,0} = \{h \in \hat{\mathcal{D}}_1^t \mid rank(\mathcal{K}_{(x_2,h)} \to \mathcal{O}^{\tilde{n}}_{(x_2,h)}) = \tilde{n} - r - 1\}$$

of $\hat{\mathcal{D}}_1^t$ is irreducible. $\hat{\mathcal{D}}_1^{t,0}$ is the open subset of sheaves of the form $\widetilde{E} \oplus {}_{x_2}\mathbb{C}$ with $\widetilde{E}$ generated by global sections and having vanishing $H^1(\widetilde{E})$. It is now straightforward to imitate the proof of Remark 5.5 of [Ne].

We have shown (see Remark 3.1) that $\mathcal{H}$ is Gorenstein; so it has a canonical sheaf. Before closing this section, we will prove a formula for expressing the canonical sheaf of $\mathcal{H}$. Let

$$\mathcal{O}^{\tilde{n}} \to \mathcal{E} \to 0, \quad \mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2} \to \mathcal{Q} \to 0$$

be the universal quotients on $\widetilde{X} \times \mathcal{H}$ and $\mathcal{H}$. We write down an obvious lemma at first.

Lemma 3.4. *Let $\omega_{\widetilde{X}} = \mathcal{O}(\sum_q q)$ be the canonical sheaf of $\widetilde{X}$ and let $\omega_{\widetilde{\mathcal{R}}'_F}$ denote the canonical bundle of $\widetilde{\mathcal{R}}'_F$. Then*

$$\omega^{-1}_{\widetilde{\mathcal{R}}'_F} = (det\, R\pi_{\widetilde{\mathcal{R}}'_F}\mathcal{E})^{2r} \otimes \bigotimes_{x\in I}\left\{(det\,\mathcal{E}_x)^{n_{l_x+1}-r} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)}\right\} \otimes$$
$$\bigotimes_q (det\,\mathcal{E}_q)^{1-r} \otimes (det\, R\pi_{\widetilde{\mathcal{R}}'_F} det\mathcal{E})^{-2} \otimes (det\,\mathcal{Q})^{2r} \otimes (det\,\mathcal{E}_{x_1})^{-r} \otimes (det\,\mathcal{E}_{x_2})^{-r}.$$

Proof. $\widetilde{\mathcal{R}}'_F \to \widetilde{\mathcal{R}}_F$ is a grassmannian bundle over $\widetilde{\mathcal{R}}_F$. Then use Proposition 2.2.

We will give an extension of the right-hand side of the above formula to $\mathcal{H}$ as a line $PGL(\tilde{n})$-bundle, then prove that the extension gives the canonical bundle of $\mathcal{H}$. Note that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}^{\tilde{n}} \to \mathcal{E} \to 0$$

on $\widetilde{X} \times \mathcal{H}$, and $\mathcal{K}$ is flat over $\mathcal{H}$ since $\mathcal{E}$ is so. One proves that $\mathcal{K}$ is locally free on $\widetilde{X} \times \mathcal{H}$ (by using Lemma 5.4 of [Ne]). For $x \in \widetilde{X} \setminus \{x_1, x_2\}$, we have the identity $det\,(\mathcal{K}_x)^{-1} = det\,\mathcal{E}_x$ on $\mathcal{H}$. It is clear that

$$\Omega^{-1} := (det\, R\pi_{\mathcal{H}}\mathcal{E})^{2r} \otimes \bigotimes_{x\in I}\left\{(det\,\mathcal{E}_x)^{n_{l_x+1}-r} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)}\right\} \otimes$$
$$\bigotimes_q (det\,\mathcal{E}_q)^{1-r} \otimes (det\, R\pi_{\mathcal{H}} det\,\mathcal{K}^{-1})^{-2} \otimes (det\,\mathcal{Q})^{2r} \otimes (det\,\mathcal{K}_{x_1})^{r} \otimes (det\,\mathcal{K}_{x_2})^{r}$$

is an extension of the line bundle in Lemma 3.4. We now prove that it is the dual of the canonical sheaf of $\mathcal{H}$.

Proposition 3.4. *Let $\mathcal{K}$ be the kernel of the universal surjection $\mathcal{O}^{\tilde{n}} \to \mathcal{E}$ on $\widetilde{X} \times \mathcal{H}$, and let $\omega_{\mathcal{H}}$ denote the canonical bundle of $\mathcal{H}$. Then*

$$\omega_{\mathcal{H}}^{-1} = \Omega^{-1} = (det\, R\pi_{\mathcal{H}}\mathcal{E})^{2r} \otimes$$
$$\bigotimes_{x\in I}\left\{(det\,\mathcal{E}_x)^{n_{l_x+1}-r} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)}\right\}\otimes$$
$$\bigotimes_q (det\,\mathcal{E}_q)^{1-r}\otimes(det\, R\pi_{\mathcal{H}} det\,\mathcal{K}^{-1})^{-2}\otimes(det\,\mathcal{Q})^{2r}\otimes(det\,\mathcal{K}_{x_1})^r\otimes(det\,\mathcal{K}_{x_2})^r.$$

Proof. By Lemma 3.4, $\omega_{\mathcal{H}}^{-1} = \Omega^{-1}$ holds outside the $\hat{\mathcal{D}}_j^t$. We will check that it extends to each $\hat{\mathcal{D}}_j^t$. For definiteness take $j = 1$ and for simplicity of notation suppose there is no ordinary parabolic point. We will determine $\omega_{\mathcal{H}}$ in a neighbourhood of a suitable point of $\hat{\mathcal{D}}_1^t$. Since $\hat{\mathcal{D}}_1^t$ is irreducible, it will be enough to show that $\omega_{\mathcal{H}}^{-1} = \Omega^{-1}$ holds in one such neighbourhood.

We consider a point $(\mathcal{O}^{\tilde{n}} \to E \to 0, Q)$ in $\mathcal{H}$ satisfying
(1) E has torsion at x_2 (i.e., the point lies on $\hat{\mathcal{D}}_1^t$),
(2) E is locally free at x_1, and
(3) the maps $E_{x_j} \to Q$ are surjective for both $j = 1, 2$.
The conditions (2) and (3) will hold in a neighourhood U of the point. On $\widetilde{X} \times U$, we define a locally free sheaf $\widetilde{\mathcal{E}}$ by the exact sequence

$$0 \to \widetilde{\mathcal{E}} \to \mathcal{E} \to {}_{x_2}\mathcal{Q} \to 0, \tag{3.3}$$

where ${}_{x_2}\mathcal{Q}$ is the sheaf on $\widetilde{X}\times\mathcal{H}$ obtained by pulling back $\mathcal{Q}$ from $\mathcal{H}$ and then restricting to $\{x_2\}\times\mathcal{H}$. We can assume that for any $u \in U$ the vector bundle $\widetilde{\mathcal{E}}_u$ is generated by global sections and $H^1(\widetilde{\mathcal{E}}_u) = 0$. Thus $\pi_{U*}\widetilde{\mathcal{E}}$ is a locally free sheaf of rank $\tilde{n} - r$ and commutes with any base change (π_U denotes the projection $\widetilde{X} \times U \to U$). Let $p : F_U \to U$ denote the frame bundle of $\pi_{U*}\widetilde{\mathcal{E}}$. We will use the same notation $\widetilde{\mathcal{E}}$, $\mathcal{E}$ and $\mathcal{Q}$ to denote their pulling back to $\widetilde{X} \times F_U$ and F_U.

Let $\widetilde{\mathbf{Q}}$ be the Quot scheme of rank r, degree $d - r$ quotients

$$\mathcal{O}^{\tilde{n}-r} \to \widetilde{E}' \to 0$$

and let $\widetilde{\mathbf{Q}}_F$ be the open subset of locally free quotients generated by global sections with $H^1(\widetilde{E}') = 0$. Let $\mathcal{O}^{\tilde{n}-r} \to \widetilde{E}' \to 0$ induce the isomorphism $\mathbb{C}^{\tilde{n}-r} \cong H^0(\widetilde{E}')$. Let

$$\mathcal{O}^{\tilde{n}-r}_{\widetilde{X}\times\widetilde{\mathbf{Q}}_F} \to \widetilde{\mathcal{E}}' \to 0$$

be the universal quotient. Then there is a morphism $f_1 : F_U \to \widetilde{\mathbf{Q}}_F$ such that $(1 \times f_1)^*\widetilde{\mathcal{E}}' = \widetilde{\mathcal{E}}$. Let $\pi_{\widetilde{\mathbf{Q}}_F} : \widetilde{X} \times \widetilde{\mathbf{Q}}_F \to \widetilde{\mathbf{Q}}_F$ be the projection and

$\mathbb{E} = Ext^1_{\pi_{\widetilde{\mathbf{Q}}_F}}({}_{x_2}\widetilde{\mathcal{E}}'_{x_1}, \widetilde{\mathcal{E}}')$ (see [La] for the definition of the sheaf), where ${}_{x_2}\widetilde{\mathcal{E}}'_{x_1}$ is the sheaf on $\widetilde{X} \times \widetilde{\mathbf{Q}}_F$ obtained by pulling back $\widetilde{\mathcal{E}}'_{x_1}$ from $\widetilde{\mathbf{Q}}_F$ and then restricting to $\{x_2\} \times \widetilde{\mathbf{Q}}_F$. Then there exists a universal extension

$$0 \to (1 \times q_1)^*\widetilde{\mathcal{E}}' \to \mathcal{E}' \to {}_{x_2}q_1^*\widetilde{\mathcal{E}}'_{x_1} \to 0$$

on $\widetilde{X} \times V(\mathbb{E}^\vee)$, where $q_1 : V(\mathbb{E}^\vee) \to \widetilde{\mathbf{Q}}_F$ is the projection. Note that $\widetilde{\mathcal{E}}_{x_1} = \mathcal{E}_{x_1} \cong \mathcal{Q}$ on U and, by (3.3), we have a morphism $f_2 : F_U \to V(\mathbb{E}^\vee)$ such that $f_1 = q_1 \cdot f_2$ and $(1 \times f_2)^*\mathcal{E}' = \mathcal{E}$. Let $q : F_V \to V(\mathbb{E}^\vee)$ be the frame bundle of $\pi_{V*}\mathcal{E}'$, where $\pi_V : \widetilde{X} \times V(\mathbb{E}^\vee) \to V(\mathbb{E}^\vee)$ is the projection and $\mathcal{O}^{\tilde{n}}_{F_V} \xrightarrow{u} q^*(\pi_{V*}\mathcal{E}')$ is the universal frame. Then, if we denote the pulling back of $\widetilde{\mathcal{E}}'$ and $\mathcal{E}'$ still by $\widetilde{\mathcal{E}}'$ and $\mathcal{E}'$, there is a morphism $f : F_U \to F_V$ such that $(1 \times f)^*\widetilde{\mathcal{E}}' = \widetilde{\mathcal{E}}$, $(1 \times f)^*\mathcal{E}' = \mathcal{E}$, and

$$\begin{array}{ccc} \mathcal{O}^{\tilde{n}}_{F_U} & \xrightarrow{f^*(u)} & f^*(\pi_{F_V*}\mathcal{E}') \\ \| & & \| \\ \mathcal{O}^{\tilde{n}}_{F_U} & \longrightarrow & \pi_{F_U*}\mathcal{E} \end{array}$$

is commutative, where $\mathcal{O}^{\tilde{n}}_{F_U} \to \pi_{F_U*}\mathcal{E}$ is the induced isomorphism obtained by taking the direct image of $\mathcal{O}^{\tilde{n}}_{\widetilde{X}\times F_U} \to \mathcal{E} \to 0$.

It is not difficult to check that $f : F_U \to F_V$ is unramified. In fact, the universal frame $\mathcal{O}^{\tilde{n}}_{F_V} \xrightarrow{u} q^*(\pi_{V*}\mathcal{E}')$ induces a quotient

$$\mathcal{O}^{\tilde{n}}_{\widetilde{X}\times F_V} \xrightarrow{1\otimes u} (1 \times q)^*(\pi_V^*\pi_{V*}\mathcal{E}') \to (1 \times q)^*\mathcal{E}' := \mathcal{E}' \to 0$$

on $\widetilde{X} \times F_V$, which gives a morphism $g_1 : F_V \to \widetilde{\mathcal{R}}$ such that $\mathcal{E}' = (1 \times g_1)^*\mathcal{E}$. The universal extension gives a quotient

$$\mathcal{E}'_{x_1} \oplus \mathcal{E}'_{x_2} \to \widetilde{\mathcal{E}}'_{x_1} \to 0$$

on F_V, and thus a morphism $g_2 : F_V \to U$ such that $g_2^*\mathcal{Q} = \widetilde{\mathcal{E}}'_{x_1}$ and $(1 \times g_2)^*\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}'$. Finally, the universal quotient $\mathcal{O}^{\tilde{n}-r} \to \widetilde{\mathcal{E}}' \to 0$ induces an isomorphism $\mathcal{O}^{\tilde{n}-r}_{F_V} \cong \pi_{F_V*}\widetilde{\mathcal{E}}'$. Thus we have a morphism $g : F_V \to F_U$, which can be checked to be a section of $f : F_U \to F_V$. Hence f is actually an isomorphism if $dim(F_U) = dim(F_V)$.

Now we check that $dim(F_U) = dim(F_V)$. It is easy to check that

$$dim(F_U) - dim(F_V) = r^2 - rank(\mathbb{E}).$$

Thus we need to determine the locally free sheaf $\mathbb{E} = Ext^1_{\pi_{\widetilde{\mathbf{Q}}_F}}({}_{x_2}\widetilde{\mathcal{E}}'_{x_1}, \widetilde{\mathcal{E}}')$. Using the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}\times\widetilde{\mathbf{Q}}_F}(-\{x_2\}\times\widetilde{\mathbf{Q}}_F)\otimes\pi^*_{\widetilde{\mathbf{Q}}_F}\widetilde{\mathcal{E}}'_{x_1} \to \pi^*_{\widetilde{\mathbf{Q}}_F}\widetilde{\mathcal{E}}'_{x_1} \to {}_{x_2}\widetilde{\mathcal{E}}'_{x_1} \to 0$$

and using that $Ext^1_{\pi_{\widetilde{\mathbf{Q}}_F}}(\pi^*_{\widetilde{\mathbf{Q}}_F}\widetilde{\mathcal{E}}'_{x_1}, \widetilde{\mathcal{E}}') = R^1\pi_{\widetilde{\mathbf{Q}}_F*}(\widetilde{\mathcal{E}}'\otimes\pi^*_{\widetilde{\mathbf{Q}}_F}\widetilde{\mathcal{E}}'^{\vee}_{x_1}) = 0$, we have

$$(3.4)\quad 0 \to \pi_{\widetilde{\mathbf{Q}}_F*}(\widetilde{\mathcal{E}}')\otimes\widetilde{\mathcal{E}}'^{\vee}_{x_1} \to \pi_{\widetilde{\mathbf{Q}}_F*}(\widetilde{\mathcal{E}}'\otimes\mathcal{O}_{\widetilde{X}\times\widetilde{\mathbf{Q}}_F}(\{x_2\}\times\widetilde{\mathbf{Q}}_F))\otimes\widetilde{\mathcal{E}}'^{\vee}_{x_1} \to \mathbb{E} \to 0.$$

One can see easily that $rank(\mathbb{E}) = r^2$, and thus $\omega_{F_U} = f^*\omega_{F_V} = f^*q^*\omega_V$.

Since $\omega_V^{-1} = q_1^*\omega^{-1}_{\widetilde{\mathbf{Q}}_F}\otimes det(q_1^*\mathbb{E})$ and $det(\mathbb{E}) = (det\,\widetilde{\mathcal{E}}'_{x_2})^r\otimes(det\,\widetilde{\mathcal{E}}'_{x_1})^{-r}$ (by using (3.4) and the Riemann-Roch theorem), we have, using Lemma 2.10 and the pulling back of (3.3),

$$(3.5)\quad \begin{aligned}\omega_{F_U}^{-1} &= (det\,R\pi_{F_U}\mathcal{E})^{2r}\otimes\bigotimes_q(det\,\mathcal{E}_q)^{1-r}\otimes(det\,R\pi_{F_U}det\,\widetilde{\mathcal{E}})^{-2}\\ &\otimes(det\,\mathcal{Q})^{2r}\otimes(det\,\widetilde{\mathcal{E}}_{x_2})^r\otimes(det\,\widetilde{\mathcal{E}}_{x_1})^{-r}.\end{aligned}$$

On $\widetilde{X}\times F_U$, let $\mathcal{K}'$ be the kernel of $\mathcal{O}^{\tilde{n}} \to \mathcal{E} \to {}_{x_2}\mathcal{Q}$. Then we have the commutative diagram

$$(3.6)\quad \begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{K}' & \longrightarrow & \widetilde{\mathcal{E}} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{O}^{\tilde{n}} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & {}_{x_2}\mathcal{Q} & = & {}_{x_2}\mathcal{Q} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

One sees easily that $\mathcal{K}'$ is a vector bundle of rank $\tilde{n}$, and (note that F_U is smooth)

$$det\,\mathcal{K}' = \mathcal{O}_{\widetilde{X}\times F_U}(-r\cdot\{x_2\}\times F_U).$$

Thus we can compute easily that

$$\begin{aligned} det\,R\pi_{F_U}det\,\widetilde{\mathcal{E}} &= det\,R\pi_{F_U}(det\,\mathcal{K}'\otimes det\,\mathcal{K}^{-1})\\ &= det\,R\pi_{F_U}(det\,\mathcal{K}^{-1})\otimes(det\,\mathcal{K}_{x_2})^{-r}.\end{aligned}$$

Noting that $det\,\widetilde{\mathcal{E}}_{x_2} = (det\,\mathcal{K}_{x_2})^{-1} \otimes (det\,\mathcal{K}'_{x_2})$ and that $det\,\mathcal{K}'_{x_2}$ is trivial, we have

$$
\begin{aligned}
\omega_{F_U}^{-1} &= (det\,R\pi_{F_U}\mathcal{E})^{2r} \otimes \bigotimes_q (det\,\mathcal{E}_q)^{1-r} \otimes (det\,R\pi_{F_U} det\,\mathcal{K}^{-1})^{-2} \\
&\quad \otimes (det\,\mathcal{Q})^{2r} \otimes (det\,\mathcal{K}_{x_2})^r \otimes (det\,\mathcal{K}_{x_1})^r.
\end{aligned}
\tag{3.7}
$$

Thus, by (3.7), we have $p^*\omega_U^{-1} = p^*\Omega^{-1}$, which shows clearly that $\omega_{\mathcal{H}}^{-1} = \Omega^{-1}$ holds on U since $p : F_U \to U$ is locally trivial for the Zariski's topology, and we are done.

§4. Seminormality and the decomposition theorem

Let I_Z denote the ideal sheaf of a closed subscheme Z in a scheme X. When Z is of codimension one (not necessarily a Cartier divisor), we set $\mathcal{O}_X(-Z) := I_Z$. If $\mathcal{L}$ is a line bundle on X and if Y is a closed subscheme of X, we denote $\mathcal{L} \otimes I_Z$ and the restriction $I_Z \otimes \mathcal{O}_Y$ of I_Z on Y by $\mathcal{L}(-Z)$ and $\mathcal{O}_Y(-Z)$. Now we collect some general facts at first.

Lemma 4.1. *Let V be a projective scheme on which a reductive group G acts, $\widetilde{\mathcal{L}}$ an ample line bundle linearising the G-action, and V^{ss} the open subscheme of semistable points. Let V' be a G-invariant closed subscheme of V^{ss}, $\bar{V}'$ its schematic closure in V. Then*

(1) *$\bar{V}'^{ss} = V'$, and $V'//G$ is a closed subscheme of $V^{ss}//G$.*

(2) *$H^0(V^{ss}, \widetilde{\mathcal{L}})^{inv} = H^0(W, \widetilde{\mathcal{L}})^{inv}$, where W is an open G-invariant (irreducible) normal subscheme of V containing V^{ss} and $(\ \ )^{inv}$ denotes the invariant subspace for an action of G.*

Proof. See Lemma 4.14 and Lemma 4.15 of [NR].

Lemma 4.2. *Let V be a normal variety with a G-action, where G is a reductive algebraic group. Suppose a good quotient $\pi : V \to U$ exists. Let $\widetilde{\mathcal{L}}$ be a G-line bundle on V, and suppose it descends as a line bundle $\mathcal{L}$ on U. Let $V'' \subset V' \subset V$ be open G-invariant subvarieties of V such that V' maps onto U and $V'' = \pi^{-1}(U'')$ for some nonempty open subset U'' of U. Then any invariant section of $\widetilde{\mathcal{L}}$ on V' extends to V.*

Proof. See Lemma 4.16 of [NR].

Lemma 4.3. *Suppose there is given a seminormal variety V with normalization $\sigma : \widetilde{V} \to V$. Let the non-normal locus be W, endowed with its reduced structure. Let $\widetilde{W}$ be the set-theoretic inverse image of W in $\widetilde{V}$, endowed with its reduced structure. Let N be a line bundle on V, and let $\widetilde{N}$ be its pull-back to $\widetilde{V}$ ($\widetilde{N} = \sigma^*N$). Suppose $H^0(\widetilde{V}, \widetilde{N}) \to H^0(\widetilde{W}, \widetilde{N})$ is surjective. Then*

(1) *there is an exact sequence*

$$0 \to H^0(\widetilde{V}, \widetilde{N} \otimes I_{\widetilde{W}}) \to H^0(V, N) \to H^0(W, N) \to 0.$$

(2) *If* $H^1(W,N) \to H^1(\widetilde{W},\widetilde{N})$ *is injective, so is* $H^1(V,N) \to H^1(\widetilde{V},\widetilde{N})$.

Proof. See Proposition 5.8 of [NR].

Lemma 4.4. *Let G be a reductive group and P a parabolic subgroup. Let $\mathcal{L}$ be an ample line bundle on G/P, and X a union of Schubert varieties with the reduced structure. Then*

(1) *the restriction map* $H^0(G/P,\mathcal{L}) \to H^0(X,\mathcal{L})$ *is surjective, and*

(2) $H^i(G/P,\mathcal{L})$ *and* $H^i(X,\mathcal{L})$ *vanish for* $i > 0$.

Proof. See Theorem 3 of [MRa].

Remark 4.1. Let F_1 and F_2 be two vector spaces of dimension r, and let Gr denote the grassmannian $Grass_r(F_1 \oplus F_2)$ of r-dimensional quotients. Let E_j be the vector bundle on Gr generated by F_j, and let $E_1 \oplus E_2 \to Q$ be the universal quotient. Write $l_j = (det\, E_j)^{-1} \otimes det\, Q$ and

$$D_j(a) = \{q \in Gr \mid rank(E_{jq} \to Q_q) \leq a\}.$$

The action of $GL(F_1) \times GL(F_2)$ on Gr lifts to the line bundles l_j, and the subvarieties $D_j(a)$ are invariant under the action of $GL(F_1) \times GL(F_2)$. Thus $H^0(l_j^k)$ and $H^0(l_1^k|_{D_1(a)\cap D_2 \cup D_1(a-1)})$ are $GL(F_1)\times GL(F_2)$ modules. By Lemma 4.4,

(4.1)
$$0 \to H^0(l_1^k \otimes I_{D_1(a)\cap D_2\cup D_1(a-1)}) \to H^0(l_1^k) \to H^0(l_1^k|_{D_1(a)\cap D_2\cup D_1(a-1)}) \to 0$$

is an exact sequence of $GL(F_1) \times GL(F_2)$ modules. Thus it is splitting.

Proposition 4.1. *The following maps are surjective for any* $1 \leq a \leq r$:

(1) $H^0(\mathcal{D}_1(a),\Theta_{\mathcal{P}}) \to H^0(\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1),\Theta_{\mathcal{P}})$;

(2) $H^0(\mathcal{D}_1(a),\Theta_{\mathcal{P}}) \to H^0(\mathcal{D}_1(a)\cap\mathcal{D}_2,\Theta_{\mathcal{P}})$.

Proof. It is clear that the following Proposition 4.2 implies Proposition 4.1.

Proposition 4.2. *The following maps are surjective for any* $1 \leq a \leq r$:

(1) $H^0(\mathcal{P},\Theta_{\mathcal{P}}) \to H^0(\mathcal{D}_1(a),\Theta_{\mathcal{P}})$,

(2) $H^0(\mathcal{P},\Theta_{\mathcal{P}}) \to H^0(\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1),\Theta_{\mathcal{P}})$,

(3) $H^0(\mathcal{P},\Theta_{\mathcal{P}}) \to H^0(\mathcal{D}_1(a)\cap\mathcal{D}_2,\Theta_{\mathcal{P}})$.

Proof. We will deal with (2) in detail; the other statements will follow the same proof. Let us consider the diagram

$$\begin{tikzcd}
H^0(\widetilde{\mathcal{R}}'^{ss},\hat{\Theta}')^{inv} \arrow[r, "a"] & H^0(\hat{\mathcal{D}}_1(a)^{ss}\cap\hat{\mathcal{D}}_2^{ss}\cup\hat{\mathcal{D}}_1(a-1)^{ss},\hat{\Theta}')^{inv} \\
H^0(\mathcal{H},\hat{\Theta}')^{inv} \arrow[r] \arrow[u, "e"] \arrow[d, "b"] & H^0(\hat{\mathcal{D}}_1(a)\cap\hat{\mathcal{D}}_2\cup\hat{\mathcal{D}}_1(a-1),\hat{\Theta}')^{inv} \arrow[u, "f"] \arrow[d, "d"] \\
H^0(\widetilde{\mathcal{R}}'_F,\hat{\Theta}')^{inv} \arrow[r, "c"] & H^0(\hat{\mathcal{D}}_{F,1}(a)\cap\hat{\mathcal{D}}_{F,2}\cup\hat{\mathcal{D}}_{F,1}(a-1),\hat{\Theta}')^{inv}.
\end{tikzcd}$$

We need to prove that a is surjective. The map e is an isomorphism by Lemma 4.1. To prove that b is an isomorphism, it is enough to check that

$$H^0(\widetilde{\mathcal{R}}'^{ss}, \hat{\Theta}')^{inv} \to H^0(\widetilde{\mathcal{R}}'^{ss} \cap \widetilde{\mathcal{R}}'_F, \hat{\Theta}')^{inv}$$

is an isomorphism. We use Lemma 4.2 with the identification $V = \widetilde{\mathcal{R}}'^{ss}$, $U = \mathcal{P}$, $\pi = \tilde{\psi}'$, $V' = \widetilde{\mathcal{R}}'^{ss} \cap \widetilde{\mathcal{R}}'_F$ and $U'' = \mathcal{P} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$. (One can show that $U'' = \mathcal{P} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$ is nonempty, for example, by Corollary 2.1.) Lemma 2.5 shows that $V' = \widetilde{\mathcal{R}}'^{ss} \cap \widetilde{\mathcal{R}}'_F$ maps onto $U = \mathcal{P}$. Thus b is also an isomorphism.

Given a section s of $H^0(\hat{\mathcal{D}}_1(a)^{ss} \cap \hat{\mathcal{D}}_2^{ss} \cup \hat{\mathcal{D}}_1(a-1)^{ss}, \hat{\Theta}')^{inv}$, it extends to sections s_1, s_2 on $\hat{\mathcal{D}}_1(a) \cap \hat{\mathcal{D}}_2$ and $\hat{\mathcal{D}}_1(a-1)$ by Lemma 4.1 since $\hat{\mathcal{D}}_1(a) \cap \hat{\mathcal{D}}_2$ and $\hat{\mathcal{D}}_1(a-1)$ are normal, which are equal on $\hat{\mathcal{D}}_1(a)^{ss} \cap \hat{\mathcal{D}}_2^{ss} \cap \hat{\mathcal{D}}_1(a-1)^{ss}$. For any point $x \in \hat{\mathcal{D}}_1(a) \cap \hat{\mathcal{D}}_2 \cap \hat{\mathcal{D}}_1(a-1) = \hat{\mathcal{D}}_1(a-1) \cap \hat{\mathcal{D}}_2$, we have $s_1(x) = s_2(x)$ if x is semistable, and $s_1(x) = s_2(x) = 0$ if x is nonsemistable (by the definition of semistability). Thus the sections s_1 and s_2 yield a section on $\hat{\mathcal{D}}_1(a) \cap \hat{\mathcal{D}}_2 \cup \hat{\mathcal{D}}_1(a-1)$, which is an extension of s. This proves that f is an isomorphism. Hence we only need to prove that

$$H^0(\widetilde{\mathcal{R}}'_F, \hat{\Theta}')^{inv} \xrightarrow{c} H^0(\hat{\mathcal{D}}_{F,1}(a) \cap \hat{\mathcal{D}}_{F,2} \cup \hat{\mathcal{D}}_{F,1}(a-1), \hat{\Theta}')^{inv}$$

is surjective. Recall that $\rho : \widetilde{\mathcal{R}}'_F \to \widetilde{\mathcal{R}}_F$ is a grassmannian bundle over $\widetilde{\mathcal{R}}_F$. Lemma 4.4 implies that

$$\begin{aligned} 0 \to \rho_*(\hat{\Theta}' \otimes I_{\hat{\mathcal{D}}_{F,1}(a) \cap \hat{\mathcal{D}}_{F,2} \cup \hat{\mathcal{D}}_{F,1}(a-1)}) &\to \rho_* \hat{\Theta}' \\ \to \rho_*(\hat{\Theta}'|_{\hat{\mathcal{D}}_{F,1}(a) \cap \hat{\mathcal{D}}_{F,2} \cup \hat{\mathcal{D}}_{F,1}(a-1)}) &\to 0 \end{aligned}$$

is exact. In fact, we claim that the above sequence is splitting. Noting that

$$\hat{\Theta}' = \rho^* \Theta_{\widetilde{\mathcal{R}}} \otimes (\det \mathcal{Q})^k \otimes (\det \mathcal{E}_y)^{-k}$$

and that $\mathcal{E}$ is the pull-back of $\mathcal{F}$ by ρ, we can rewrite

$$\hat{\Theta}' = \rho^*(\Theta_{\widetilde{\mathcal{R}}} \otimes \det \mathcal{F}_y^{-k} \otimes \det \mathcal{F}_{x_1}^k) \otimes (\det \mathcal{Q} \otimes \det \mathcal{E}_{x_1}^{-1})^k.$$

Let $X = \hat{\mathcal{D}}_{F,1}(a) \cap \hat{\mathcal{D}}_{F,2} \cup \hat{\mathcal{D}}_{F,1}(a-1)$ and $\eta_{x_1} = (\det \mathcal{Q})(\det \mathcal{E}_{x_1})^{-1}$. Then it is enough to show that

$$0 \to \rho_*(\eta_{x_1}^k \otimes I_X) \to \rho_* \eta_{x_1}^k \to \rho_*(\eta_{x_1}^k \otimes \mathcal{O}_X) \to 0 \tag{4.2}$$

is splitting. The above direct image sheaves can be thought of as vector bundles associated to representations of $GL(r) \times GL(r)$ in (4.1) of Remark 4.1 (see Remark 5.10 of [NR]). Thus, by Remark 4.1, (4.2) is splitting and we proved the proposition.

In order to prove the following proposition, we need to show that $\mathcal{W}_a$ is seminormal for any $0 \leq a \leq r$. However, we will admit this fact, and prove it later, so that we can prove the decomposition theorem as soon as possible.

Proposition 4.3. *We have a (noncanonical) isomorphism*

$$H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \cong H^0(\mathcal{P}, \Theta_{\mathcal{P}}(-\mathcal{D}_2)).$$

Proof. If we take, in Lemma 4.3, $V = \mathcal{W}_a$, $\widetilde{V} = \mathcal{D}_1(a)$, $\sigma = \phi|_{\mathcal{D}_1(a)}$ and $N = \Theta_{\mathcal{U}_X}|_{\mathcal{W}_a}$, then we have $W = \mathcal{W}_{a-1}$, $\widetilde{W} = \mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1)$ and $\widetilde{N} = \Theta_{\mathcal{P}}|_{\mathcal{D}_1(a)}$ by Proposition 2.1. Using Proposition 4.1(1) and Lemma 4.3, we have

$$\begin{aligned} 0 &\to H^0(\mathcal{D}_1(a), \Theta_{\mathcal{P}} \otimes I_{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)}) \\ &\to H^0(\mathcal{W}_a, \Theta_{\mathcal{U}_X}) \to H^0(\mathcal{W}_{a-1}, \Theta_{\mathcal{U}_X}) \to 0. \end{aligned}$$

Thus we have a noncanonical isomorphism
(4.3)

$$H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \cong H^0(\mathcal{W}_0, \Theta_{\mathcal{U}_X}) \oplus \bigoplus_{a=1}^{r} H^0(\mathcal{D}_1(a), \Theta_{\mathcal{P}} \otimes I_{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)}).$$

If we define $\mathcal{D}_1(-1) = \varnothing$, note that $\mathcal{D}_1(0) \cong \mathcal{W}_0$ and $\mathcal{D}_1(0) \cap \mathcal{D}_2 = \varnothing$ (by Lemma 2.4 (1) and Lemma 2.5), we can rewrite (4.3) into

$$H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \cong \bigoplus_{a=0}^{r} H^0(\mathcal{D}_1(a), \Theta_{\mathcal{P}} \otimes I_{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)}). \tag{4.4}$$

By Proposition 4.1, we have

$$\begin{aligned} &H^0(\mathcal{D}_1(a), \Theta_{\mathcal{P}} \otimes I_{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)}) \oplus H^0(\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1), \Theta_{\mathcal{P}}) \\ &\cong H^0(\mathcal{D}_1(a), \Theta_{\mathcal{P}}), \end{aligned}$$

which isomorphism and (4.4) imply that
(4.5)

$$\bigoplus_{a=0}^{r} H^0(\mathcal{D}_1(a), \Theta_{\mathcal{P}}) \cong H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \oplus \bigoplus_{a=0}^{r} H^0(\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1), \Theta_{\mathcal{P}}).$$

By using the following exact sequence

$$0 \to \mathcal{O}_{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)} \to \mathcal{O}_{\mathcal{D}_1(a)\cap\mathcal{D}_2} \oplus \mathcal{O}_{\mathcal{D}_1(a-1)} \to \mathcal{O}_{\mathcal{D}_1(a-1)\cap\mathcal{D}_2} \to 0$$

and Proposition 4.1, we get

$$\begin{aligned} &H^0(\mathcal{D}_1(a) \cap \mathcal{D}_2, \Theta_{\mathcal{P}}) \oplus H^0(\mathcal{D}_1(a-1), \Theta_{\mathcal{P}}) \\ &\cong H^0(\mathcal{D}_1(a-1) \cap \mathcal{D}_2, \Theta_{\mathcal{P}}) \oplus H^0(\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1), \Theta_{\mathcal{P}}). \end{aligned} \tag{4.6}$$

By (4.5) and (4.6), we have a noncanonical isomorphism

$$H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \oplus H^0(\mathcal{D}_1(r) \cap \mathcal{D}_2, \Theta_{\mathcal{P}}) \cong H^0(\mathcal{D}_1(r), \Theta_{\mathcal{P}}).$$

On the other hand, by Proposition 4.2, we have the exact sequence

$$0 \to H^0(\mathcal{P}, \Theta_{\mathcal{P}}(-\mathcal{D}_2)) \to H^0(\mathcal{P}, \Theta_{\mathcal{P}}) \to H^0(\mathcal{D}_2, \Theta_{\mathcal{P}}) \to 0.$$

Thus we proved our proposition if one remarks that $\mathcal{D}_1(r) = \mathcal{P}$.

We recall some facts about the representation of $GL(n)$ (see [FH]). For any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, we have the so-called Schur functor $\mathbb{S}_\lambda$ and Schur polynomial S_λ. One gets all of the irreducible representations of $GL(n)$ by applying Schur functors $\mathbb{S}_\lambda$ to the standard representation V of $GL(n)$. We denote these representations $\mathbb{S}_\lambda(V)$ by $R_\lambda := R_{\lambda_1,\cdots,\lambda_n}$ and $D_k = (\Lambda^n V)^{\otimes k}$. Then

$$R_{\lambda_1+k,\cdots,\lambda_n+k} = R_{\lambda_1,\cdots,\lambda_n} \otimes D_k,$$

and the dual of $R_{\lambda_1,\cdots,\lambda_n}$, which is isomorphic to $\mathbb{S}_\lambda(V^*)$, is the representation $R_{-\lambda_n,\cdots,-\lambda_1}$. In a more fantastic language, R_λ is the irreducible representation with highest weight

$$(\lambda_1 - \lambda_2)\omega_1 + (\lambda_2 - \lambda_3)\omega_2 + \cdots + (\lambda_{n-1} - \lambda_n)\omega_{n-1} + \lambda_n\omega_n,$$

where $\omega_1, \cdots, \omega_n$ are the fundamental weights defined by

$$\omega_i(diag(a_1, \cdots, a_n)) = a_1 + \cdots + a_i.$$

Let $N_{\mu v\lambda}$ denote the Littlewood-Richardson number. Then we have a general decomposition over $GL(V) \times GL(W)$

$$\mathbb{S}_\lambda(V \oplus W) = \bigoplus N_{\mu v\lambda}(\mathbb{S}_\mu V \otimes \mathbb{S}_v W) \tag{4.7}$$

where the sum is over all partitions μ, v such that the sum of the numbers partitioned by μ and v is the number partitioned by λ.

For $j = 1, 2$, let E_j be r-dimensional vector spaces, and Gr denote the grassmannian of r-dimensional quotients $E_1 \oplus E_2 \to Q$. We still use E_j to denote the vector bundle on Gr generated by E_j, and use Q to denote the universal quotient $E_1 \oplus E_2 \to Q$ on Gr.

Lemma 4.5. *Let l_j denote the line bundle $(det\, E_j)^{-1} \otimes det\, Q$ on Gr. Then we have a natural isomorphism of $GL(E_1) \times GL(E_2)$ modules*

$$H^0(Gr, l_2^m) = \bigoplus_\mu \mathbb{S}_\mu(E_1) \otimes \mathbb{S}_\mu(E_2^*),$$

where $\mu = (\mu_1, \cdots, \mu_r)$ runs through the integers $0 \le \mu_r \le \cdots \le \mu_1 \le m$.

Proof. It is clear that $H^0(Gr, l_2^m) = (\Lambda^r E_2)^{-m} \otimes H^0(Gr, (det\, Q)^m)$, where the space $H^0(Gr, (det\, Q)^m)$ is an irreducible representation of $GL(2r)$ with highest weight $m\omega_r$ (see §15.4 of [FH]). Thus

$$H^0(Gr, l_2^m) = (\Lambda^r E_2)^{-m} \otimes \mathbb{S}_\lambda(E_1 \oplus E_2),$$

where $\lambda = (m, \cdots, m)$. Using (4.7), we have

$$\mathbb{S}_\lambda(E_1 \oplus E_2) = \bigoplus N_{\mu v \lambda}(\mathbb{S}_\mu E_1 \otimes \mathbb{S}_v E_2).$$

Clearly, if $N_{\mu v \lambda} \neq 0$, $\mu = (\mu_1, \cdots, \mu_r)$ must satisfy that $0 \le \mu_r \le \cdots \le \mu_1 \le m$. The skew Schur function $S_{\lambda/\mu} = |H_{\lambda_i - \mu_j - i + j}|$ $(i, j = 1, \cdots, r)$ can be written as

$$S_{\lambda/\mu} = \sum N_{\mu v \lambda} S_v$$

in terms of ordinary Schur polynomials S_v (see §6 of [FH]), where $S_v = |H_{v_i + j - i}|$. On the other hand, for given $\mu = (\mu_1, \cdots, \mu_r)$ and $\lambda = (m, \cdots, m)$,

$$S_{\lambda/\mu} = |H_{m - \mu_j - i + j}| = |H_{m - \mu_i + i - j}| = |H_{m - \mu_{r-i+1} + j - i}| = S_v$$

where $v = (m - \mu_r, \cdots, m - \mu_1)$. Thus $N_{\mu v \lambda} = 0$ when $v \neq (m - \mu_r, \cdots, m - \mu_1)$ and $N_{\mu v \lambda} = 1$ when $v = (m - \mu_r, \cdots, m - \mu_1)$. Note that

$$\mathbb{S}_{(m-\mu_r, \cdots, m-\mu_1)}(E_2) = (\Lambda^r E_2)^m \otimes \mathbb{S}_\mu(E_2^*). \tag{4.8}$$

We have

$$\mathbb{S}_\lambda(E_1 \oplus E_2) = \bigoplus_\mu (\Lambda^r E_2)^m \otimes \mathbb{S}_\mu(E_1) \otimes \mathbb{S}_\mu(E_2^*),$$

which proves that

$$H^0(Gr, l_2^m) = \bigoplus_\mu \mathbb{S}_\mu(E_1) \otimes \mathbb{S}_\mu(E_2^*) \tag{4.9}$$

where μ runs through the integers $0 \le \mu_r \le \cdots \le \mu_1 \le m$.

Given $\mu = (\mu_1, \cdots, \mu_r)$, $\mathbb{S}_\mu(E_1)$ is the irreducible representation of $GL(r)$ with highest weight

$$(\mu_1 - \mu_2)\omega_1 + \cdots + (\mu_{r-1} - \mu_r)\omega_{r-1} + \mu_r \omega_r.$$

We can rewrite it as (forgetting into the zero terms)

$$(\mu_{r_1(x_1)} - \mu_{r_1(x_1)+1})\omega_{r_1(x_1)} + \cdots + (\mu_{r_l(x_1)} - \mu_{r_l(x_1)+1})\omega_{r_l(x_1)} + \mu_r \omega_r.$$

Let $d_i(x_1) = \mu_{r_i(x_1)} - \mu_{r_i(x_1)+1}$ $(i = 1, \cdots, l)$ and let $Flag_{\vec{n}(x_1)}(E_1)$ be the flag variety of type $\vec{n}(x_1) = (n_1(x_1), \cdots, n_l(x_1))$, where $n_i(x_1) = r_i(x_1) - r_{i-1}(x_1)$ (we set $n_1(x_1) = r_1(x_1)$). If we denote the universal flag on $Flag_{\vec{n}(x_1)}(E_1)$ by

$$E_1 = F_0(E_1) \supset F_1(E_1) \supset \cdots \supset F_l(E_1) \supset F_{l+1}(E_1) = 0$$

and the quotient $E_1/F_i(E_1)$ by $\mathcal{Q}_{x_1,i}$, then we have

$$H^0(Flag_{\vec{n}(x_1)}(E_1), \bigotimes_{i=1}^{l}(det\mathcal{Q}_{i,x_1})^{d_i(x_1)}) \otimes (\Lambda^r E_1)^{\mu_r} = \mathbb{S}_\mu(E_1). \tag{4.10}$$

Similarly, if we set $r_i(x_2) = r - r_{l-i+1}(x_1)$ and $d_i(x_2) = d_{l-i+1}(x_1)$, we have

$$H^0(Flag_{\vec{n}(x_2)}(E_2), \bigotimes_{i=1}^{l}(det\mathcal{Q}_{i,x_2})^{d_i(x_2)}) \otimes (\Lambda^r E_2)^{-\mu_1} = \mathbb{S}_\mu(E_2^*). \tag{4.11}$$

We remark that l may be zero, namely, $\mu = (\mu_r, \cdots, \mu_r)$ and $\mathbb{S}_\mu(E_1)$ is the one-dimensional irreducible representation $(\Lambda^r E_1)^{\mu_r}$ in this case.

Recall that $\mathcal{O}^{\tilde{n}} \to \mathcal{F} \to 0$ is the universal quotient on $\widetilde{X} \times \widetilde{\mathbf{Q}}_F$ and

$$Grass_r(\mathcal{F}_{x_1} \oplus \mathcal{F}_{x_2}) \xrightarrow{f} \widetilde{\mathbf{Q}}_F, \quad \widetilde{\mathcal{R}}_F = \times_{\widetilde{\mathbf{Q}}_F, x\in I} Flag_{\vec{n}(x)}(\mathcal{F}_x) \xrightarrow{g} \widetilde{\mathbf{Q}}_F.$$

We will use $\mathcal{E}$ to denote the various pull-backs of $\mathcal{F}$. Let $g_j : Flag_{\vec{n}(x_j)}(\mathcal{F}_{x_j}) \to \widetilde{\mathbf{Q}}_F$ $(j = 1, 2)$ be the relative flag scheme of type $\vec{n}(x_j)$ and

$$\mathcal{E}_{x_j} = F_0(\mathcal{E}_{x_j}) \supset F_1(\mathcal{E}_{x_j}) \supset \cdots \supset F_l(\mathcal{E}_{x_j}) \supset F_{l+1}(\mathcal{E}_{x_j}) = 0$$

the universal flag on $Flag_{\vec{n}(x_j)}(\mathcal{F}_{x_j})$. If we set $\mathcal{Q}_{x_j,i} = \mathcal{E}_{x_j}/F_i(\mathcal{E}_{x_j})$ and

$$\mathcal{L}_1^\mu = (det\,\mathcal{E}_{x_1})^{\mu_r} \otimes \bigotimes_{i=1}^{l}(det\mathcal{Q}_{x_1,i})^{d_i(x_1)},$$

$$\mathcal{L}_2^\mu = (det\,\mathcal{E}_{x_2})^{-\mu_1} \otimes \bigotimes_{i=1}^{l}(det\mathcal{Q}_{x_2,i})^{d_i(x_2)}$$

where the integers l, $n_i(x_j)$ and $d_i(x_j)$ $(i = 1, \cdots, l)$ were defined in (4.10) and (4.11) (determined by μ), then we have

Lemma 4.6. *Let $\mathcal{E}_{x_1}\oplus\mathcal{E}_{x_2} \to \mathcal{Q}$ be the universal r-quotient on $Grass(\mathcal{E}_{x_1}\oplus\mathcal{E}_{x_2})$, and let*

$$h^\mu : Flag_{\vec{n}(x_1)}(\mathcal{F}_{x_1}) \times_{\widetilde{\mathbf{Q}}_F} Flag_{\vec{n}(x_2)}(\mathcal{F}_{x_2}) \to \widetilde{\mathbf{Q}}_F.$$

Write $\eta_x := (det\,\mathcal{Q})(det\,\mathcal{E}_x)^{-1}$ for a point $x \in \widetilde{X}$. Then we have

$$f_*(\eta_{x_2}^m) = \bigoplus_\mu h_*^\mu(\mathcal{L}_1^\mu \otimes \mathcal{L}_2^\mu)$$

where $\mu = (\mu_1, \cdots, \mu_r)$ runs through the integers $0 \le \mu_r \le \cdots \le \mu_1 \le m$.

Proof. This is the immediate corollary of Lemma 4.5 and (4.10)-(4.11).

For $\mu = (\mu_1, \cdots, \mu_r)$, let $\mathcal{U}_{\widetilde{X}}^\mu$ be the moduli space of semistable parabolic bundles on $\widetilde{X}$ with parabolic structures at points $I \cup \{x_1, x_2\}$ and weights $\vec{a}(x)$ for $x \in I$ (see Definition 1.1 in §1) and for $j = 1, 2$

$$\vec{a}(x_j) = (\mu_r, \mu_r + d_1(x_j), \cdots, \mu_r + \sum_{i=1}^{l-1} d_i(x_j), \mu_r + \sum_{i=1}^{l} d_i(x_j)).$$

Let

$$\Theta_{\mathcal{U}_{\widetilde{X}}^\mu} = \Theta(k, \ell, \vec{a}, \vec{\alpha}, I \cup \{x_1, x_2\})$$

be the line bundle defined in Theorem 1.2 with $\alpha_{x_1} = \mu_r$ and $\alpha_{x_2} = k - \mu_1$. Then we have the decomposition theorem

Theorem 4.1. *There exists a (noncanonical) isomorphism*

$$H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \cong \bigoplus_\mu H^0(\mathcal{U}_{\widetilde{X}}^\mu, \Theta_{\mathcal{U}_{\widetilde{X}}^\mu})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ runs through the integers $0 \le \mu_r \le \cdots \le \mu_1 \le k-1$.

Proof. We consider the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{R}}'_F = Grass_r(\mathcal{F}_{x_1} \oplus \mathcal{F}_{x_2}) \times_{\widetilde{\mathbf{Q}}_F} \widetilde{\mathcal{R}}_F & \xrightarrow{\rho} & \widetilde{\mathcal{R}}_F \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\ Grass_r(\mathcal{F}_{x_1} \oplus \mathcal{F}_{x_2}) & \xrightarrow{f} & \widetilde{\mathbf{Q}}_F \end{array}$$

and note that $\hat{\Theta}' = \rho^*\Theta_{\widetilde{\mathcal{R}}_F} \otimes (det\mathcal{Q})^k \otimes (det\mathcal{E}_y)^{-k}$. Then

$$\hat{\Theta}' \otimes \mathcal{O}(-\hat{\mathcal{D}}_2) = \rho^*\Theta_{\widetilde{\mathcal{R}}_F} \otimes (det\mathcal{E}_y)^{-k} \otimes (det\mathcal{E}_{x_2})^k \otimes p_1^*\eta_{x_2}^{k-1}.$$

Thus

$$\rho_*(\hat{\Theta}' \otimes \mathcal{O}(-\hat{\mathcal{D}}_2)) = \Theta_{\widetilde{\mathcal{R}}_F} \otimes (det\mathcal{E}_y)^{-k} \otimes (det\mathcal{E}_{x_2})^k \otimes g^*(f_*\eta_{x_2}^{k-1}).$$

By Lemma 4.6, we have

$$(4.12)\quad \rho_*(\hat{\Theta}'\otimes\mathcal{O}(-\hat{\mathcal{D}}_2)) = \bigoplus_{\mu}(\Theta_{\widetilde{\mathcal{R}}_F}\otimes(det\mathcal{E}_y)^{-k}\otimes(det\mathcal{E}_{x_2})^k)\otimes g^*h_*^{\mu}(\mathcal{L}_1^{\mu}\otimes\mathcal{L}_2^{\mu})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ runs through the integers $0 \le \mu_1 \le \cdots \le \mu_r \le k-1$. Let

$$\widetilde{\mathcal{R}}^{\mu} := \widetilde{\mathcal{R}} \times_{\widetilde{\mathbf{Q}}} Flag_{\vec{n}(x_1)}(\mathcal{F}_{x_1}) \times_{\widetilde{\mathbf{Q}}} Flag_{\vec{n}(x_2)}(\mathcal{F}_{x_2}) = \underset{x\in I\cup\{x_1,x_2\}}{\times_{\widetilde{\mathbf{Q}}}} Flag_{\vec{n}(x)}(\mathcal{F}_x)$$

and

$$\hat{\Theta}_{\mu} = \Theta_{\widetilde{\mathcal{R}}_F} \otimes (det\mathcal{E}_y)^{-k} \otimes (det\mathcal{E}_{x_2})^k) \otimes \mathcal{L}_1^{\mu} \otimes \mathcal{L}_2^{\mu}.$$

Recalling that (see §1)

$$\Theta_{\widetilde{\mathcal{R}}_F} = (det\, R\pi_{\widetilde{\mathcal{R}}}\mathcal{E})^k \otimes \bigotimes_{x\in I}\{(det\,\mathcal{E}_x)^{\alpha_x} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{d_i(x)}\} \otimes (det\,\mathcal{E}_y)^{\tilde{\ell}}$$

and using the definition of $\mathcal{L}_1^{\mu}$ and $\mathcal{L}_2^{\mu}$, one has

$$\hat{\Theta}_{\mu} = (det\, R\pi_{\widetilde{\mathcal{R}}^{\mu}}\mathcal{E})^k \otimes \bigotimes_{x\in I\cup\{x_1,x_2\}}\{(det\,\mathcal{E}_x)^{\alpha_x} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{d_i(x)}\} \otimes (det\,\mathcal{E}_y)^{\ell}$$

with $\alpha_{x_1} = \mu_r$, $\alpha_{x_2} = k - \mu_1$, and $l_{x_1} = l_{x_2} = l$. $\hat{\Theta}_{\mu}$ is the restriction to $\widetilde{\mathcal{R}}_F^{\mu}$ of a line bundle linearising the $SL(\tilde{n})$-action on the projective variety $\widetilde{\mathcal{R}}^{\mu}$ and $\mathcal{U}_{\widetilde{X}}^{\mu}$ is the GIT quotient of the semistable points $(\widetilde{\mathcal{R}}^{\mu})^{ss} \subset \widetilde{\mathcal{R}}_F^{\mu}$. Noting that $r_i(x_2) = r - r_{l-i+1}(x_1)$ and $d_i(x_2) = d_{l-i+1}(x_1)$, we can check that

$$\sum_{x\in I\cup\{x_1,x_2\}}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\sum_{x\in I\cup\{x_1,x_2\}}\alpha_x + r\ell = k\tilde{n}.$$

Thus $\hat{\Theta}_{\mu}$ descends to the line bundle $\Theta_{\mathcal{U}_{\widetilde{X}}^{\mu}}$ on $\mathcal{U}_{\widetilde{X}}^{\mu}$, and

$$H^0(\mathcal{U}_{\widetilde{X}}^{\mu}, \Theta_{\mathcal{U}_{\widetilde{X}}^{\mu}}) = H^0(\widetilde{\mathcal{R}}^{\mu\, ss}, \hat{\Theta}_{\mu})^{inv} = H^0(\widetilde{\mathcal{R}}_F^{\mu}, \hat{\Theta}_{\mu})^{inv}.$$

Let $p^\mu : \widetilde{\mathcal{R}}^\mu_F \to \widetilde{\mathcal{R}}_F$ be the projection. Then (4.12) can be written as

$$\rho_*(\hat{\Theta}' \otimes \mathcal{O}(-\hat{\mathcal{D}}_2)) = \bigoplus_\mu p^\mu_* \hat{\Theta}_\mu. \tag{4.13}$$

Thus

$$H^0(\widetilde{\mathcal{R}}'_F, \hat{\Theta}' \otimes \mathcal{O}(-\hat{\mathcal{D}}_2))^{inv} = \bigoplus_\mu H^0(\widetilde{\mathcal{R}}^\mu_F, \hat{\Theta}_\mu)^{inv} = \bigoplus_\mu H^0(\mathcal{U}^\mu_{\widetilde{X}}, \Theta_{\mathcal{U}^\mu_{\widetilde{X}}}).$$

On the other hand, since a section of $\hat{\Theta}' \otimes \mathcal{O}(-\hat{\mathcal{D}}_2)$ is also a section of $\hat{\Theta}'$, we have

$$H^0(\widetilde{\mathcal{R}}'^{ss}, \hat{\Theta}' \otimes \mathcal{O}(-\hat{\mathcal{D}}_2))^{inv} = H^0(\widetilde{\mathcal{R}}'^{ss} \cap \widetilde{\mathcal{R}}'_F, \hat{\Theta}' \otimes \mathcal{O}(-\hat{\mathcal{D}}_2))^{inv}$$

by Lemma 4.2 (see the proof of Proposition 4.2 for details), and

$$H^0(\widetilde{\mathcal{R}}'^{ss} \cap \widetilde{\mathcal{R}}'_F, \hat{\Theta}' \otimes \mathcal{O}(-\hat{\mathcal{D}}_2))^{inv} = H^0(\widetilde{\mathcal{R}}'_F, \hat{\Theta}' \otimes \mathcal{O}(-\hat{\mathcal{D}}_2))^{inv}$$

by Lemma 4.1. Thus one gets a canonical decomposition

$$H^0(\mathcal{P}, \Theta_{\mathcal{P}}(-\mathcal{D}_2)) = \bigoplus_\mu H^0(\mathcal{U}^\mu_{\widetilde{X}}, \Theta_{\mathcal{U}^\mu_{\widetilde{X}}}). \tag{4.14}$$

The theorem follows Proposition 4.3 and the proof is completed.

Remark 4.2. The proof of the above theorem gives also a decomposition of $\rho_*(\hat{\Theta}')$

$$\rho_*(\hat{\Theta}') = \bigoplus_\mu p^\mu_* \hat{\Theta}_\mu,$$

where $\mu = (\mu_1, \cdots, \mu_r)$ runs through the integers $0 \le \mu_1 \le \cdots \le \mu_r \le k$.

Now we are in a position to deal with the seminormality of subvarieties $\mathcal{W}_a$, which was actually hidden in some literature ([Fa], [S2] and [Tr]). Our task here is to reveal the fact in this literature. In order to make our paper self-contained, we begin with the definition of seminormality (see [Sw] or [NR]) and we also assume that $|I| = 0$ for simplicity.

Definition 4.1. An extension $A \subset B$ of reduced rings is subintegral if

(1) B is integral over A;

(2) $Spec(B) \to Spec(A)$ is a bijection;

(3) $k_{A\cap\mathfrak{p}} \to k_\mathfrak{p}$ is an isomorphism for any $\mathfrak{p} \in Spec(B)$, where $k_\mathfrak{p} = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$.

Definition 4.2. Let $A \subset B$ be reduced rings. We say that A is seminormal in B if there is no extension $A \subset C \subset B$ with $C \neq A$ and $A \subset C$ subintegral. We say that A is seminormal if it is seminormal in its total ring of quotients. A variety V is seminormal if its local ring at any point is seminormal.

Proposition 4.4. *Let V be a variety and let $\hat{\mathcal{O}}_p$ denote the completion of $\mathcal{O}_p$. Let I_1 and I_2 be two radical ideals in a ring A such that $I_1 + I_2$ is radical. Then we have*

(1) *V is seminormal if, for any $p \in V$, $\hat{\mathcal{O}}_p[[u_1, \cdots, u_n]]$ is seminormal for some n;*
(2) *$A/(I_1 \cap I_2)$ is seminormal if A/I_1 and A/I_2 are seminormal;*
(3) *A GIT quotient of a seminormal variety is seminormal.*

Proof. See §3 of [NR].

Let $\mathbf{Q}$ be the Quot scheme of semistable torsion free sheaves of rank r and degree d, and let $\mathcal{F}$ be a universal sheaf on $\mathbf{Q} \times X$. For any $q \in \hat{\mathcal{W}}_a \subset \mathbf{Q}$, we will prove that $\hat{\mathcal{O}}_{\hat{\mathcal{W}}_a, q}[[u_1, \cdots, u_n]]$ is seminormal for some n, which will imply that $\hat{\mathcal{W}}_a$, thus $\mathcal{W}_a$, is seminormal by (1) and (3) of Proposition 4.4. Without loss of generality, we can assume that q is the point of $\hat{\mathcal{W}}_a$ such that

$$\mathcal{F}_{q,x} \cong m_x^{\oplus r}.$$

To work out the local model of $\hat{\mathcal{W}}_a$ at q, we have to recall the local model of $\mathbf{Q}$ at q (see Huitième Partie III of [S2]). It is known that there is a subspace $W \subset H^0(X, \mathcal{F}_q^*(m))$ of dimension r such that $\mathcal{F}_q(-m) \to \mathcal{O} \otimes W^*$ is injective and induces the canonical inclusion $m_x^{\oplus r} \subset \mathcal{O}_x^{\oplus r}$ for some m (Proposition 21 of [S2]). Let Λ be the category of Artinian local $\mathbb{C}$-algebras, and let $X_A = X \times Spec(A)$ for any object A of Λ. Let $\mathcal{O}_{X_A}^p \to \mathcal{F}_A \to 0$ be an exact sequence, which induces $\mathcal{O}_X^p \to \mathcal{F}_q \to 0$ on X, and let $W_A \subset H^0(X_A, \mathcal{F}_A^*(m))$ be a free A-module of rank r such that $W_A \otimes_A A/m_A = W$. Then

$$\mathcal{F}_A(-m) \to W_A^* = Hom_{\mathcal{O}_{X_A}}(\mathcal{O}_{X_A} \otimes W_A, \mathcal{O}_{X_A})$$

is an injective morphism (see Lemma 19 of [S2]). Writing $T = \mathcal{O} \otimes W^* / \mathcal{F}_q(-m)$ and letting T_x be the restriction of T on $\{x\}$, one has the following functors:

$$F, G, H : \Lambda \to \mathbf{Set},$$

$$F(A) := \{\text{isomorphic classes of } (\mathcal{O}_{X_A}^p \to \mathcal{F}_A \to 0, W_A)\},$$

$$G(A) := \{\text{isomorphic classes of } (\mathcal{O}_{X_A}^p \to \mathcal{F}_A \to 0)\},$$

$$H(A) := \{\text{isomorphic classes of } (\mathcal{O}_{X_A}^r \to T_A \to 0)\},$$

where $\mathcal{F}_A$ and T_A are A-flat, T_A has support $\{x\} \times Spec(A)$, and the functors satisfy

$$F(A/m_A) = \{(\mathcal{O}_X^p \to \mathcal{F}_q \to 0, W)\},$$

$$G(A/m_A) = \{(\mathcal{O}_X^p \to \mathcal{F}_q \to 0)\},$$

$$H(A/m_A) = \{(\mathcal{O}_X^r \to T_x \to 0)\}.$$

We have two morphisms $f_1 : F \to G$ and $f_2 : F \to H$ defined by

$$f_1((\mathcal{O}_{X_A}^p \to \mathcal{F}_A \to 0, W_A)) = (\mathcal{O}_{X_A}^p \to \mathcal{F}_A \to 0),$$

$$f_2((\mathcal{O}_{X_A}^p \to \mathcal{F}_A \to 0, W_A)) = (W_A^* \to W_A^*/\mathcal{F}_A(-m)|_{\{x\}\times Spec(A)} \to 0).$$

Lemma 4.7. *The morphisms $f_1 : F \to G$ and $f_2 : F \to H$ are formally smooth.*

Proof. See Lemma 23 and Lemma 24 of [S2].

Suppose that $R = \hat{\mathcal{O}}_x \cong \mathbb{C}[T_1, T_2]/(T_1 \cdot T_2)$, and $u = \bar{T}_1$, $v = \bar{T}_2$ are the elements of R. Then the matrices

$$\alpha = \begin{pmatrix} u \cdot \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & v \cdot \mathbf{I}_r \end{pmatrix}, \quad \beta = \begin{pmatrix} v \cdot \mathbf{I}_r & u \cdot \mathbf{I}_r \end{pmatrix}$$

determine an exact sequence

$$(*) \qquad R^{2r} \xrightarrow{\alpha} R^{2r} \xrightarrow{\beta} R^r \to \mathbb{C}^r \to 0.$$

We define the functor $\Phi : \Lambda \to \mathbf{Set}$ by associating with an object $A \in \Lambda$ the set of isomorphic classes

$$(R \otimes_{\mathbb{C}} A)^{2r} \xrightarrow{\alpha_A} (R \otimes_{\mathbb{C}} A)^{2r} \xrightarrow{\beta_A} (R \otimes_{\mathbb{C}} A)^r \to T_A \to 0$$

of deformations of $(*)$, with $T_A = Coker(\beta_A)$ A-flat. One proved that Φ is isomorphic to H (see Proposition 29 of [S2]). On the other hand, we consider the variety

$$Z = \{(X, Y) \in M(r) \times M(r) \mid X \cdot Y = Y \cdot X = 0\}.$$

For any $(X, Y) \in Z(A)$, where $X = (x_{ij})_{r\times r}$, $Y = (y_{ij})_{r\times r}$, the matrices

$$\alpha_{A(X,Y)} = \begin{pmatrix} u \cdot \mathbf{I}_r & X \\ Y & v \cdot \mathbf{I}_r \end{pmatrix}, \quad \beta_{A(X,Y)} = \begin{pmatrix} v \cdot \mathbf{I}_r - Y & u \cdot \mathbf{I}_r - X \end{pmatrix}$$

determine, if $x_{ij} \in m_A$ and $y_{ij} \in m_A$ $(i, j = 1, 2, \cdots, r)$, a deformation

$$((*)_{(X,Y)}) \quad (R \otimes_{\mathbb{C}} A)^{2r} \xrightarrow{\alpha_{A(X,Y)}} (R \otimes_{\mathbb{C}} A)^{2r} \xrightarrow{\beta_{A(X,Y)}} (R \otimes_{\mathbb{C}} A)^r \to T_A \to 0$$

of $(*)$, which gives an element of $\Phi(A)$. In fact, $\hat{\mathcal{O}}_{Z,(0,0)}$ represents the functor Φ (see Proposition 28 of [S2]). Thus we get the local model Z of $\mathbf{Q}$ at q. It is not difficult to see that the local model of $\hat{\mathcal{W}}_a$ at q is

$$Z' = \{(X, Y) \in Z \mid rk(X) + rk(Y) \le a\}$$

if we remark that $Im(\beta_{A(X,Y)}) \otimes R/m_R$ has rank $2r - rank(X) - rank(Y)$. Namely,

$$\hat{\mathcal{O}}_{\hat{\mathcal{W}}_a, q}[[u_1, \cdots, u_n]] \cong \hat{\mathcal{O}}_{Z',(0,0)}[[v_1, \cdots, v_t]]$$

for some n and t. To prove the seminormality of $\hat{\mathcal{W}}_a$, we will need

Lemma 4.8. *Let R be a ring, $X = (x_{ij})_{r\times r}$, $Y = (y_{ij})_{r\times r}$ and*

$$W(k_1, k_2) = \{(X, Y) \mid XY = YX = 0, rank(X) \le k_1, rank(Y) \le k_2\},$$

$$I(k_1, k_2) = (XY, YX, I_{k_1}(X), I_{k_2}(Y))R[X, Y].$$

If $B(k_1, k_2)$ is the reduced coordinate ring of $W(k_1, k_2)$, then

$$B(k_1, k_2) = R[X, Y]/I(k_1, k_2).$$

Moreover, if R is Cohen-Macaulay and normal, the $W(k_1, k_2)$ are Cohen-Macaulay and normal if $k_1 + k_2 \le r$.

Proof. See Theorem 2.9 and Theorem 2.14 of [St].

Theorem 4.2. *The varieties $\hat{\mathcal{W}}_a$ $(0 \le a \le r)$ are seminormal. In particular, the varieties $\mathcal{W}_a$ $(0 \le a \le r)$ are seminormal.*

Proof. By Proposition 4.4, we only need to check that $\hat{\mathcal{O}}_{Z',(0,0)}[[v_1, \cdots, v_t]]$ is seminormal. It is clear that

$$Z' = \bigcup_{k=0}^{a} W(k, a-k), \quad \mathcal{O}_{Z'} = \mathbb{C}[X, Y]/\bigcap_{k=0}^{a} I(k, a-k).$$

It is easy to check that, for any $0 < l \le a$,

$$\bigcap_{k=0}^{a-l} I(k, a-k) + I(a-l+1, l-1) = \bigcap_{k=0}^{a-l} I(k, l-1). \tag{4.15}$$

Thus one can use (2) of Proposition 4.4 to prove that $\mathcal{O}_{Z'}$ is seminormal because of the normality of $W(k, a-k)$. But (4.15) and the normality are unchanged under completion, a classic fact (for example, see §13 of [ZS]), and so we have proved the seminormality of $\hat{\mathcal{O}}_{Z',(0,0)}[[v_1, \cdots, v_t]]$ by the same reason.

§5. Codimension computations and the vanishing theorems

We are going to prove the vanishing theorems in this section. For this purpose, we need some computations of codimensions, which may have some independent interest. Let V be a vector space of dimension r and $V' \subset V$ an r_1-dimensional subspace. We denote the flag variety of type $\vec{n} = (n_1, \cdots, n_{l+1})$ by $Flag_{\vec{n}}(V)$, and its closed point

$$(V = V_0 \supset V_1 \supset \cdots \supset V_l \supset V_{l+1} = 0)$$

by $F(V)$. We begin the story by the following lemma.

Lemma 5.1. *For any partition $r_1 = m_1 + \cdots + m_{l+1}$ with $m_i \geq 0$, let*

$$\Omega(V') = \{F(V) \in Flag_{\vec{n}}(V) \mid dim(V' \cap V_i) \geq r_1 - (m_1 + \cdots + m_i)\}.$$

Then we have

$$codim(\Omega(V')) = \sum_{j=1}^{l+1}(n_j - m_j)(r_1 - \sum_{i=1}^{j} m_i).$$

Proof. The closed points of $Flag_{\vec{n}}(V)$ can be expressed as the quotients

$$(V = V/V_{l+1} \to V/V_l \to \cdots \to V/V_1 \to 0)$$

with $dim(V/V_i) = n_1 + \cdots + n_i$ and the closed points of $\Omega(V')$ are the points of $Flag_{\vec{n}}(V)$ such that

$$rank(V' \to V/V_i) \leq m_1 + \cdots + m_i.$$

By Proposition 9.6 and Remark 9.16 of [Fu], there exists, for any $n \geq r$, a unique permutation $\omega \in \mathbb{S}_n$ such that

$$r_\omega(n_1 + \cdots + n_i, r_1) = m_1 + \cdots + m_i.$$

(We take $a_i = r_1$, $1 \leq i \leq l+1$ and $b_i = n_1 + \cdots + n_{l+2-i}$ in Proposition 9.6 of [Fu].) Thus the codimension of $\Omega(V')$ is $\ell(\omega)$ (see Proposition 8.1 of [Fu]). By Proposition 9.6 (c) of [Fu], we compute that

$$\ell(\omega) = \sum_{j=1}^{l+1}(n_j - m_j)(r_1 - \sum_{i=1}^{j} m_i),$$

which proves the lemma.

Proposition 5.1. *With the same notation as before, we have the following estimations of codimensions:*

(1) *$codim(\widetilde{\mathcal{R}}^{ss} \setminus \widetilde{\mathcal{R}}^s) \geq (r-1)(\tilde{g}-1)+1$ if $|I| > 0$, and $codim(\widetilde{\mathcal{R}}^{ss} \setminus \widetilde{\mathcal{R}}^s) \geq (r-1)(\tilde{g}-1)$ if $|I| = 0$;*

(2) *$codim(\widetilde{\mathcal{R}}_F \setminus \widetilde{\mathcal{R}}^{ss}) \geq (r-1)(\tilde{g}-1)+1$.*

Proof. Recall that $\widetilde{\mathcal{R}}_F = \times_{\widetilde{\mathbf{Q}}_F, x\in I} Flag_{\vec{n}(x)}(\mathcal{F}_x)$ and the tangent space of $\widetilde{\mathbf{Q}}_F$ at the point $(0 \to K \to \mathcal{O}^{\tilde{n}} \to E \to 0) \in \widetilde{\mathbf{Q}}_F$ is $H^0(\widetilde{X}, K^\vee \otimes E)$. Since $\mathbb{C}^{\tilde{n}} \cong H^0(E)$ and $H^1(E) = 0$ (by the definition of $\widetilde{\mathbf{Q}}_F$), we have the exact sequence

$$(5.1) \quad 0 \to H^0(E^\vee \otimes E) \to \mathbb{C}^{\tilde{n}} \otimes \mathbb{C}^{\tilde{n}} \to H^0(K^\vee \otimes E) \to H^1(E^\vee \otimes E) \to 0.$$

The Riemann-Roch theorem implies that $dim\, H^0(K^\vee \otimes E) = r^2(\tilde{g}-1)+1+ dim\, PGL(\tilde{n})$. Thus

$$dim\, \widetilde{\mathcal{R}}_F = r^2(\tilde{g}-1)+1+\sum_{x\in I} dim\, Flag_{\vec{n}(x)}(\mathcal{F}_x) + dim\, PGL(\tilde{n}).$$

We will deal with (1) in detail. Consider a point $E \in \widetilde{\mathcal{R}}^{ss} \setminus \widetilde{\mathcal{R}}^{s}$. It is an extension

$$0 \to E_1 \to E \to E_2 \to 0$$

by two vector bundles E_j of rank r_j and degree d_j such that

$$pardeg(E_1) = \frac{r_1}{r} pardeg(E), \tag{5.2}$$

where we take the induced parabolic structures on E_j. Let

$$E_x = F_0(E)_x \supset F_1(E)_x \supset \cdots \supset F_{l_x}(E)_x \supset F_{l_x+1}(E)_x = 0$$

be the quasi-parabolic structure of E at $x \in I$, with weight

$$0 \le a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) \le k,$$

and let $m_i(x) = dim(E_{1\,x} \cap F_{i-1}(E)_x / E_{1\,x} \cap F_i(E)_x)$. Then we rewrite (5.2) as

$$rd_1 - r_1 d = \frac{1}{k}\sum_{x\in I}\sum_{i=1}^{l_x+1}(r_1 n_i(x) - r m_i(x))a_i(x). \tag{5.3}$$

We will now describe a (countable) number of quasi-projective varieties parametrising such parabolic bundles.

For $j=1,2$, let d_j, r_j and $\tilde{n}_j$ be integers such that $d_1+d_2=d$, $r_1+r_2=r$, and $\tilde{n}_1+\tilde{n}_2=\tilde{n}$. For each $x\in I$, let $m_1(x), \cdots, m_{l_x+1}$ be nonnegative integers such that $r_1 = m_1(x)+\cdots+m_{l_x+1}(x)$ and

$$rd_1 - r_1 d = \frac{1}{k}\sum_{x\in I}\sum_{i=1}^{l_x+1}(r_1 n_i(x) - r m_i(x))a_i(x).$$

Let $\widetilde{\mathbf{Q}}^j$ $(j=1,2)$ be the Quot scheme of rank r_j, degree d_j quotients

$$\mathcal{O}^{\tilde{n}_j} \to E_j \to 0$$

and let $\widetilde{\mathbf{Q}}^j_F$ be the open subset of locally free quotients with vanishing $H^1(E_j)$ such that $\mathbb{C}^{\tilde{n}_j} \cong H^0(E_j)$. Let $\mathcal{F}_j$ be the universal quotient on $\widetilde{X} \times \widetilde{\mathbf{Q}}^j_F$, $V = \widetilde{\mathbf{Q}}^1_F \times \widetilde{\mathbf{Q}}^2_F$ and $\mathcal{F} = \mathcal{F}_2^\vee \otimes \mathcal{F}_1$ on $\widetilde{X} \times V$. If we set $f : \widetilde{X} \times V \to V$ and

$$V_{h^1} = \{y \in V \mid h^1(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) = h^1\},$$

then V_{h^1} are locally closed subschemes (with the reduced structure) of V, and

$$V = \bigcup_{h^1 \geq 0} V_{h^1}.$$

$R^1 f_*(\mathcal{F})$ is locally free of rank h^1 on V_{h^1}. We define varieties P_{h^1} as follows:

(1) if $h^1 = 0$, we set $P_{h^1} = V$ and $\mathcal{F}^{h^1} = \mathcal{F}_1 \oplus \mathcal{F}_2$ on $\widetilde{X} \times V$;

(2) if $h^1 > 0$, we define $P_{h^1} = \mathbb{P}((R^1 f_* \mathcal{F})^\vee)$ to be the projective bundle on V_{h^1}, and $\mathcal{F}^{h^1}$ to be the universal extension

$$0 \to \mathcal{F}_1 \otimes \mathcal{O}_{P_h}(1) \to \mathcal{F}^{h^1} \to \mathcal{F}_2 \to 0$$

on $\widetilde{X} \times P_{h^1}$.

For any $x \in I$ and $v(x) = (r_1, d_1, h^1, m_1(x), \cdots, m_{l_x+1}(x))$, we define a locally closed subscheme of $Flag_{\vec{n}(x)}(\mathcal{F}^{h^1}_x)$ to be

$$X^0_{v(x)} = \left\{ \begin{array}{l} (E_x = F_0(E)_x \supset F_1(E)_x \supset \cdots \supset F_{l_x}(E)_x \supset F_{l_x+1}(E) = 0) \\ \in Flag_{\vec{n}(x)}(\mathcal{F}^{h^1}_x) \mid dim(F_i(E)_x \cap E_1) = r_1 - \sum_{j=1}^{i} m_j(x) \end{array} \right\}$$

and let

$$X_v = \times_{P_{h^1}, x \in I} X^0_{v(x)}.$$

Each X_v parametrises a family of parabolic bundles E, which occur as extensions $0 \to E_1 \to E \to E_2 \to 0$ (the extension being split if $h^1 = 0$), with parabolic structures at $x \in I$ of type $\vec{n}(x) = (n_1(x), \cdots, n_{l_x+1}(x))$, whose induced parabolic structures on E_1 are of type $(m_1(x), \cdots, m_{l_x+1}(x))$ (we will forget $m_i(x)$ if it is zero). The dimension of X_v is not bigger than

$$\begin{cases} (\tilde{g}-1)\sum r_i^2 + \sum dim\, PGL(\tilde{n}_i) + 2 + h^1 - 1 + \sum_{x \in I} dim\, X_{v(x)} & \text{if } h^1 > 0, \\ (\tilde{g}-1)\sum r_i^2 + \sum dim\, PGL(\tilde{n}_i) + 2 + \sum_{x \in I} dim\, X_{v(x)} & \text{if } h^1 = 0, \end{cases}$$

where $i = 1, 2$. Let X_v^{ss} be the open set of semistable parabolic bundles, and let $F(v)$ be the frame-bundle of the direct image of $\mathcal{F}(v)$ (the pull-back of

$\mathcal{F}^{h^1}$) on X_v^{ss}. There is a map from each $F(v)$ to $\widetilde{\mathcal{R}}^{ss} \setminus \widetilde{\mathcal{R}}^{s}$, and the union of the images covers $\widetilde{\mathcal{R}}^{ss} \setminus \widetilde{\mathcal{R}}^{s}$

$$dim(Im\, F(v)) = dim(F(v)) - e$$

where e is the infimum of the dimensions of the irreducible components of the fibres. If a vector bundle E is an extension

$$0 \to E_1 \to E \to E_2 \to 0, \tag{5.4}$$

then it is easy to see that

$$dim(Aut(E)) \geq \begin{cases} 2 + dim(H^0(E_2^\vee \otimes E_1)) & \text{if (5.4) is splitting,} \\ 1 + dim(H^0(E_2^\vee \otimes E_1)) & \text{if (5.4) is not splitting.} \end{cases}$$

Since the E are generated by sections and any automorphism of E acts non-trivially on the frames of $H^0(E)$, we have

(1) $e \geq dim(H^0(E_2^\vee \otimes E_1)) + \tilde{n}_1^2 + \tilde{n}_2^2$ if $h^1 = 0$, and

(2) $e \geq dim(H^0(E_2^\vee \otimes E_1)) + \tilde{n}_1^2 + \tilde{n}_2^2 - 1$ if $h^1 > 0$.

Thus, by using the Riemann-Roch theorem, the codimension of the images are bounded below by

$$r_1 r_2(\tilde{g} - 1) + \sum_{x \in I} codim(X^0_{v(x)}) + rd_1 - r_1 d.$$

By Lemma 5.1 and (5.3), noting that $r_1 r_2 = r_1(r - r_1) \geq r - 1$, we have

$$codim(\widetilde{\mathcal{R}}^{ss} \setminus \widetilde{\mathcal{R}}^{s}) \geq (r-1)(\tilde{g}-1) + \sum_{x \in I} \left\{ \begin{array}{l} \sum_{j=1}^{l_x+1} (r_1 - \sum_{i=1}^{j} m_i(x))(n_j(x) - m_j(x)) \\ + \sum_{j=1}^{l_x+1} (r_1 n_j(x) - r m_j(x)) \frac{a_j(x)}{k} \end{array} \right\}.$$

Since $r_1 = \sum_{i=1}^{l_x+1} m_i(x)$ and $r = \sum_{i=1}^{l_x+1} n_i(x)$, the first statement of the lemma follows the following Lemma 5.2.

Now we prove (2) of the lemma, the arguments being word by word as above, except that we replace the equality (5.3) by an inequality

$$rd_1 - r_1 d > \frac{1}{k} \sum_{x \in I} \sum_{i=1}^{l_x+1} (r_1 n_i(x) - r m_i(x)) a_i(x).$$

Lemma 5.2. *For any integers $n_j > 0$ and $m_j \geq 0$ $(j = 1, \cdots, l+1)$ with $n_j \geq m_j$, let $0 < a_1 < \cdots < a_{l+1} \leq 1$ be rational numbers. Then*

$$\sum_{j=1}^{l+1} m_j \sum_{j=1}^{l+1}(n_j - m_j) + \sum_{j=1}^{l+1} m_j \sum_{j=1}^{l+1} n_j a_j \geq \sum_{j=1}^{l+1}(\sum_{i=1}^{j} m_i)(n_j - m_j) + \sum_{j=1}^{l+1} n_j \sum_{j=1}^{l+1} m_j a_j.$$

Moreover, if $\sum_{j=1}^{l+1} n_j > \sum_{j=1}^{l+1} m_j > 0$, we have the strict inequality

$$\sum_{j=1}^{l+1} m_j \sum_{j=1}^{l+1}(n_j - m_j) + \sum_{j=1}^{l+1} m_j \sum_{j=1}^{l+1} n_j a_j > \sum_{j=1}^{l+1}(\sum_{i=1}^{j} m_i)(n_j - m_j) + \sum_{j=1}^{l+1} n_j \sum_{j=1}^{l+1} m_j a_j.$$

Proof. We check it by induction on l, letting $(*)$ denote the inequality and $LHS(*)$ and $RHS(*)$ denote the "left (right)-hand side of $(*)$". When $l = 1$, we have

$$LHS(*) - RHS(*) = m_2(n_1 - m_1) + (m_1 n_2 - m_2 n_1)(a_2 - a_1),$$

which satisfies the lemma. Assume that $(*)$ is true for $l - 1$. Then

$$LHS(*) - RHS(*) \geq m_{l+1} \sum_{j=1}^{l}(n_j - m_j) - \sum_{j=1}^{l}(m_{l+1}n_j - n_{l+1}m_j)(a_{l+1} - a_j),$$

which is a strict inequality if $\sum_{j=1}^{l} n_j > \sum_{j=1}^{l} m_j > 0$. When $m_{l+1} = 0$, $LHS(*) - RHS(*) \geq n_{l+1} \sum_{j=1}^{l}(a_{l+1} - a_j)m_j \geq 0$, which is strict if $\sum_{j=1}^{l+1} n_j > \sum_{j=1}^{l+1} m_j > 0$. When $m_{l+1} > 0$, we have that

$$\begin{aligned} LHS(*) - RHS(*) &\geq m_{l+1} \left\{ \sum_{j=1}^{l}(n_j - m_j) - \sum_{j=1}^{l}(n_j - \frac{n_{l+1}}{m_{l+1}} m_j)(a_{l+1} - a_j) \right\} \\ &\geq 0, \end{aligned}$$

which is strict if $n_{l+1} > m_{l+1}$. The lemma is proved.

Proposition 5.2. *Let $\mathcal{D}_1^f = \hat{\mathcal{D}}_1(r-1) \cup \hat{\mathcal{D}}_1^t$ and $\mathcal{D}_2^f = \hat{\mathcal{D}}_2(r-1) \cup \hat{\mathcal{D}}_2^t$. Then*

(1) *$Codim(\mathcal{H} \smallsetminus \widetilde{\mathcal{R}}'^{ss}) \geq (r-1)\tilde{g} + 1$.*
(2) *The complement in $\widetilde{\mathcal{R}}'^{ss} \smallsetminus \{\mathcal{D}_1^f \cup \mathcal{D}_2^f\}$ of the set $\widetilde{\mathcal{R}}'^{s}$ of stable points has codimension $\geq (r-1)\tilde{g} + 1$ if $|I| > 0$, and codimension $\geq (r-1)\tilde{g}$ if $|I| = 0$.*

Proof. We will prove (1) in detail, and (2) will follow similarly. For any $(E,Q) \in \mathcal{H} \setminus \widetilde{\mathcal{R}}'^{ss}$ with $E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0$, there exists a nontrivial subsheaf $E_1 \subset E$, of $rank(E_1) = r_1 > 0$, such that E/E_1 is torsion free outside $\{x_1, x_2\}$ and

$$pardeg(E_1) - dim(Q^{E_1}) > \frac{r_1}{r}(pardeg(E) - r). \tag{5.5}$$

In fact, we can choose E_1 such that $E/E_1 = E_2$ is torsion free. If E_2 has torsion ${}_{x_1}\tau_1 \oplus {}_{x_2}\tau_2$, then let $\widetilde{E}_1 \supset E_1$ be the inverse image in E of ${}_{x_1}\tau_1 \oplus {}_{x_2}\tau_2$. Then $pardeg(\widetilde{E}_1) = pardeg(E_1) + dim(\tau_1) + dim(\tau_2)$ and

$$dim(Q^{\widetilde{E}_1}) - dim(Q^{E_1}) \leq dim(\tau_1) + dim(\tau_2) = pardeg(\widetilde{E}_1) - pardeg(E_1),$$

which shows that $\widetilde{E}_1$ satisfies (5.5), and we can choose $\widetilde{E}_1$ instead of E_1. Thus E is an extension

$$0 \to E_1 \to E \to E_2 \to 0$$

with E_2 torsion free (note that $r_2 = rank(E_2) > 0$) and with E_1 satisfying (5.5).

We can write $E = E' \oplus {}_{x_1}\mathbb{C}^{s_1} \oplus {}_{x_2}\mathbb{C}^{s_2}$ and $E_1 = E_1' \oplus {}_{x_1}\mathbb{C}^{s_1} \oplus {}_{x_2}\mathbb{C}^{s_2}$ with E' and E_1' torsion free. Thus (E,Q) is a GPS such that $E' = E/\mathrm{Tor}(E)$ occurs as an extension

$$0 \to E_1' \to E' \to E_2' \to 0 \quad (\text{where } E_2' = E_2)$$

with $pardeg(E_1') > dim(Q^{E_1}) + \frac{r_1}{r} pardeg(E) - r_1 - s_1 - s_2$. When $d = deg(E) = deg(E') + s_1 + s_2$ is large enough (so is $deg(E')$ since $s_1 + s_2 \leq r$), we can assume that E_1' and E_2' are generated by global sections and $H^1(E_1') = H^1(E_2') = 0$.

Let $d_j = deg(E_j')$, $r_j = rank(E_j')$ $(j = 1, 2)$ and, for any $x \in I$,

$$m_i(x) = dim(E'_{1\,x} \cap F_{i-1}(E)_x / E'_{1\,x} \cap F_i(E)_x)$$

where $E_x = F_0(E)_x \supset F_1(E)_x \supset \cdots \supset F_{l_x}(E)_x \supset F_{l_x+1}(E)_x = 0$ is the quasi-parabolic structure of E at $x \in I$ of type $(n_1(x), \cdots, n_{l_x+1}(x))$, with weights

$$0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) \leq k.$$

Let $t = dim(Q^{E_1})$ and $s = s_1 + s_2$. Then $s \leq t \leq r$ and

$$rd_1 - r_1 d > r(t - s - r_1) + \frac{1}{k} \sum_{x \in I} \sum_{i=1}^{l_x+1} (r_1 n_i(x) - r m_i(x)) a_i(x). \tag{5.6}$$

Let $v=(d_1,r_1,s_1,s_2,t,\{m_1(x),\cdots,m_{l_x+1}(x)\}_{x\in I},h)$, where $h\geq 0$ is an integer. We will construct a variety $F(v)$ with a morphism $F(v)\to\mathcal{H}\setminus\widetilde{\mathcal{R}}'^{ss}$ such that its image contains the point (E,Q).

For $j=1,2$, let $\tilde{n}_j=dim\,H^0(E'_j)$, $\widetilde{\mathbf{Q}}^j$ the Quot scheme of rank r_j, degree d_j with quotients $\mathcal{O}^{\tilde{n}_j}\to E'_j\to 0$ and $\widetilde{\mathbf{Q}}^j_F$ the open subset of locally free quotients with vanishing $H^1(E'_j)$ and E'_j generated by global sections. Let $\mathcal{E}'_j$ be the universal quotient on $\widetilde{X}\times\widetilde{\mathbf{Q}}^j_F$, $V=\widetilde{\mathbf{Q}}^1_F\times\widetilde{\mathbf{Q}}^2_F$ and $\mathcal{F}=\mathcal{E}'^{\vee}_2\otimes\mathcal{E}'_1$ on $\widetilde{X}\times V$. We have $V=\bigcup_{h\geq 0}V_h$ and $R^1f_*(\mathcal{F})$ is locally free of rank h on V_h (see the proof of Proposition 5.1), where $f:\widetilde{X}\times V\to V$ is the projection. Let $P_h=\mathbb{P}((R^1f_*\mathcal{F})^{\vee})$ be the projective bundle on V_h and let

$$0\to\mathcal{E}'_1\otimes\mathcal{O}_{P_h}(-1)\to\mathcal{E}'(h)\to\mathcal{E}'_2\to 0 \tag{5.7}$$

be the universal extension on $\widetilde{X}\times P_h$ (we set $P_h=V$ and $\mathcal{E}'(h)=\mathcal{E}'_1\oplus\mathcal{E}'_2$ if $h=0$). For $v'=(d_1,r_1,\{m_1(x),\cdots,m_{l_x+1}(x)\}_{x\in I},h)$, as in the proof of Proposition 5.1, we can define a variety $X(v')\to P_h$. It parametrises a family of parabolic bundles E', which occur as extensions $0\to E'_1\to E'\to E'_2\to 0$ (the extension being split if $h=0$), with parabolic structures at $x\in I$ of type $\vec{n}(x)=(n_1(x),\cdots,n_{l_x+1}(x))$, whose induced parabolic structures on E'_1 are of type $(m_1(x),\cdots,m_{l_x+1}(x))$ (we will forget $m_i(x)$ if it is zero). Let $0\to\mathcal{E}'_1(-1)\to\mathcal{E}'(v')\to\mathcal{E}'_2\to 0$ be the pull-back of (5.7) on $\widetilde{X}\times X(v')$, and $\mathcal{E}(v')=\mathcal{E}'(v')\oplus{}_{x_1}\mathcal{O}^{s_1}\oplus{}_{x_2}\mathcal{O}^{s_2}$. We consider

$$G_{v'}=Grass_r(\mathcal{E}(v')_{x_1}\oplus\mathcal{E}(v')_{x_2})\to X(v')$$

and define a subvariety of $G_{v'}$ by

$$X(v):=\left\{\begin{array}{l}(E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q\to 0)\in G_{v'}\text{ with }dim(ker(q)\cap(\mathbb{C}^{s_1}\oplus\mathbb{C}^{s_2}))=0\\ \text{and }dim(ker(q)\cap(E'_{1\,x_1}\oplus\mathbb{C}^{s_1}\oplus E'_{1\,x_2}\oplus\mathbb{C}^{s_2}))=2r_1+s-t\end{array}\right\}.$$

Then $X(v)$ parametrises a family of GPS $(E=E'\oplus{}_{x_1}\mathbb{C}^{s_1}\oplus{}_{x_2}\mathbb{C}^{s_2},\,Q)$, where E' occurs as an extension $0\to E'_1\to E'\to E'_2\to 0$ (it is split if $h=0$) with parabolic structures at $x\in I$ of type $\vec{n}(x)$, whose induced parabolic structures on E'_1 are of type $(m_1(x),\cdots,m_{l_x+1}(x))$ (we will forget $m_i(x)$ if it is zero), such that ${}_{x_1}\mathbb{C}^{s_1}\oplus{}_{x_2}\mathbb{C}^{s_2}\to Q$ is injective and

$$\operatorname{rank}(E'_{1\,x_1}\oplus\mathbb{C}^{s_1}\oplus E'_{1\,x_2}\oplus\mathbb{C}^{s_2}\to Q)=t.$$

One computes $dim\,X(v)=dim\,X(v')+r(r+s)-(r-t)(2r_1+s-t)$. Let $\mathcal{E}(v)$ be the pull-back of $\mathcal{E}(v')$ on $\widetilde{X}\times X(v)\to\widetilde{X}\times X(v')$, and let $F(v)$ be the frame bundle of the direct image of $\mathcal{E}(v)$ on $X(v)$. Then there is a morphism $F(v) \to \mathcal{H} \setminus \widetilde{\mathcal{R}}'^{ss}$ whose image contains (E, Q).

Therefore we have a (countable) number of quasi-projective varieties $F(v)$ and morphisms $F(v) \to \mathcal{H} \setminus \widetilde{\mathcal{R}}'^{ss}$ such that the union of the images covers $\mathcal{H} \setminus \widetilde{\mathcal{R}}'^{ss}$. Since the sheaf $E' \oplus {}_{x_1}\mathbb{C}^{s_1} \oplus {}_{x_2}\mathbb{C}^{s_2}$ has an automorphism group of dimension at least $dim\, Aut(E') + rs + s^2$, and the dimension of $\mathcal{H}$ is

$$r^2(\tilde{g}-1) + 1 + r^2 + \sum_{x \in I} dim\, Flag_{\vec{n}(x)}(\mathcal{F}_x) + dim\, PGL(\tilde{n}),$$

we find that the codimension of $\mathcal{H} \setminus \widetilde{\mathcal{R}}'^{ss}$ is bounded below by

$$\begin{aligned} & r_1 r_2(\tilde{g}-1) + s^2 + r_1 s + (r-t)(2r_1 + s - t) \\ & \quad + rd_1 - r_1 d + \sum_{x \in I} \sum_{j=1}^{l_x+1} (r_1 - \sum_{i=1}^{j} m_i(x))(n_j(x) - m_j(x)). \end{aligned}$$

By using (5.6), we get

$$\begin{aligned} codim(\mathcal{H} \setminus \widetilde{\mathcal{R}}'^{ss}) &> r_1 r_2 \tilde{g} + (r_1 - t)^2 + (r_1 - t + s)s \\ &\quad + \sum_{x \in I} \left\{ \begin{aligned} & \sum_{j=1}^{l_x+1} (r_1 - \sum_{i=1}^{j} m_i(x))(n_j(x) - m_j(x)) \\ & + \sum_{j=1}^{l_x+1} (r_1 n_j(x) - r m_j(x)) \frac{a_j(x)}{k} \end{aligned} \right\}. \end{aligned} \tag{5.8}$$

It is clear that $(r_1 - t)^2 + (r_1 - t + s)s \geq 0$ when $t \leq r_1 + s$. Otherwise, if $t > r_1 + s$, we have $(r_1 - t)^2 + (r_1 - t + s)s = s^2 + (t - r_1)(t - r_1 - s) > s^2$. Thus

$$codim(\mathcal{H} \setminus \widetilde{\mathcal{R}}'^{ss}) \geq (r-1)\tilde{g} + 1,$$

and we have proved (1) of the proposition.

For any $(E, Q) \in \widetilde{\mathcal{R}}'^{ss} \setminus \{\mathcal{D}_1^f \cup \mathcal{D}_2^f\} \setminus \widetilde{\mathcal{R}}'^{s}$ with $E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0$, there is a subsheaf $\widetilde{E} \subset E$ such that $E/\widetilde{E}$ is torsion free outside $\{x_1, x_2\}$, contradicting the stability. One can show that $\widetilde{E}$ has to be of rank $0 < r_1 < r$. Otherwise, $\widetilde{E}$ must satisfy the exact sequence

$$0 \to \widetilde{E} \to E \to {}_{x_1}\tau_1 \oplus {}_{x_2}\tau_2 \to 0$$

with $dim(\tau_1 \oplus \tau_2) = dim(Q/Q^{\widetilde{E}})$, and we have the diagram

$$\begin{array}{ccccccccc} & & \widetilde{E}_{x_1} \oplus \widetilde{E}_{x_2} & \longrightarrow & E_{x_1} \oplus E_{x_2} & \longrightarrow & \tau_1 \oplus \tau_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle q} & & \| & & \\ 0 & \longrightarrow & Q^{\widetilde{E}} & \longrightarrow & Q & \longrightarrow & Q/Q^{\widetilde{E}} & \longrightarrow & 0. \end{array}$$

Since $q_j : E_{x_j} \to Q$ $(j = 1, 2)$ are isomorphisms, τ_j have to be zero. Thus (2) now follows the same proof except that we replace the inequality (5.6) by an equality.

Remark 5.1. It is not true that $\widetilde{\mathcal{R}}'^{ss} \setminus \widetilde{\mathcal{R}}'^{s}$ has codimension > 1. Points on $\mathcal{D}_1^f = \hat{\mathcal{D}}_1(r-1) \cup \hat{\mathcal{D}}_1^t$ and $\mathcal{D}_2^f = \hat{\mathcal{D}}_2(r-1) \cup \hat{\mathcal{D}}_2^t$ are never stable (see Remark 1.2). The above codimension bound breaks down because, for $(E, Q) \in \mathcal{D}_1^f \cup \mathcal{D}_2^f$, we cannot assume that the subsheaf contradicting stability is of rank $0 < r_1 < r$.

We denote the Jacobian of degree d line bundles on $\widetilde{X}$ by $J^d_{\widetilde{X}}$ and the Poincaré line bundle on $\widetilde{X} \times J^d_{\widetilde{X}}$ by $\mathcal{L}$. Let

$$\Theta_y := (det\, R\pi_J \mathcal{L}) \otimes (det\, \mathcal{L}_y)^{d+1-\tilde{g}}$$

and $Det : \widetilde{\mathcal{R}}_F \to J^d_{\widetilde{X}}$ be the morphism given by the determinant of the universal quotient bundle. This induces a morphism $\mathcal{U}_{\widetilde{X}} \to J^d_{\widetilde{X}}$, which will also be denoted by Det. On $\widetilde{\mathcal{R}}_F$, one sees easily that

$$(det\, R\pi_{\widetilde{\mathcal{R}}_F} det\, \mathcal{E})^{-2} = (det\, \mathcal{E}_y)^{2\tilde{n}+2(r-1)(\tilde{g}-1)} \otimes Det^* \Theta_y^{-2}. \tag{5.9}$$

Lemma 5.3. *Let $Det : \mathcal{U}_{\widetilde{X}} \to J^d_{\widetilde{X}}$ be the induced morphism by $Det : \widetilde{\mathcal{R}}_F \to J^d_{\widetilde{X}}$. Then*

$$\Theta_{\mathcal{U}_{\widetilde{X}}} \otimes (Det^* \Theta_y)^{-2}$$

is ample if $k > 2r$.

Proof. Let $\mathcal{U}^L_{\widetilde{X}}$ be the fibre of $Det : \mathcal{U}_{\widetilde{X}} \to J^d_{\widetilde{X}}$ at $L \in J^d_{\widetilde{X}}$. One has an $r^{2\tilde{g}}$-fold covering

$$f : \mathcal{U}^L_{\widetilde{X}} \times J^0_{\widetilde{X}} \to \mathcal{U}_{\widetilde{X}}$$

given by $f(E, L_0) = E \otimes L_0$. We will show that $\Theta_{\mathcal{U}_{\widetilde{X}}} \otimes (Det^*\Theta_y)^{-2}$ is ample when pulled back to this finite cover.

One can show that $\mathcal{U}^L_{\widetilde{X}}$ is unirational, which implies that

$$Pic(\mathcal{U}^L_{\widetilde{X}} \times J^0_{\widetilde{X}}) = Pic(\mathcal{U}^L_{\widetilde{X}}) \times Pic(J^0_{\widetilde{X}}).$$

Hence it suffices to check that the restriction to each factor is ample. The restriction to the first factor $\mathcal{U}^L_{\widetilde{X}}$ is $\Theta_{\mathcal{U}_{\widetilde{X}}}|_{\mathcal{U}^L_{\widetilde{X}}}$, which is clearly ample.

The restriction to the second factor is $f^*(\Theta_{\mathcal{U}_{\widetilde{X}}})|_{J^0_{\widetilde{X}}} \otimes f^*(Det^*\Theta_y)^{-2}|_{J^0_{\widetilde{X}}}$. Writing $M_1 = f^*(\Theta_{\mathcal{U}_{\widetilde{X}}})|_{J^0_{\widetilde{X}}}$ and $M_2 = f^*(Det^*\Theta_y)^{-2}|_{J^0_{\widetilde{X}}}$, we are left with the task of proving that $M_1 \otimes M_2$ is ample. It is easy to see that $M_1 = f_1^*\Theta_{\mathcal{U}_{\widetilde{X}}}$ and $M_2 = f_2^*(\Theta_y^{-2})$, where $f_1 : J^0_{\widetilde{X}} \to \mathcal{U}_{\widetilde{X}}$ and $f_2 : J^0_{\widetilde{X}} \to J^d_{\widetilde{X}}$ are given by $f_1(L_0) = E \otimes L_0$ (for a fixed E) and $f_2(L_0) = L_0^r \otimes L$. If we identify $J^0_{\widetilde{X}}$ with $J^d_{\widetilde{X}}$

by the isomorphism $J^0_{\widetilde{X}} \xrightarrow{\otimes L} J^d_{\widetilde{X}}$ and work up to algebraic equivalence, then $M_2 = [r]^*(\Theta_y^{-2})$ is algebraically equivalent to $\Theta_y^{-2r^2}$, where $[r] : J^0_{\widetilde{X}} \to J^0_{\widetilde{X}}$ is the finite cover given by $[r](L_0) = L_0^r$. To figure M_1 out, we consider the commutative diagram

$$\begin{array}{ccc} \widetilde{X} \times J^0_{\widetilde{X}} & \xrightarrow{1\times f_1} & \widetilde{X} \times \mathcal{U}_{\widetilde{X}} \\ {\scriptstyle \pi_J}\downarrow & & \downarrow \\ J^0_{\widetilde{X}} & \xrightarrow{f_1} & \mathcal{U}_{\widetilde{X}}. \end{array}$$

By the base change theorem, if $\mathcal{L}$ denotes a Poincaré bundle on $\widetilde{X} \times J^0_{\widetilde{X}}$, then

$$M_1 = (det\, R\pi_J E \otimes \mathcal{L})^k \otimes \bigotimes_{x\in I} \{(det\, \mathcal{L}_x)^{r\alpha_x} \otimes \bigotimes_{i=1}^{l_x} (det\, \mathcal{L}_x)^{r_i(x)d_i(x)}\} \otimes (det\, \mathcal{L}_y)^{r\tilde{\ell}},$$

which is clearly algebraically equivalent to

$$\begin{aligned} &(det\, R\pi_J E \otimes \mathcal{L})^k \otimes (det\, \mathcal{L}_y)^{\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\sum_{x\in I}\alpha_x + r\tilde{\ell}} \\ &\qquad = (det\, R\pi_J E \otimes \mathcal{L})^k \otimes (det\, \mathcal{L}_y)^{k\tilde{n}}. \end{aligned}$$

On the other hand, since E is generated by sections and $det(E) = L$, we have

$$0 \to \mathcal{O}_{\widetilde{X}} \otimes \mathbb{C}^{r-1} \to E \to L \to 0.$$

Thus $(det\, R\pi_J E \otimes \mathcal{L}) = (det\, R\pi_J \mathcal{L})^{r-1} \otimes det\, R\pi_J(L \otimes \mathcal{L})$, and M_1 is algebraically equivalent to

$$\{det\, R\pi_J \mathcal{L} \otimes (det\, \mathcal{L}_y)^{1-\tilde{g}}\}^{(r-1)k} \otimes \{det\, R\pi_J(L\otimes\mathcal{L}) \otimes (det(L\otimes\mathcal{L})_y)^{d+1-\tilde{g}}\}^k.$$

After identifying $J^d_{\widetilde{X}}$ with $J^0_{\widetilde{X}}$, we see that M_1 is algebraically equivalent to Θ_y^{rk}. Thus $M_1 \otimes M_2$ is algebraically equivalent to $\Theta_y^{rk-2r^2}$, which is clearly ample when $k > 2r$.

The next lemma is a copy of Lemma 4.17 of [NR] (one can see [Kn] for its detailed proof).

Lemma 5.4. *Let X be a normal, Cohen-Macaulay variety on which a reductive group G acts such that a good quotient $\pi : X \to Y$ exists. Suppose that the action is generically free and that $dim(G) = dim(X) - dim(Y)$, and further suppose that*

(1) *the subset where the action is not free has codimension ≥ 2, and*
(2) *for every prime divisor D in X, $\pi(D)$ has codimension ≤ 1, where D need not be invariant.*

Then $\omega_Y = (\pi_*\omega_X)^G$ where ω_X, ω_Y are the respective dualising sheaves and the superscript $(\ \ )^G$ denotes the G-invariant direct image.

Fix an ample line bundle $\mathcal{O}(1)$ on $\widetilde{X}$, and a set of data

$$\omega = (d, r, k, \tilde{\ell}, \{d_i(x)\}_{x\in I, 1\le i\le l_x}, \{\alpha_x\}_{x\in I}, I)$$

satisfying

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\sum_{x\in I}\alpha_x + r\tilde{\ell} = k\tilde{n}. \tag{5.10}$$

Then ω determines a polarisation (for fixed $\mathcal{O}(1)$)

$$\frac{\tilde{\ell}}{m} \times \prod_{x\in I}\{\alpha_x, d_1(x), \cdots, d_{l_x}(x)\}.$$

We denote the set of semistable points for the $SL(\tilde{n})$ action under this polarisation by $\widetilde{\mathcal{R}}_\omega^{ss} \subset \widetilde{\mathcal{R}}_F$, and its good quotient by $\mathcal{U}_{\widetilde{X},\omega}$.

$$\Theta_{\widetilde{\mathcal{R}}_\omega^{ss}} = (det\, R\pi_{\widetilde{\mathcal{R}}_\omega^{ss}}\mathcal{E})^k \otimes \bigotimes_{x\in I}\{(det\,\mathcal{E}_x)^{\alpha_x} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{d_i(x)}\} \otimes (det\,\mathcal{E}_y)^{\tilde{\ell}}$$

descends to an ample line bundle $\Theta_{\mathcal{U}_{\widetilde{X},\omega}}$ on $\mathcal{U}_{\widetilde{X},\omega}$, and we need to prove that

$$H^1(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}}) = 0.$$

Theorem 5.1. *Assume that $\tilde{g} \ge 2$. Then, for any set of data ω satisfying (5.10),*

$$H^1(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}}) = 0.$$

Proof. We can assume that $r > 2$ since the vanishing theorem for $r = 2$ is known (see [NR] and [Ra]). Let $\bar{\omega} = (d, r, \bar{k}, \bar{\ell}, \{\bar{d}_i(x)\}_{x\in I, 1\le i\le l_x}, \{\bar{\alpha}_x\}_{x\in I}, I)$ be a new set of data with $\bar{k} = k + 2r$, $\bar{\ell} = 2\tilde{n} + \tilde{\ell} - r|I|$, $\bar{d}_i(x) = d_i(x) + n_i(x) + n_{i+1}(x)$ and $\bar{\alpha}_x = \alpha_x + n_{l_x+1}(x)$. Let

$$\begin{aligned}\hat{\Theta}_{\bar{\omega}} := (det\, R\pi_{\widetilde{\mathcal{R}}_F}\mathcal{E})^{\bar{k}} \otimes \bigotimes_{x\in I}\left\{(det\,\mathcal{E}_x)^{\bar{\alpha}_x} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{\bar{d}_i(x)}\right\} \\ \otimes \bigotimes_{x\in I}(det\,\mathcal{E}_x)^{-r} \otimes \bigotimes_q (det\,\mathcal{E}_q)^{1-r} \otimes (det\,\mathcal{E}_y)^{2\tilde{n}+2(r-1)(\tilde{g}-1)+\tilde{\ell}}.\end{aligned}$$

One can check that

$$\sum_{x\in I}\sum_{i=1}^{l_x}\bar{d}_i(x)r_i(x)+r\sum_{x\in I}\bar{\alpha}_x+r\bar{\ell}=\bar{k}\tilde{n}. \tag{5.11}$$

$\bar{\omega}$ determines a new polarisation

$$\frac{\bar{\ell}}{m}\times\prod_{x\in I}\{\bar{\alpha}_x,\bar{d}_1(x),\cdots,\bar{d}_{l_x}(x)\}.$$

We denote the set of semistable points for the $SL(\tilde{n})$ action under the new polarisation by $\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}\subset\widetilde{\mathcal{R}}_F$, and its good quotient

$$\psi_{\bar{\omega}}:\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}\to\mathcal{U}_{\widetilde{X},\bar{\omega}}.$$

$\hat{\Theta}_{\bar{\omega}}$ descends to an ample line bundle $\Theta_{\bar{\omega}}$ (see Remark 1.1 (2)). By Proposition 2.2 and (5.9), we have

$$\Theta_{\widetilde{\mathcal{R}}_F}\otimes\omega^{-1}_{\widetilde{\mathcal{R}}_F}=\hat{\Theta}_{\bar{\omega}}\otimes Det^*\Theta_y^{-2}. \tag{5.12}$$

Since we assumed that $\tilde{g}\geq 2$ and $r>2$, the codimension of $\widetilde{\mathcal{R}}_F\setminus\widetilde{\mathcal{R}}^{ss}_{\omega}$ for any ω is at least 3 (see Proposition 5.1 (2)). Thus, by local cohomology theory, we have

$$H^1(\widetilde{\mathcal{R}}^{ss}_{\omega},\Theta_{\widetilde{\mathcal{R}}^{ss}_{\omega}})^{inv}=H^1(\widetilde{\mathcal{R}}_F,\Theta_{\widetilde{\mathcal{R}}_F})^{inv}=H^1(\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}},\Theta_{\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}})^{inv}. \tag{5.13}$$

Since $codim(\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}\setminus\widetilde{\mathcal{R}}^{s}_{\bar{\omega}})\geq 2$ (see Proposition 5.1 (1)), by using Lemma 5.4, we have

$$(\psi_{\bar{\omega}*}\omega_{\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}})^{inv}=\omega_{\mathcal{U}_{\widetilde{X},\bar{\omega}}}$$

(see Lemma 6.3 of [NR]). By (5.12), we can write

$$\Theta_{\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}}=\psi^*_{\bar{\omega}}(\Theta_{\bar{\omega}}\otimes Det^*\Theta_y^{-2})\otimes\omega_{\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}}. \tag{5.14}$$

One uses the fact that for good quotients the space of invariants of the cohomology of an invariant line bundle is the same as the cohomology of the invariant direct image and (5.14) to prove that

$$\begin{aligned}H^1(\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}},\Theta_{\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}})^{inv}&=H^1(\mathcal{U}_{\widetilde{X},\bar{\omega}},\Theta_{\bar{\omega}}\otimes Det^*\Theta_y^{-2}\otimes(\psi_{\bar{\omega}*}\omega_{\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}})^{inv})\\&=H^1(\mathcal{U}_{\widetilde{X},\bar{\omega}},\Theta_{\bar{\omega}}\otimes Det^*\Theta_y^{-2}\otimes\omega_{\mathcal{U}_{\widetilde{X},\bar{\omega}}}).\end{aligned}$$

Now since $\Theta_{\bar{\omega}} \otimes Det^*\Theta_y^{-2}$ is an ample line bundle by Lemma 5.3 (note that $\bar{k} > 2r$) and $\mathcal{U}_{\widetilde{X},\bar{\omega}}$ has only rational singularities, we can apply a Kodaira-type vanishing theorem (see Theorem 7.80(f) of [SS]) and conclude that

$$H^1(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}}) = H^1(\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}, \Theta_{\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}}})^{inv} = 0.$$

Remark 5.2. (1) To check (5.11), one has to show that for any $x \in I$

$$r(n_{l_x+1}(x) - r) + \sum_{i=1}^{l_x} r_i(x)(n_i(x) + n_{i+1}(x)) = 0.$$

Noting that $r = n_{l_x+1}(x) + \sum_{i=1}^{l_x} n_i(x)$ and $n_i(x) = r_i(x) - r_{i-1}(x)$, we have to show that

$$\sum_{i=1}^{l_x} \{r_{i-1}(x)(r - r_i(x)) - r_i(x)(r - r_{i+1}(x))\} = 0,$$

which is clearly true since $r_0(x) = 0$.

(2) Since $R^i\rho_*(\hat{\Theta}') = 0$ for all $i > 0$ (note that $\rho : \widetilde{\mathcal{R}}'_F \to \widetilde{\mathcal{R}}_F$ is a grassmannian bundle over $\widetilde{\mathcal{R}}_F$), we have $H^1(\widetilde{\mathcal{R}}'_F, \hat{\Theta}')^{inv} = H^1(\widetilde{\mathcal{R}}_F, \rho_*\hat{\Theta}')^{inv}$. By using the canonical decomposition (see Remark 4.2) and the vanishing Theorem 5.1, we can show that $H^1(\widetilde{\mathcal{R}}'_F, \hat{\Theta}')^{inv} = 0$.

Next we will show the vanishing theorem for the moduli space of semistable parabolic torsion free sheaves on a nodal curve X.

Theorem 5.2. *Assume that $g \geq 3$. Then $H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) = 0$.*

Proof. It will be reduced to proving a vanishing theorem for $\mathcal{P}$ given the following lemma.

Lemma 5.5. *For $0 \leq a \leq r$, the natural maps $H^1(\mathcal{W}_a, \Theta_{\mathcal{U}_X}) \to H^1(\mathcal{D}_1(a), \Theta_{\mathcal{P}})$ are injective. In particular, $H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \to H^1(\mathcal{P}, \Theta_{\mathcal{P}})$ is injective.*

Proof. It is known that $\phi_a := \phi|_{\mathcal{D}_1(a)} : \mathcal{D}_1(a) \to \mathcal{W}_a$ is the normalisation of $\mathcal{W}_a$ (see Proposition 2.1). If $\mathcal{W}_{a-1}$ is empty, ϕ_a is an isomorphism and the lemma is clear. If $\mathcal{W}_{a-1}$ is not empty, $\mathcal{W}_{a-1}$ is the non-normal locus of $\mathcal{W}_a$ and we are reduced to proving that

$$H^1(\mathcal{W}_{a-1}, \Theta_{\mathcal{U}_X}) \to H^1(\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1), \Theta_{\mathcal{P}})$$

is injective by Lemma 4.3 (2). Thus it is enough to show that

$$H^1(\mathcal{W}_{a-1}, \Theta_{\mathcal{U}_X}) \to H^1(\mathcal{D}_1(a-1), \Theta_{\mathcal{P}})$$

is injective, and we are done by induction since $\phi_0 : \mathcal{D}_1(0) \to \mathcal{W}_0$ is always an isomorphism.

In order to prove the vanishing theorem for $\mathcal{P}$, we have to prepare some lemmas.

Lemma 5.6. *Assume $\tilde{g} \geq 2$. Then $(\tilde{\psi}'_* \omega_{\mathcal{H}})^{inv} = \omega_{\mathcal{P}}$ where $\omega_{\mathcal{P}}$ is the canonical (dualising) sheaf of $\mathcal{P}$.*

Proof. We will check the conditions of Lemma 5.4. By Proposition 5.1 (2), $\mathcal{U}_{\widetilde{X}}(d-r)$ contains a stable bundle, and thus $\mathcal{W}_0$ contains a stable parabolic sheaf by Lemma 2.8, which shows that there exist stable parabolic bundles on X since stability is an open condition. Thus there exist stable generalised parabolic bundles on $\widetilde{X}$ by Lemma 2.2 (2), and the action of $PGL(\tilde{n})$ on $\mathcal{H}$ is therefore generically free. We now check conditions (1) and (2) of Lemma 5.4.

(1) By Proposition 5.2 (2), the nonstable locus in $\widetilde{\mathcal{R}}'^{ss} \smallsetminus \{\mathcal{D}_1^f \cup \mathcal{D}_2^f\}$ has codimension ≥ 2. We need to show that each of the $\hat{\mathcal{D}}_j(r-1)$ and $\hat{\mathcal{D}}_j^t$ contains GPS with no automorphism except scales. Take $j = 1$ for definiteness, let $\widetilde{E}$ be a stable parabolic bundle on $\widetilde{X}$ of degree $d-r$, let $E = \widetilde{E} \oplus {}_{x_2}\mathbb{C}^r$ and define the GPS structure on E as follows. We take $Q = \mathbb{C}^r$, the map $E_{x_2} \to Q$ to be the obvious projection, and the map $E_{x_1} \to Q$ any isomorphism. This yields, after an identification $H^0(E) \cong \mathbb{C}^{\tilde{n}}$, a point on $\hat{\mathcal{D}}_1^t$ as required. Next consider $E = \widetilde{E} \otimes \mathcal{O}_{\widetilde{X}}(x_2)$, the GPS structure being given by taking $Q = \widetilde{E}_{x_2}$, the map $E_{x_1} \to Q$ being zero, and the map $E_{x_2} \to Q$ the residue $\mathcal{O}_{x_2}(x_2) \cong \mathbb{C}$. This yields a point on $\hat{\mathcal{D}}_1(r-1)$ with only automorphisms by scales.

(2) If a prime divisor is not contained in the nonstable locus, its image in $\mathcal{P}$ will have codimension one. If it is contained in the nonstable locus, then, by (2) of Proposition 5.2, it has to be one of the $(\hat{\mathcal{D}}_j(r-1))^{ss}$ and $(\hat{\mathcal{D}}_j^t)^{ss}$. We have already seen that the respective images of these in $\mathcal{P}$ are the $\mathcal{D}_j$ by Proposition 3.3.

Lemma 5.7. *There is a morphism $Det : \mathcal{H} \to J^d_{\widetilde{X}}$ that extends the determinant morphism on the open set $\widetilde{\mathcal{R}}'_F$. Moreover, it yields a flat morphism $Det : \mathcal{P} \to J^d_{\widetilde{X}}$.*

Proof. Note that, on $\widetilde{X} \times \mathcal{H}$, we have an exact squence

$$0 \to \mathcal{K} \to \mathcal{O}^{\tilde{n}} \to \mathcal{E} \to 0,$$

and $\mathcal{K}$ is flat over $\mathcal{H}$ since $\mathcal{E}$ is so. One proves that $\mathcal{K}$ is locally free on $\widetilde{X} \times \mathcal{H}$ (by using Lemma 5.4 of [Ne]). Thus $det(\mathcal{K})^{-1}$ is a line bundle on $\widetilde{X} \times \mathcal{H}$ and gives a morphism

$$Det : \mathcal{H} \to J^d_{\widetilde{X}},$$

which is clearly an extension of the determinant morphism on the open set $\widetilde{\mathcal{R}}'_F$. Restricted to $\widetilde{\mathcal{R}}'^{ss}$ the map Det clearly factors through the quotient by the $SL(\tilde{n})$ action and yields a morphism

$$Det : \mathcal{P} \to J^d_{\widetilde{X}},$$

which we will prove to be a flat morphism. J^0_X acts on $\mathcal{P}$ by

$$(E,Q) \mapsto L\cdot(E,Q) := (E\otimes \pi^*L, Q\otimes L_{x_0}).$$

One checks that $Det(L\cdot(E,Q)) = Det(E,Q)\otimes(\pi^*L)^r$. Noting that the pull-back map $J^0_X \to J^0_{\widetilde{X}}$ and the r-power map $J^0_{\widetilde{X}} \to J^0_{\widetilde{X}}$ are surjective and that $J^0_{\widetilde{X}}$ acts transitively on $J^d_{\widetilde{X}}$, we can see that $Det : \mathcal{P} \to J^d_{\widetilde{X}}$ is flat by generic flatness.

Let $\mathcal{H}^L$ denote the (reduced) fibre over $L \in J^d_{\widetilde{X}}$, and let $\mathcal{P}^L$ denote the (reduced) fibre of Det above L. Clearly $\mathcal{P}^L$ is the GIT quotient of $\mathcal{H}^L$, and all of the properties of $\mathcal{H}$ and $\mathcal{P}$ continue to be valid for $\mathcal{H}^L$ and $\mathcal{P}^L$. From the proof of the above lemma, one sees that all of the fibres of $Det : \mathcal{P} \to J^d_{\widetilde{X}}$ are reduced. Thus $\mathcal{P}^L$ is also the scheme-theoretic fibre over L, and we have

Proposition 5.3. *The canonical (dualising) sheaf of $\mathcal{P}^L$ is the restriction of $\omega_{\mathcal{P}}$ to $\mathcal{P}^L$.*

Proof. The following general fact can be proved by repeated use of Bertini (on U) and the adjunction formula: Suppose $f : V \to U$ is a flat map of varieties, with U smooth and V Gorenstein. Let V_p be the scheme-theoretic fibre over $p \in U$. Then the dualising sheaf of V_p is the restriction of the dualising sheaf of V.

Proposition 5.4. *Assume $\tilde{g} \geq 2$. Then $H^1(\mathcal{P}^L, \Theta_{\mathcal{P}}) = 0$ for any $L \in J^d_{\widetilde{X}}$.*

Proof. Let $\omega^L_{\mathcal{H}}$ denote the restriction of $\omega_{\mathcal{H}}$ to $\mathcal{H}^L$. Then $(\tilde{\psi}'_*\omega^L_{\mathcal{H}})^{inv} = \omega_{\mathcal{P}^L}$ by Lemma 5.6 and Proposition 5.3. Recall that, for the polarisation

$$\frac{(\tilde{\ell}-k)}{m} \times k \times \prod_{x\in I}\{\alpha_x, d_1(x), \cdots, d_{l_x}(x)\},$$

the line bundle $\hat{\Theta}'$ was defined to be

$$(det\, R\pi_{\mathcal{H}}\mathcal{E})^k \otimes \bigotimes_{x\in I}\{(det\,\mathcal{E}_x)^{\alpha_x} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{d_i(x)}\} \otimes (det\,\mathcal{E}_y)^{\tilde{\ell}}$$
$$\otimes\,(det\,\mathcal{Q})^k \otimes (det\,\mathcal{E}_y)^{-k},$$

which descends to the ample line bundle $\Theta_{\mathcal{P}}$ if the polarisation satisfies

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\sum_{x\in I}\alpha_x + r\tilde{\ell} = k\tilde{n}.$$

Noting that $(1\times Det)^*\mathcal{L} = (det\,\mathcal{K})^{-1}\otimes \pi^*_{\mathcal{H}}\mathcal{N}$ for a suitable line bundle $\mathcal{N}$ on $\mathcal{H}$, one sees that $\mathcal{N} \cong det\,\mathcal{K}_x$ on $\mathcal{H}^L$ for any $x \in \widetilde{X}$ (x may be x_1 and x_2), and

$$(det\, R\pi_{\mathcal{H}^L}\, det\,\mathcal{K}^{-1})^{-2} = (det\,\mathcal{E}_y)^{2\tilde{n}+(r-1)(2\tilde{g}-2)}.$$

Thus, on $\mathcal{H}^L$, we have $\hat{\Theta}' = \hat{\Theta}'_\omega \otimes \omega^L_{\mathcal{H}}$ by Proposition 3.4, where

$$\hat{\Theta}'_\omega = (det\, R\pi_{\mathcal{H}^L}\mathcal{E})^{\bar{k}} \otimes \bigotimes_{x\in I}\{(det\,\mathcal{E}_x)^{\bar{\alpha}_x} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{\bar{d}_i(x)}\} \otimes (det\,\mathcal{E}_y)^{\bar{\ell}}$$
$$\otimes\,(det\,\mathcal{Q})^{\bar{k}} \otimes (det\,\mathcal{E}_y)^{-\bar{k}}$$

with $\bar{k} = k + 2r$, $\bar{\ell} = 2\tilde{n} + \tilde{\ell} - r|I|$, $\bar{d}_i(x) = d_i(x) + n_i(x) + n_{i+1}(x)$, and $\bar{\alpha}_x = \alpha_x + n_{l_x+1}(x)$. One checks that

$$\sum_{x\in I}\sum_{i=1}^{l_x}\bar{d}_i(x)r_i(x) + r\sum_{x\in I}\bar{\alpha}_x + r\bar{\ell} = \bar{k}\tilde{n}.$$

The rest of the proof proceeds as Theorem 5.1 except that an analogue of Lemma 5.3 is not needed. The Kodaira-type vanishing theorem and Hartogs-type extension theorem for cohomology are applicable since $\mathcal{H}^L$ and $\mathcal{P}^L$ are Cohen-Macaulay and have only rational singularities.

Theorem 5.3. *Assume $\tilde{g} \geq 2$. Then $H^1(\mathcal{P}, \Theta_{\mathcal{P}}) = 0$.*

Proof. We consider the flat morphism $Det : \mathcal{P} \to J^d_{\widetilde{X}}$ and try to decompose the direct image $(Det)_*\Theta_{\mathcal{P}}$. One can see that $(Det)_*\Theta_{\mathcal{P}} = \{(Det_{\widetilde{\mathcal{R}}'^{ss}})_*\hat{\Theta}'\}^{inv}$ and the equalities

$$\{(Det_{\widetilde{\mathcal{R}}'^{ss}})_*\hat{\Theta}'\}^{inv} = \{(Det_{\mathcal{H}})_*\hat{\Theta}'\}^{inv} = \{(Det_{\widetilde{\mathcal{R}}'_F})_*\hat{\Theta}'\}^{inv}$$

hold by using Lemma 4.1 and Lemma 4.2, where $Det_{\widetilde{\mathcal{R}}'_F} : \widetilde{\mathcal{R}}'_F \to J^d_{\widetilde{X}}$ is clearly factorized through the projection $\rho : \widetilde{\mathcal{R}}'_F \to \widetilde{\mathcal{R}}_F$. Thus $(Det_{\widetilde{\mathcal{R}}'_F})_*\hat{\Theta}' = (Det_{\widetilde{\mathcal{R}}_F})_*\rho_*\hat{\Theta}'$ and, by Remark 4.2, we have

$$(Det_{\widetilde{\mathcal{R}}'_F})_*\hat{\Theta}' = \bigoplus_{\mu}(Det_{\widetilde{\mathcal{R}}^\mu_F})_*\hat{\Theta}_\mu,$$

where $Det_{\widetilde{\mathcal{R}}^\mu_F} = Det_{\widetilde{\mathcal{R}}_F}\cdot p^\mu : \widetilde{\mathcal{R}}^\mu_F \to J^d_{\widetilde{X}}$, which restricting to $(\widetilde{\mathcal{R}}^\mu)^{ss}$ induces a morphism $Det_\mu : \mathcal{U}^\mu_{\widetilde{X}} \to J^d_{\widetilde{X}}$. It is now clear that we have the decomposition

$$(Det)_*\Theta_{\mathcal{P}} = \bigoplus_{\mu}(Det_\mu)_*\Theta_\mu,$$

which implies that $H^1(J^d_{\widetilde{X}}, (Det)_*\Theta_{\mathcal{P}}) = 0$ by Theorem 5.1 since $\tilde{g} \geq 2$. On the other hand, $R^1(Det)_*(\Theta_{\mathcal{P}}) = 0$ by Proposition 5.4. Hence we are done by using spectral sequence.

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