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Research Article

Some New Nonlinear Weakly Singular Integral Inequalities of Wendroff Type with Applications

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Some new weakly singular integral inequalities of Wendroff type are established, which generalized some known weakly singular inequalities for functions in two variables and can be used in the analysis of various problems in the theory of certain classes of integral equations and evolution equations. Application examples are also given.

1. Introduction

In the study of differential and integral equations, one often deals with certain integral inequalities. The Gronwall-Bellman inequality and its various linear and nonlinear generalizations are crucial in the discussion of existence, uniqueness, continuation, boundedness, oscillation, and stability properties of solutions. The literature on such inequalities and their applications is vast; see [1–6], and the references are given therein. Usually, the integrals concerning such inequalities have regular or continuous kernels, but some problems arising from theoretical or practical phenomena require us to solve integral inequalities with singular kernels. For example, Henry [7] used this type of integral inequalities to prove global existence and exponential decay results for a parabolic Cauchy problem; Sano and Kunimatsu [8] gave a sufficient condition for stabilization of semilinear parabolic distributed systems by making use of a modification of Henry-type inequalities; Ye et al. [9] proved a generalization of this type of inequalities and used it to study the dependence of the solution on the order and the initial condition of a fractional differential equation. All such inequalities are proved by an iteration argument, and the estimation formulas are expressed by a complicated power series which are sometimes not very convenient for applications. To avoid this shortcoming, Medveď [10] presented a new method for studying Henry-type inequalities and established explicit bounds with relatively simple formulas which are similar to the classic Gronwall-Bellman inequalities. Very recently, Ma and Pečarić [11] used a modification of Medveď's method to study certain class of nonlinear inequalities of Henry-type, which generalized some known results and were used as handy and effective tools in the study of the solutions' boundedness of some fractional differential and integral equations.

In this paper, by applying Medveď's method of desingularization of weakly singular inequalities we establish some new singular version of the Wendroff inequality (see [1, 12]) for functions in two variables. An example is included to illustrate the usefulness of our results.

2. Main result

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0,+\infty)$. As usual, $C^i(M,S)$ denotes the class of all i -times continuously differentiable functions defined on a set M with range in a set $S(i = 1,2,\ldots)$, and $C^0(M,S) = C(M,S)$.

For convenience, before giving our main results, we first cite some useful lemmas and definitions here.

Lemma 2.1 (see [13]). *Let $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{q/p} \leq \frac{q}{p}K^{(q-p)/p}a + \frac{p-q}{p}K^{q/p} \tag{2.1}$$

for any $K > 0$.

Definition 2.2 (see [14]). Let $[x,y,z]$ be an ordered parameter group of nonnegative real numbers. The group is said to belong to the first class distribution and denoted by $[x,y,z] \in I$ if conditions $x \in (0,1]$, $y \in (1/2,1)$ and $z \geq 3/2 - y$ are satisfied; it is said to belong to the second-class distribution and denoted by $[x,y,z] \in II$ if conditions $x \in (0,1]$, $y \in (0,1/2]$, and $z > (1-2y^2)/(1-y^2)$ are satisfied.

Lemma 2.3 (see [15, page 296]). *Let α, β, γ, and p be positive constants. Then,*

$$\int_0^t (t^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\gamma-1)} ds = \frac{t^\theta}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in \mathbb{R}_+, \tag{2.2}$$

where $B[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}\,ds$ ($\xi,\eta \in \mathbb{C}$, $\operatorname{Re}\xi > 0$, $\operatorname{Re}\eta > 0$) is the well-known beta function and $\theta = p[\alpha(\beta-1)+\gamma-1]+1$.

Lemma 2.4 (see [14]). *Suppose that the positive constants α, β, γ, p_1, and p_2 satisfy*

(a) *if $[\alpha,\beta,\gamma] \in I$, $p_1 = 1/\beta$;*

(b) *if $[\alpha,\beta,\gamma] \in II$, $p_2 = (1+4\beta)/(1+3\beta)$, then*

$$\begin{gathered} B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right] \in (0,+\infty), \\ \theta_i = p_i\left[\alpha(\beta-1)+\gamma-1\right]+1 \geq 0 \end{gathered} \tag{2.3}$$

are valid for $i = 1,2$.

Lemma 2.5 (see [6, page 329]). *Let $u(x,y)$, $p(x,y)$, $q(x,y)$, and $k(x,y)$ be nonnegative continuous functions defined for $x, y \in \mathbb{R}_+$. If*

$$u(x,y) \le p(x,y) + q(x,y)\int_0^x\int_0^y k(s,t)u(s,t)ds\,dt \tag{2.4}$$

for $x, y \in \mathbb{R}_+$, then

$$u(x,y) \le p(x,y) + q(x,y)\left(\int_0^x\int_0^y k(s,t)p(s,t)ds\,dt\right)\exp\left(\int_0^x\int_0^y k(s,t)q(s,t)ds\,dt\right) \tag{2.5}$$

for $x, y \in \mathbb{R}_+$.

We also need the following well-known consequence of the Jensen inequality:

$$(A_1 + A_2 + \cdots + A_n)^r \le n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r) \tag{JI}$$

for $A_i \ge 0$ $(i = 1, 2, \dots, n)$ and $r \ge 1$.

Theorem 2.6. *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, and $f(x,y)$ be nonnegative continuous functions for $(x,y) \in D = [0,T) \times [0,T)$ $(0 < T \le \infty)$. Let p and q be constants with $p \ge q > 0$. If $u(x,y)$ satisfies*

$$\begin{aligned} &u^p(x,y) \\ &\quad\le a(x,y) + b(x,y)\int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1}f(s,t)u^q(s,t)ds\,dt, \quad (x,y) \in D, \end{aligned} \tag{2.6}$$

then for any $K > 0$ one has the following.

(i) *If $[\alpha, \beta, \gamma] \in I$,*

$$\begin{aligned} u(x,y) \le \Bigg\{ a(x,y) + \Bigg[&P_1(x,y) + Q_1(x,y)\left(\int_0^x\int_0^y f^{1/(1-\beta)}(s,t)P_1(s,t)ds\,dt\right) \\ &\times \exp\left(\int_0^x\int_0^y f^{1/(1-\beta)}(s,t)Q_1(s,t)ds\,dt\right)\Bigg]^{1-\beta}\Bigg\}^{1/p} \end{aligned} \tag{2.7}$$

for $(x, y) \in D$*, where*

$$M_1 = \frac{1}{\alpha} B\left[\frac{\beta+\gamma-1}{\alpha\beta}, \frac{2\beta-1}{\beta}\right],$$

$$A(x, y) = \frac{q}{p} K^{(q-p)/p} a(x, y) + \frac{p-q}{p} K^{q/p},$$

$$\mathscr{A}_1(x, y) = \int_0^x \int_0^y f^{1/(1-\beta)}(s, t) A^{1/(1-\beta)}(s, t) ds\, dt,$$

$$P_1(x, y) = 2^{\beta/(\beta-1)} M_1^{2\beta/(1-\beta)} (xy)^{((\alpha+1)(\beta-1)+\gamma)/(1-\beta)} \mathscr{A}_1(x, y) b^{1/(1-\beta)}(x, y),$$

$$Q_1(x, y) = 2^{\beta/(\beta-1)} K^{(q-p)/p(1-\beta)} M_1^{2\beta/(1-\beta)} \left(\frac{q}{p}\right)^{1/(1-\beta)} (xy)^{((\alpha+1)(\beta-1)+\gamma)/(1-\beta)} b^{1/(1-\beta)}(x, y). \tag{2.8}$$

(ii) *If* $[\alpha, \beta, \gamma] \in II$*,*

$$\begin{aligned} u(x, y) \leq \Bigg\{ a(x, y) + \Bigg[P_2(x, y) + Q_2(x, y) \left(\int_0^x \int_0^y f^{(1+4\beta)/\beta}(s, t) P_2(s, t) ds\, dt \right) \\ \times \exp\left(\int_0^x \int_0^y f^{(1+4\beta)/\beta}(s, t) Q_2(s, t) ds\, dt \right) \Bigg]^{\beta/(1+4\beta)} \Bigg\}^{1/p}, \end{aligned} \tag{2.9}$$

for $(x, y) \in D$*, where*

$$M_2 = \frac{1}{\alpha} B\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)}, \frac{4\beta^2}{1+3\beta}\right],$$

$$\mathscr{A}_2(x, y) = \int_0^x \int_0^y f^{(1+4\beta)/\beta}(s, t) A^{(1+4\beta)/\beta}(s, t) ds\, dt,$$

$$P_2(x, y) = 2^{(1+3\beta)/\beta} M_2^{2(1+3\beta)/\beta} (xy)^{((1+4\beta)[\alpha(\beta-1)+\gamma]-\beta)/\beta} \mathscr{A}_2(x, y) b^{(1+4\beta)/\beta}(x, y),$$

$$Q_2(x, y) = 2^{(1+3\beta)/\beta} K^{(q-p)(1+4\beta)/p\beta} M_2^{2(1+3\beta)/\beta} \left(\frac{q}{p}\right)^{(1+4\beta)/\beta} (xy)^{((1+4\beta)[\alpha(\beta-1)+\gamma]-\beta)/\beta} b^{(1+4\beta)/\beta}(x, y). \tag{2.10}$$

Proof. Define a function $v(x, y)$ by

$$v(x, y) = b(x, y) \int_0^x \int_0^y (x^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} (y^\alpha - t^\alpha)^{\beta-1} t^{\gamma-1} f(s, t) u^q(s, t) ds\, dt, \quad (x, y) \in D, \tag{2.11}$$

then

$$u^p(x,y) \leq a(x,y) + v(x,y), \tag{2.12}$$

or

$$u(x,y) \leq (a(x,y) + v(x,y))^{1/p}, \quad (x,y) \in D. \tag{2.13}$$

By Lemma 2.1 and inequality (2.13), for any $K > 0$, we have

$$u^q(x,y) \leq \left(a(x,y) + v(x,y)\right)^{q/p} \leq \frac{q}{p}K^{(q-p)/p}\left(a(x,y) + v(x,y)\right) + \frac{p-q}{p}K^{q/p}. \tag{2.14}$$

Substituting the last relation into (2.11), we get

$$\begin{aligned} v(x,y) &\leq b(x,y)\int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1} \\ &\qquad \times f(s,t)\left[\frac{q}{p}K^{(q-p)/p}(a(s,t) + v(s,t)) + \frac{p-q}{p}K^{q/p}\right]ds\,dt \\ &= b(x,y)\int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1}f(s,t)A(s,t)ds\,dt \\ &\quad + \frac{q}{p}K^{(q-p)/p}b(x,y)\int_0^x\int_0^y (x^\alpha - s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha - t^\alpha)^{\beta-1}t^{\gamma-1}f(s,t)v(s,t)ds\,dt, \end{aligned} \tag{2.15}$$

where $A(x,y) = (q/p)K^{(q-p)/p}a(x,y) + ((p-q)/p)K^{q/p}$.

If $[\alpha,\beta,\gamma] \in I$, let $p_1 = 1/\beta$, $q_1 = 1/(1-\beta)$; if $[\alpha,\beta,\gamma] \in II$, let $p_2 = (1+4\beta)/(1+3\beta)$, $q_2 = (1+4\beta)/\beta$, then $1/p_i + 1/q_i = 1$ for $i = 1,2$. By applying Hölder's inequality with indices p_i, q_i to (2.15), we get

$$\begin{aligned} v(x,y) &\leq b(x,y)\left[\int_0^x\int_0^y (x^\alpha - s^\alpha)^{p_i(\beta-1)}s^{p_i(\gamma-1)}(y^\alpha - t^\alpha)^{p_i(\beta-1)}t^{p_i(\gamma-1)}ds\,dt\right]^{1/p_i} \\ &\quad \times \left[\int_0^x\int_0^y f^{q_i}(s,t)A^{q_i}(s,t)ds\,dt\right]^{1/q_i} \\ &\quad + \frac{q}{p}K^{(q-p)/p}b(x,y)\left[\int_0^x\int_0^y (x^\alpha - s^\alpha)^{p_i(\beta-1)}s^{p_i(\gamma-1)}(y^\alpha - t^\alpha)^{p_i(\beta-1)}t^{p_i(\gamma-1)}ds\,dt\right]^{1/p_i} \\ &\quad \times \left[\int_0^x\int_0^y f^{q_i}(s,t)v^{q_i}(s,t)ds\,dt\right]^{1/q_i}. \end{aligned} \tag{2.16}$$

By Lemmas 2.3 and 2.4, the last inequality can be rewritten as

$$\begin{aligned} v(x,y) \le{}& \left(M_i^2 (xy)^{\theta_i}\right)^{1/p_i} \mathcal{A}_i^{1/q_i}(x,y) b(x,y) + K^{(q-p)/p}\frac{q}{p}\left(M_i^2 (xy)^{\theta_i}\right)^{1/p_i} b(x,y) \\ &\times \left[\int_0^x\int_0^y f^{q_i}(s,t) v^{q_i}(s,t)\,ds\,dt\right]^{1/q_i} \end{aligned} \tag{2.17}$$

for $(x,y) \in D$, where

$$\begin{aligned} M_i &= \frac{1}{\alpha} B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right], \\ \mathcal{A}_i(x,y) &= \int_0^x\int_0^y f^{q_i}(s,t) A^{q_i}(s,t)\,ds\,dt, \end{aligned} \tag{2.18}$$

and θ_i is given as in Lemma 2.4 for $i = 1,2$.

Applying inequality (JI) to (2.17), we get

$$\begin{aligned} v^{q_i}(x,y) \le{}& 2^{q_i-1}\left(M_i^2 (xy)^{\theta_i}\right)^{q_i/p_i} \mathcal{A}_i(x,y) b^{q_i}(x,y) \\ &+ 2^{q_i-1}\left(\frac{q}{p}\right)^{q_i} K^{q_i((q-p)/p)}\left(M_i^2 (xy)^{\theta_i}\right)^{q_i/p_i} b^{q_i}(x,y) \int_0^x\int_0^y f^{q_i}(s,t) v^{q_i}(s,t)\,ds\,dt. \end{aligned} \tag{2.19}$$

By Lemma 2.5 and the last inequality, we have

$$v^{q_i}(x,y) \le P_{1i}(x,y) + Q_{1i}(x,y)\left(\int_0^x\int_0^y f^{q_i}(s,t) P_{1i}(s,t)\,ds\,dt\right)\exp\left(\int_0^x\int_0^y f^{q_i}(s,t) Q_{1i}(s,t)\,ds\,dt\right), \tag{2.20}$$

where

$$\begin{aligned} P_{1i}(x,y) &= 2^{q_i-1}\left(M_i^2 (xy)^{\theta_i}\right)^{q_i/p_i} \mathcal{A}_i(x,y) b^{q_i}(x,y), \\ Q_{1i}(x,y) &= 2^{q_i-1}\left(\frac{q}{p}\right)^{q_i} K^{q_i((q-p)/p)}\left(M_i^2 (xy)^{\theta_i}\right)^{q_i/p_i} b^{q_i}(x,y). \end{aligned} \tag{2.21}$$

Finally, substituting (2.20) into (2.13), considering two situations for $i = 1,2$ and using parameters α, β, and γ to denote p_i, q_i and θ_i in (2.20), we can get the desired estimations (2.7) and (2.9), respectively. □

Remark 2.7. In (2.7) and (2.9), we not only have given some bounds to a new class of nonlinear weakly singular integral inequalities of Wendroff type, but also note that function $a(x,y)$ appearing in (2.7) and (2.9) is not required to satisfy the nondecreasing condition as some known results [16].

Corollary 2.8. *Let functions $u(x,y)$, $a(x,y)$, $b(x,y)$, and $f(x,y)$ be defined as in Theorem 2.6, and let q be a constant with $0 < q \le 1$. Suppose that*

$$u(x,y) \le a(x,y) + b(x,y)\int_0^x\int_0^y (x-s)^{\beta-1}s^{\gamma-1}(y-t)^{\beta-1}t^{\gamma-1}f(s,t)u^q(s,t)\,ds\,dt \tag{2.22}$$

for $(x,y)\in D$, then one has the following.

(i) *If $\beta \in (1/2,1)$,*

$$\begin{aligned} u(x,y) \le a(x,y) + \Bigg[&\overline{P}_{11}(x,y) + \overline{Q}_{11}(x,y)\left(\int_0^x\int_0^y f^{1/(1-\beta)}(s,t)\overline{P}_{11}(s,t)\,ds\,dt\right) \\ &\times \exp\left(\int_0^x\int_0^y f^{1/(1-\beta)}(s,t)\overline{Q}_{11}(s,t)\,ds\,dt\right)\Bigg]^{1-\beta} \end{aligned} \tag{2.23}$$

for $(x,y)\in D$, where

$$\begin{aligned} M_{11} &= B\left[\frac{\beta+\gamma-1}{\beta}, \frac{2\beta-1}{\beta}\right], \\ A_1(x,y) &= qK^{q-1}a(x,y) + (1-q)K^q, \\ \mathscr{A}_{11}(x,y) &= \int_0^x\int_0^y f^{1/(1-\beta)}(s,t)A_1^{1/(1-\beta)}(s,t)\,ds\,dt, \\ \overline{P}_{11}(x,y) &= 2^{\beta/(\beta-1)}M_{11}^{2\beta/(1-\beta)}(xy)^{(2\beta+\gamma-2)/(1-\beta)}\mathscr{A}_{11}(x,y)b^{1/(1-\beta)}(x,y), \\ \overline{Q}_{11}(x,y) &= 2^{\beta/(\beta-1)}K^{(q-1)/(1-\beta)}M_{11}^{2\beta/(1-\beta)}q^{1/(1-\beta)}(xy)^{(2\beta+\gamma-2)/(1-\beta)}b^{1/(1-\beta)}(x,y). \end{aligned} \tag{2.24}$$

(ii) *If $\beta \in (0,1/2]$,*

$$\begin{aligned} u(x,y) \le a(x,y) + \Bigg[&\overline{P}_{12}(x,y) + \overline{Q}_{12}(x,y)\left(\int_0^x\int_0^y f^{(1+4\beta)/\beta}(s,t)\overline{P}_{12}(s,t)\,ds\,dt\right) \\ &\times \exp\left(\int_0^x\int_0^y f^{(1+4\beta)/\beta}(s,t)\overline{Q}_{12}(s,t)\,ds\,dt\right)\Bigg]^{\beta/(1+4\beta)}, \end{aligned} \tag{2.25}$$

where

$$M_{12} = B\left[\frac{\gamma(1+4\beta)-\beta}{1+3\beta}, \frac{4\beta^2}{1+3\beta}\right],$$

$$\mathcal{A}_{12}(x,y) = \int_0^x \int_0^y f^{(1+4\beta)/\beta}(s,t) A_1^{(1+4\beta)/\beta}(s,t)\,ds\,dt,$$

$$\overline{P}_{12}(x,y) = 2^{(1+3\beta)/\beta} M_{12}^{2(1+3\beta)/\beta} (xy)^{((4\beta+1)(\gamma-1)+4\beta^2)/\beta} \mathcal{A}_{12}(x,y) b^{(1+4\beta)/\beta}(x,y),$$

$$\overline{Q}_{12}(x,y) = 2^{(1+3\beta)/\beta} K^{(q-1)(1+4\beta)/\beta} M_{12}^{2(1+3\beta)/\beta} q^{(1+4\beta)/\beta} (xy)^{((4\beta+1)(\gamma-1)+4\beta^2)/\beta} b^{(1+4\beta)/\beta}(x,y). \tag{2.26}$$

Proof. Inequalities (2.23) and (2.25) follow by letting $p = \alpha = 1$ and $0 < q \le 1$ in Theorem 2.6 and by simple computation. Details are omitted here. □

Remark 2.9. When $b(x,y) \equiv 1$, the inequality (2.22) has been studied in [16], but here we not only have given some new estimates for $u(x,y)$ (which are unfortunately incomparable with the results in [16]), but also eliminated the nondecreasing condition for function $a(x,y)$.

Let $p = 2$, $q = \alpha = 1$, we get the following interesting Henry-Ou-Iang type singular integral inequality. For a more detailed account of Ou-Iang type inequalities and their applications, one is referred to [6] and references cited therein.

Corollary 2.10. *Let functions $u(x,y)$, $a(x,y)$, $b(x,y)$, and $f(x,y)$ be defined as in Theorem 2.6. Suppose that*

$$u^2(x,y) \le a(x,y) + b(x,y)\int_0^x\int_0^y (x-s)^{\beta-1}s^{\gamma-1}(y-t)^{\beta-1}t^{\gamma-1} f(s,t)u(s,t)\,ds\,dt \tag{2.27}$$

for $(x,y) \in D$, then for any $K > 0$, one has the following.

(i) *If $\beta \in (1/2, 1)$,*

$$\begin{aligned} u^2(x,y) \le a(x,y) + \Bigg[&\overline{P}_{21}(x,y) + \overline{Q}_{21}(x,y)\left(\int_0^x\int_0^y f^{1/(1-\beta)}(s,t)\overline{P}_{21}(s,t)\,ds\,dt\right) \\ &\times \exp\left(\int_0^x\int_0^y f^{1/(1-\beta)}(s,t)\overline{Q}_{21}(s,t)\,ds\,dt\right)\Bigg]^{1-\beta} \end{aligned} \tag{2.28}$$

for $(x,y) \in D$, where

$$M_{11} = B\left[\frac{\beta+\gamma-1}{\beta}, \frac{2\beta-1}{\beta}\right],$$

$$A_2(x,y) = \frac{1}{2}K^{-1/2}a(x,y) + \frac{1}{2}K^{1/2},$$

$$\mathscr{A}_{21}(x,y)=\int_0^x\int_0^y f^{1/(1-\beta)}(s,t)A_2^{1/(1-\beta)}(s,t)ds\,dt,$$
$$\overline{P}_{21}(x,y)=2^{\beta/(\beta-1)}M_{11}^{2\beta/(1-\beta)}(xy)^{(2\beta+\gamma-2)/(1-\beta)}\mathscr{A}_{21}(x,y)b^{1/(1-\beta)}(x,y),$$
$$\overline{Q}_{21}(x,y)=2^{(\beta+1)/(\beta-1)}K^{-1/2(1-\beta)}M_{11}^{2\beta/(1-\beta)}(xy)^{(2\beta+\gamma-2)/(1-\beta)}b^{1/(1-\beta)}(x,y). \tag{2.29}$$

(ii) *If* $\beta\in(0,1/2]$,

$$u^2(x,y)\le a(x,y)+\left[\overline{P}_{22}(x,y)+\overline{Q}_{22}(x,y)\left(\int_0^x\int_0^y f^{(1+4\beta)/\beta}(s,t)\overline{P}_{22}(s,t)ds\,dt\right)\right.$$
$$\left.\times\exp\left(\int_0^x\int_0^y f^{(1+4\beta)/\beta}(s,t)\overline{Q}_{22}(s,t)ds\,dt\right)\right]^{\beta/(1+4\beta)} \tag{2.30}$$

for $(x,y)\in D$, *where*

$$M_{12}=B\left[\frac{\gamma(1+4\beta)-\beta}{1+3\beta},\frac{4\beta^2}{1+3\beta}\right],$$
$$\mathscr{A}_{22}(x,y)=\int_0^x\int_0^y f^{(1+4\beta)/\beta}(s,t)A_2^{(1+4\beta)/\beta}(s,t)ds\,dt,$$
$$\overline{P}_{22}(x,y)=2^{(1+3\beta)/\beta}M_{12}^{2(1+3\beta)/\beta}(xy)^{((4\beta+1)(\gamma-1)+4\beta^2)/\beta}\mathscr{A}_{22}(x,y)b^{(1+4\beta)/\beta}(x,y),$$
$$\overline{Q}_{22}(x,y)=2^{-1}K^{(q-1)(1+4\beta)/\beta}M_{12}^{2(1+3\beta)/\beta}(xy)^{((4\beta+1)(\gamma-1)+4\beta^2)/\beta}b^{(1+4\beta)/\beta}(x,y). \tag{2.31}$$

Proof. Inequalities (2.28) and (2.30) follow by letting $p=2$, $q=\alpha=1$ in Theorem 2.6 and by simple computation. Details are omitted. □

Theorem 2.11. *Let* $u(x,y)$, $a(x,y)$, $b(x,y)$, *and* $f(x,y)$ *be defined as in Theorem 2.6, let* $p\ge 1$ *be a constant, and let* $L: D\times\mathbb{R}_+\to\mathbb{R}_+$ *be a continuous function which satisfies the condition*

$$0\le L(x,y,v)-L(x,y,w)\le N(x,y,w)(v-w) \tag{C}$$

for $(x,y)\in D$ *and* $v\ge w\ge 0$, *where* $N: D\times\mathbb{R}_+\to\mathbb{R}_+$ *is a continuous function.*

If $u(x,y)$ *satisfies that*

$$u^p(x,y)\le a(x,y)+b(x,y)\int_0^x\int_0^y\left(x^\alpha-s^\alpha\right)^{\beta-1}s^{\gamma-1}\left(y^\alpha-t^\alpha\right)^{\beta-1}t^{\gamma-1}f(s,t)L\big(s,t,u(s,t)\big)ds\,dt \tag{2.32}$$

for $(x,y)\in D$, *then for any* $K>0$ *one has the following.*

(i) *If* $[\alpha,\beta,\gamma] \in I$,

$$u(x,y) \le \Bigg\{ a(x,y) + \Bigg[P_1^*(x,y) + Q_1^*(x,y) \left(\int_0^x \int_0^y f^{1/(1-\beta)}(s,t) P_1^*(s,t)\,ds\,dt \right) \times \exp\left(\int_0^x \int_0^y f^{1/(1-\beta)}(s,t) N^{1/(1-\beta)}\left(s,t,\frac{1}{p}a(s,t) + \frac{p-1}{p}\right) \times Q_1^*(s,t)\,ds\,dt \right) \Bigg]^{1-\beta} \Bigg\}^{1/p} \tag{2.33}$$

for $(x,y) \in D$, *where*

$$\begin{gathered} M_1 = \frac{1}{\alpha} B\left[\frac{\beta+\gamma-1}{\alpha\beta}, \frac{2\beta-1}{\beta}\right], \\ \mathcal{L}_1(x,y) = \int_0^x \int_0^y f^{1/(1-\beta)}(s,t) L^{1/(1-\beta)}\left(s,t,\frac{1}{p}a(s,t)+\frac{p-1}{p}\right) ds\,dt, \\ P_1^*(x,y) = 2^{\beta/(\beta-1)} M_1^{2\beta/(1-\beta)} (xy)^{((\alpha+1)(\beta-1)+\gamma)/(1-\beta)} \mathcal{L}_1(x,y) b^{1/(1-\beta)}(x,y), \\ Q_1^*(x,y) = 2^{\beta/(\beta-1)} M_1^{2\beta/(1-\beta)} (xy)^{((\alpha+1)(\beta-1)+\gamma)/(1-\beta)} \left(\frac{b(x,y)}{p}\right)^{1/(1-\beta)}. \end{gathered} \tag{2.34}$$

(ii) *If* $[\alpha,\beta,\gamma] \in II$,

$$u(x,y) \le \Bigg\{ a(x,y) + \Bigg[P_2^*(x,y) + Q_2^*(x,y) \left(\int_0^x \int_0^y f^{(1+4\beta)/\beta}(s,t) P_2^*(s,t)\,ds\,dt \right) \times \exp\left(\int_0^x \int_0^y f^{(1+4\beta)/\beta}(s,t) N^{(1+4\beta)/\beta}\left(s,t,\frac{1}{p}a(s,t) + \frac{p-1}{p}\right) \times Q_2^*(s,t)\,ds\,dt \right) \Bigg]^{\beta/(1+4\beta)} \Bigg\}^{1/p} \tag{2.35}$$

for $(x,y) \in D$, *where*

$$\begin{gathered} M_2 = \frac{1}{\alpha} B\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)}, \frac{4\beta^2}{1+3\beta}\right], \\ \mathcal{L}_2(x,y) = \int_0^x \int_0^y f^{(1+4\beta)/\beta}(s,t) L^{(1+4\beta)/\beta}\left(s,t,\frac{1}{p}a(s,t)+\frac{p-1}{p}\right) ds\,dt, \\ P_2^*(x,y) = 2^{(1+3\beta)/\beta} M_2^{2(1+3\beta)/\beta} (xy)^{((1+4\beta)[\alpha(\beta-1)+\gamma]-\beta)/\beta} \mathcal{L}_2(x,y) b^{(1+4\beta)/\beta}(x,y), \\ Q_2^*(x,y) = 2^{(1+3\beta)/\beta} M_2^{2(1+3\beta)/\beta} (xy)^{((1+4\beta)[\alpha(\beta-1)+\gamma]-\beta)/\beta} \left(\frac{b(x,y)}{p}\right)^{(1+4\beta)/\beta}. \end{gathered} \tag{2.36}$$

Proof. Define a function $\overline{v}(x,y)$ by

$$\overline{v}(x,y)=b(x,y)\int_0^x\int_0^y (x^\alpha-s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha-t^\alpha)^{\beta-1}t^{\gamma-1}f(s,t)L(s,t,u(s,t))ds\,dt,\quad (x,y)\in D, \tag{2.37}$$

then

$$u^p(x,y)\le a(x,y)+\overline{v}(x,y). \tag{2.38}$$

By Lemma 2.1, we have

$$u(x,y)\le \left(a(x,y)+\overline{v}(x,y)\right)^{1/p}\le \frac{1}{p}\left(a(x,y)+\overline{v}(x,y)\right)+\frac{p-1}{p},\quad (x,y)\in D. \tag{2.39}$$

Substituting the last inequality into (2.37) and using condition (C), we get

$$\begin{aligned}\overline{v}(x,y)\le{}& b(x,y)\int_0^x\int_0^y (x^\alpha-s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha-t^\alpha)^{\beta-1}t^{\gamma-1}\\&\times f(s,t)L\left(s,t,\frac{1}{p}\left(a(s,t)+\overline{v}(s,t)\right)+\frac{p-1}{p}\right)ds\,dt\\\le{}& b(x,y)\int_0^x\int_0^y (x^\alpha-s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha-t^\alpha)^{\beta-1}t^{\gamma-1}\\&\times f(s,t)L\left(s,t,\frac{1}{p}a(s,t)+\frac{p-1}{p}\right)ds\,dt\\&+\frac{b(x,y)}{p}\int_0^x\int_0^y (x^\alpha-s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha-t^\alpha)^{\beta-1}t^{\gamma-1}\\&\times f(s,t)M\left(s,t,\frac{1}{p}a(s,t)+\frac{p-1}{p}\right)\overline{v}(s,t)ds\,dt.\end{aligned} \tag{2.40}$$

Applying similar procedures used from (2.15) to the end of the proof of Theorem 2.6 to the last inequality, we get the desired inequalities (2.33) and (2.35). □

3. Applications

In this section, we will indicate the usefulness of our main results in the study of the boundedness of certain partial integral equations with weakly singular kernel. Consider the partial integral equation:

$$z^p(x,y)=l(x,y)+h(x,y)\int_0^x\int_0^y (x^\alpha-s^\alpha)^{\beta-1}s^{\gamma-1}(y^\alpha-t^\alpha)^{\beta-1}t^{\gamma-1}F(s,t,z(s,t))ds\,dt \tag{3.1}$$

for $(x, y) \in D$, where $l(x, y)$ and $h(x, y) \in C(D, \mathbb{R})$, $F \in C(D \times \mathbb{R}, \mathbb{R})$ satisfies

$$\left|F(x, y, u)\right| \leq b(x, y)|u|^q \tag{3.2}$$

for some $b \in C(D, \mathbb{R}_+)$, and $p \geq q > 0$ are constants. Plugging (3.2) into (3.1) and by applying Theorem 2.6, we obtain a bound on the solutions $z(x, y)$ of (3.1).

Remark 3.1. (i) Obviously, the boundedness of the solutions of (3.1)-(3.2) cannot be derived by the known results in [16]. (ii) By our results and under some suitable conditions, other basic properties' solutions of (3.1) such as the uniqueness and the continuous dependence can also be derived here, but in order to save space, the details are omitted.

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