Packing Circuits in Matroids

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Abstract

The purpose of this paper is to characterize all matroids M that satisfy the following minimax relation: For any nonnegative integral weight function w defined on $E(M)$,

$$\begin{aligned} & \text{Maximum } \{k : M \text{ has } k \text{ circuits (repetition allowed) such that each element } e \text{ of } M \\ & \qquad \text{is used at most } 2w(e) \text{ times by these circuits}\} \\ = \; & \text{Minimum } \{\textstyle\sum_{x\in X} w(x) : X \text{ is a collection of elements (repetition allowed) of } M \\ & \qquad \text{such that every circuit in } M \text{ meets } X \text{ at least twice}\}. \end{aligned}$$

Our characterization contains a complete solution to a research problem on 2-edge-connected subgraph polyhedra posed by Cornuéjols, Fonlupt, and Naddef in 1985, which was independently solved by Vandenbussche and Nemhauser in [11].

Key words: Matroid, Circuit, Polyhedron, Total Dual Integrality, Traveling Salesman Problem.

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1 Introduction

For terminology on matroids, we follow Oxley [9]. Let M be a matroid with a nonnegative integral weight $w(e)$ on each element $e \in E(M)$. For any positive integer k, let

$$\nu_{k,w}(M) = \text{Maximum } \{p : M \text{ has } p \text{ circuits (repetition allowed) such that each element } e \text{ of } M \text{ is used at most } kw(e) \text{ times by these circuits}\}$$

$$\tau_{k,w}(M) = \text{Minimum } \{\textstyle\sum_{x \in X} w(x) : X \text{ is a collection of elements (repetition allowed) of } M \text{ such that every circuit in } M \text{ meets } X \text{ at least } k \text{ times}\}.$$

Clearly,

$$\nu_{k,w}(M) \le \tau_{k,w}(M). \tag{1.1}$$

However, (1.1) does not have to hold with equality in general. It is not difficult to verify that $\nu_{1,w}(M) = \tau_{1,w}(M)$ holds for every nonnegative integral weight w if and only if M is the direct sum of circuits and coloops. Let us call M *good* if the equality $\nu_{2,w}(M) = \tau_{2,w}(M)$ holds for every nonnegative integral weight w. The purpose of this paper is to characterize all good matroids.

As usual, let $M(G)$ stand for the graphic matroid of a graph G. Let $U_{2,4}$ be the uniform matroid of rank two on four elements. Let F_7 and F_7^* be the Fano matroid and its dual, respectively. Let K_n^- denote the graph obtained from K_n, the complete graph on n vertices, by deleting an edge, and let K be the following graph.

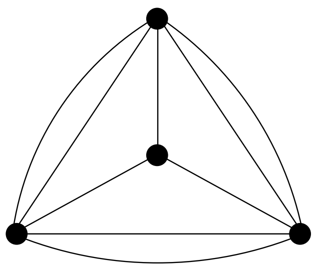

Figure 1.1: Graph K.

Theorem 1.1 *A matroid M is good if and only if none of its minors is isomorphic to $U_{2,4}$, F_7, F_7^*, $M(K_{3,3})$, $M(K_5^-)$, or $M(K)$.*

We can interpret good matroids using integer programs. Let A be the circuit-element incidence matrix of a matroid M. From the linear programming (LP) duality theorem, we see that M is good if and only if both of the following programs

$$\begin{array}{lll} \max & y^T\mathbf{1} \\ \text{s.t.} & y^TA \le w^T \\ & y \ge \mathbf{0} \end{array} \qquad \begin{array}{lll} \min & w^Tx \\ \text{s.t.} & Ax \ge \mathbf{1} \\ & x \ge \mathbf{0} \end{array} \tag{1.2}$$

have $\frac{1}{2}$-integral optimal solutions for every nonnegative integral weight w, where $\mathbf{0}$ is the zero vector and $\mathbf{1}$ is the all-one vector.

A rational linear system $Cx \ge d$, $x \ge \mathbf{0}$ is called *totally dual integral* (TDI) if the linear program $\max\{y^Td \mid y^TC \le w^T,\ y \ge \mathbf{0}\}$ has an integral optimal solution for every integral vector w for which the maximum is finite. The polyhedron $\{x : Cx \geq d,\ x \geq \mathbf{0}\}$ is call *integral* if all its vertices have integral coordinates. Equivalently, $\min\{w^T x \mid Cx \geq d,\ x \geq \mathbf{0}\}$ has an integral optimal solution for every integral vector w for which the minimum is finite. As shown by Edmonds and Giles [3], if the system $Cx \geq d$, $x \geq \mathbf{0}$ is *TDI* and d is integral, then the polyhedron $\{x : Cx \geq d,\ x \geq \mathbf{0}\}$ is integral. Therefore, M is good if and only if the linear system $Bx \geq \mathbf{1}, x \geq \mathbf{0}$ is TDI, where $B = A/2$. In general, the converse statement of the theorem of Edmonds and Giles is not true. However, we will prove the following, which is clearly a refinement of Theorem 1.1.

Theorem 1.2 *Let M be a matroid, let A be the circuit-element incidence matrix of M, and let $B = A/2$. Then the following statements are equivalent:*
(i) none of $U_{2,4}$, F_7, F_7^*, $M(K_{3,3})$, $M(K_5^-)$, and $M(K)$ is a minor of M;
(ii) the linear system $Bx \geq \mathbf{1}, x \geq \mathbf{0}$ is TDI;
(iii) the polyhedron $\{x : Bx \geq \mathbf{1}, x \geq \mathbf{0}\}$ is integral.

Observe that if M is the cographic matroid of a graph G, then circuits of M are precisely cuts of G and therefore A is the cut-edge incidence matrix of G. In this case our work is closely related to the graphical traveling salesman problem, see, for instances, Cornuéjols, Fonlupt, and Naddef [1] and Fonlupt and Naddef [4]. Given a graph G, let $P_1(G)$ be the convex hull of the incidence vectors of 2-edge-connected subgraphs of G where edges can be used several times, and let $P_2(G) = \{x : Bx \geq \mathbf{1}, x \geq \mathbf{0}\}$. Clearly, $P_1(G) \subseteq P_2(G)$ and equality need not hold in general. Cornuéjols, Fonlupt, and Naddef [1] proposed the problem of characterizing all graphs G for which $P_1(G) = P_2(G)$ (equivalently $P_2(G)$ is integral). They also showed that all series-parallel graphs G enjoy this property, where a graph is called *series-parallel* if it contains no K_4 as a minor. We point out that, when restricted to cographic matroids, the equivalence of (i) and (iii) in Theorem 1.2[1] gives a complete solution to this research problem, which was independently solved by Vandenbussche and Nemhauser in [11]. Furthermore, our approach is different from that in [11]. We determine the complete structure of matroids that satisfy (i), and we prove (ii) using this structure. Then we derive the equivalence of (i) and (iii) as a corollary. As is well known, (ii) is much stronger than (iii).

Our theorems are not the first ones on $\frac{1}{2}$-integrality. In [8], Lovász proved that the first program in (1.2) has a $\frac{1}{2}$-integral optimal solution, if A is the incidence matrix of T-cuts of a graph. A similar result is obtained by Geelen and Guenin [5] for odd cycles in signed graphs that do not have odd-K_5 minors. In addition, Gerards and Laurent [6] described all binary clutters that are box $\frac{1}{d}$-integral. (The clutters in our consideration are not binary, and box $\frac{1}{d}$-integral is much stronger than $\frac{1}{d}$-integral.)

Let us call a graph G *good* if $M^*(G)$, the cographic matroid of G, is good. Let P and K^* be the planar duals of K_5^- and K, respectively.

From Theorem 1.2 it can be seen that a graph G has an integral $P_2(G)$ if and only if it contains neither P nor K^* as a minor. With the same excluded minors, actually we can draw a much stronger statement as elaborated in the following lemma, which establishes the implication (i) $\Rightarrow$ (ii) of Theorem 1.2 when M is cographic.

Lemma 1.1 *A graph G is good if it contains neither P nor K^* as a minor.*

[1] Our result was first presented at the SIAM conference on discrete mathematics, Nashville, Tennessee, June, 2004.

Figure 1.2: P and K^*.

We finish this section by outlining the rest of the paper. In section 2, we prove Theorem 1.2 by using Lemma 1.1. In section 3, we prove that graphs without P and K^* minors can be expressed as sums of some prime graphs, which provides a structural characterization of good matroids. In section 4, we prove that being good is preserved under summing operations. In section 5, we introduce a packing property which is sufficient for being good. In sections 6, 7, and 8, we prove that each prime graph enjoys the packing property, which, together with the results established in sections 3, 4 and 5, proves Lemma 1.1 and thus completes our proof of our main theorem.

We remark that, in the last section, in order to verify our packing property on a few small graphs, we have to use computer to exhaust all the (about 2700) possibilities.

2 The easy implications

In this section, we prove Theorem 1.2, assuming Lemma 1.1. The implication (ii) $\Rightarrow$ (iii) follows instantly from the Edmonds-Giles theorem [3]. To show (iii) $\Rightarrow$ (i), it is clear that we only need to verify the following two lemmas, while the first is implied by (2.5) in [2].

Lemma 2.1 *If M satisfies (iii), then so do all its minors.*

Lemma 2.2 *None of the matroids $U_{2,4}$, F_7, F_7^*, $M(K_{3,3})$, $M(K_5^-)$, and $M(K)$ satisfies (iii).*

Proof. Clearly, we only need to find, for each of the given matroids, an integral vector w such that the optimal value of (1.2) is not $\frac{1}{2}$-integral.

If M is $U_{2,4}$, F_7, F_7^*, or $K_{3,3}$, we define $w = \mathbf{1}$. Let $m = |E(M)|$ and let g be the girth of M. Then M has exactly m circuits of length g. Moreover, each element of M belongs to exactly g of these circuits. Let $x(e) = 1/g$, for all elements e of M. Then $Ax \geq \mathbf{1}$. On the other hand, let $y(C) = 1/g$ if C is a shortest circuit, and $y(C) = 0$ otherwise. Then $y^T A = w^T$. Notice that $y^T \mathbf{1} = w^T x = m/g$, which equals 4/3, 7/3, 7/4, and 9/4, respectively. Therefore, the optimal value of (1.2) is not $\frac{1}{2}$-integral in all these cases.

If $M = M(K)$, we define $w = \mathbf{1}$. The following is an optimal solution of (1.2), which has value $\frac{15}{4}$. Let v be the unique vertex of K of degree three. Let $x(e) = 1/2$ if e is not incident with v and $x(e) = 1/4$ otherwise. Let $y(C) = 3/4$ if C is a 2-cycle; $y(C) = 1/4$ if C is a triangle using v; and $y(C) = 0$ otherwise.

Finally, suppose $M = M(K_5^-)$. Let u, v be the two degree-three vertices. Let $w(e) = 2$, if e is incident with u, and $w(e) = 1$, otherwise. The following is an optimal solution of (1.2), which has value $\frac{15}{4}$. Let $x(e) = 1/4$, if e is incident with u or v, and $x(e) = 1/2$ otherwise. Let $y(C) = 3/4$, if C is one of the three triangles that use u, and $y(C) = 1/4$ if either C is one of the three triangles that use v or C is one of the three 4-circuits that only use edges that are incident with u or v. ■

To complete our proof of Theorem 1.2, it remains to prove the implication (i) ⇒ (ii). By Lemma 1.1, it is clear that we only need to show the following.

Lemma 2.3 *If M satisfy (i), then $M = M^*(G)$ for some graph G that contains neither P nor K^* as a minor.*

Proof. Since none of $U_{2,4}$, F_7, F_7^*, $M(K_{3,3})$, and $M(K_5)$ is a minor of M, by two theorems of Tutte, Theorem 13.1.1 and Theorem 13.3.2 of [9], $M = M^*(G)$ for some graph G. Since neither $M(K_5^-)$ nor $M(K)$ is a minor of $M^*(G)$, it follows that neither $M^*(K_5^-)$ nor $M^*(K)$ is a minor of $M(G)$. Equivalently, neither P nor K^* is a minor of G, which proves the lemma. ■

3 Decomposition

The goal of this section is to show (Theorem 3.1 and Theorem 3.2) that graphs with no P and K^* minors can be constructed from some prime graphs by summing operations. By Lemma 2.3, this constitutes a structural characterization of good matroids.

Let G_1 and G_2 be two graphs. As usual, the 0-sum of G_1 and G_2 is their disjoint union. The 1-sum of G_1 and G_2 is obtained from their disjoint union by identifying a vertex in G_1 with a vertex in G_2. The 2-*sum* of G_1 and G_2 is obtained by first choosing a path $a_ic_ib_i$ ($i = 1, 2$) of length two in G_i such that c_i has degree two in G_i, then deleting c_i from G_i, and finally identifying a_1 with a_2 and identifying b_1 with b_2.

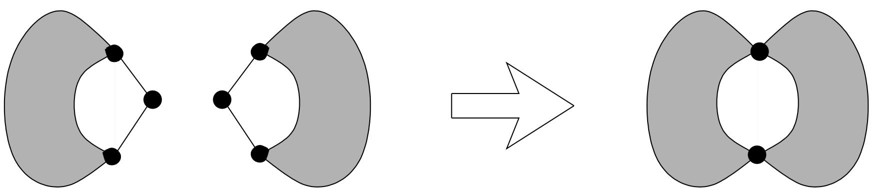

Figure 3.1: The 2-sum of two graphs.

A graph is *cyclically* 3-*connected* if it is obtained from a 3-connected simple graph by subdividing each edge at most once. The following result is well known; yet its proof is easy and hence omitted.

Lemma 3.1 *Let H be a cyclically 3-connected graph and let G be a k-sum ($k = 0, 1, 2$) of two graphs G_1 and G_2. Then H is a minor of G if and only if H is a minor of some G_i ($i = 1, 2$).* ■

A 2-*separation* of a 2-connected graph $G = (V, E)$ is a pair (G_1, G_2) of subgraphs of G, where $G_i = (V_i, E_i)$ ($i = 1, 2$), such that (E_1, E_2) is a partition of E, $|V_1 \cap V_2| = 2$, $V_1 \cup V_2 = V$, and $V_1 - V_2 \neq \emptyset \neq V_2 - V_1$. The 2-separation is *trivial* if $\min\{|E_1|, |E_2|\} = 2$.

For any $X \subseteq V$, let $G - X$ be the graph obtained from G by deleting all vertices in X and all edges that are incident with at least one vertex in X. As usual, $G - \{x\}$ will be simplified as $G - x$. In addition, let $G[X] = G - (V - X)$.

Let $\mathcal{G}_0 = \{K_1, K_2, K_3, K_4^-, C_4, C_5, K_{2,3}\}$.

Lemma 3.2 *Every simple graph can be constructed by repeatedly taking* 0-, 1-, *and* 2-*sums starting from cyclically* 3-*connected graphs and graphs in* $\mathcal{G}_0$.

Proof. Clearly, disconnected simple graphs can be constructed from connected simple graphs by 0-sums; connected simple graphs (except for K_1) can be constructed from K_2 and 2-connected simple graphs by 1-sums; 2-connected simple graphs can be constructed from those 2-connected simple graphs that have no nontrivial 2-separations by 2-sums. Therefore, to prove the lemma, we only need to prove the following.

(*) *If G is a* 2-*connected simple graph with no nontrivial* 2-*separations, then either* $G \in \mathcal{G}_0$ *or* G *is cyclically* 3-*connected.*

Let us assume that $G \notin \mathcal{G}_0$. We prove that G is cyclically 3-connected. Suppose $x \in V(G)$ has degree two. Let $y, z \in V(G)$ be the two neighbors of x. We first make a few observations.

(1) $yz \notin E(G)$.

If $e = yz \in E(G)$, let $G_1 = G[\{x, y, z\}]$ and let $G_2 = (G - x)\backslash e$. If $V(G_2) - V(G_1) = \emptyset$, then $G = K_3 \in \mathcal{G}_0$, a contradiction. If $V(G_2) - V(G_1) \neq \emptyset$, then (G_1, G_2) is a 2-separation of G, and thus it is trivial. It follows that $G = K_4^- \in \mathcal{G}_0$, a contradiction again.

(2) G has no other vertex with neighborhood $\{y, z\}$.

If $x' \neq x$ has neighborhood $\{y, z\}$, let $G_1 = G[\{x, x', y, z\}]$ and let $G_2 = G - \{x, x'\}$. If $V(G_2) - V(G_1) = \emptyset$, then $G = C_4 \in \mathcal{G}_0$, a contradiction. If $V(G_2) - V(G_1) \neq \emptyset$, then (G_1, G_2) is a 2-separation of G, and thus it is trivial. It follows that $G = K_{2,3} \in \mathcal{G}_0$, a contradiction again.

(3) Both y and z have degree at least three.

If, say, y has degree two, let the neighborhood of y be $\{x, z'\}$. By (1), $z \neq z'$. Let G_1 be the path with edges zx, xy, yz' and let $G_2 = G - \{x, y\}$. If $V(G_2) - V(G_1) = \emptyset$, then, as G is 2-connected, $G = C_4 \in \mathcal{G}_0$, a contradiction. If $V(G_2) - V(G_1) \neq \emptyset$, then (G_1, G_2) is a 2-separation of G, and thus it is trivial. It follows that $G = C_5 \in \mathcal{G}_0$, a contradiction again.

With the above three observations, we prove that G is cyclically 3-connected. Let $\mathcal{Q}$ be the set of paths Q of G such that $|V(Q)| = 3$ and the middle vertex of Q has degree two in G. From (3) we know that paths in $\mathcal{Q}$ are edge disjoint. Let $\widetilde{G}$ be obtained from G by replacing each path in $\mathcal{Q}$ by an edge with the same ends. Clearly, G can be obtained from $\widetilde{G}$ by subdividing each edge at most once, which means it is enough for us to show that $\widetilde{G}$ is simple and 3-connected. Since G is simple, by (1) and (2), $\widetilde{G}$ is simple. By (3), each vertex of $\widetilde{G}$ has degree at least three, which implies that $\widetilde{G}$ has at least four vertices and has no trivial 2-separations. Notice that each 2-separation of $\widetilde{G}$ can be extended into a 2-separation of G. Therefore, as every 2-separation of G is trivial, we conclude that $\widetilde{G}$ has no 2-separations and thus $\widetilde{G}$ is 3-connected. ∎

Let $\mathcal{G}_1$ be the class of cyclically 3-connected graphs with no minors P and K^*.

Theorem 3.1 *A simple graph has no minors P and K^* if and only if the graph can be obtained by repeatedly taking* 0-, 1-, *and* 2-*sums starting from graphs in* $\mathcal{G}_0 \cup \mathcal{G}_1$.

Proof. Since both P and K^* are cyclically 3-connected, the result follows immediately from the last two lemmas. ∎

For each integer $n \geq 3$, let W_n be the wheel with n spokes. The following is a well known result, see (10.4) in [7].

Lemma 3.3 *If a 3-connected simple graph G does not have minor P, then either $|V(G)| \leq 5$, or $G = W_n$ $(n \geq 5)$, or some three vertices meet all edges of G.* ∎

Lemma 3.4 *If xu, xv, xw are three distinct edges in a 3-connected simple graph G, then G has a subgraph H such that H is a subdivision of K_4 and H contains all these three edges.*

Proof. Since $G-x$ is 2-connected, it has a cycle C that contains both u and v. If C also contains w, then adding the three special edges to C results in a graph H that satisfies the requirement. If C does not contain w, then $G-x$ has two paths from w to C such that w is the only common vertex of these two paths. There are two subcases in this case. If at least one of these two paths, say Q, is ended on C at a vertex other than u and v, then the three special edges and Q and C form a graph H that satisfies the requirement. If the ends of these two paths on C are precisely u and v, then $G-x$ has a cycle that contains all u, v, and w, which implies, by our first case, that G has a required subgraph. ∎

Theorem 3.2 *Every graph in $\mathcal{G}_1$ is a minor of one of the graphs depicted in Figure* 3.2.

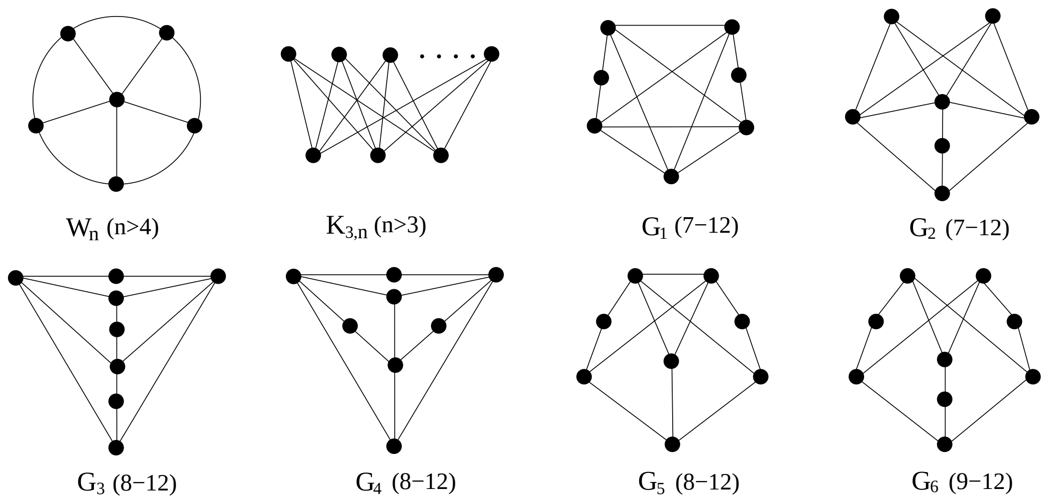

Figure 3.2: Maximal graphs in $\mathcal{G}_1$.

Proof. Let $G \in \mathcal{G}_1$ be the subdivision of a 3-connected simple graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$. If $|\widetilde{V}| \leq 5$, then $\widetilde{G}$ can only be K_4, W_4, K_5^-, or K_5. If $|\widetilde{V}| \geq 6$, by Lemma 3.3, $\widetilde{G} = W_n$ $(n \geq 5)$ or $K_{3,n}^+$ $(n \geq 3)$, which is obtained from $K_{3,n}$ by adding k edges $(0 \leq k \leq 3)$ whose ends belong to a same color class of size three. Let $F \subseteq \widetilde{E}$ be the set of edges that are subdivided to get G. Since K^* is not a minor of G, by Lemma 3.4, we may assume that no three distinct edges in F share a common vertex.

Suppose $\widetilde{G} = K_4$. Then edges in F are either all contained in a 3-cycle or all contained in a 4-cycle of $\widetilde{G}$. In the first case, G is a minor of W_6. In the second case, G is a minor of G_5.

Suppose $\widetilde{G} = W_4$. Notice that, since K^* is not a minor of G, no spoke edge in F is incident with a rim edge in F. If all edges in F are rim edges, then F is a minor of W_8. If F has two spoke edges that are contained in a triangle in $\widetilde{G}$, then G is a minor of G_5. Finally, if F has either one spoke edge or two such edges that are not contained in a triangle, then G is a minor of G_3.

Suppose $\widetilde{G} = K_5^-$. Let us call the three edges between the three degree-four vertices rim edges and the others spoke edges. If edges in F is a matching, then G is a minor of G_1. Hence we may assume that F contains two distinct incident edges, say xy and xz. Notice that, since K^* is not a minor of G, x must have degree four and the two edges are either both spoke edges or both rim edges. Therefore, G is a minor of G_3 or G_4, respectively.

Suppose $\widetilde{G} = K_5$. Since K^* is not a minor of G, edges in F must be a matching, which implies that G is a minor of G_1.

Suppose $\widetilde{G} = W_n$ $(n \geq 5)$. Since K^* is not a minor of G, no spoke edge is in F, which implies that G is a minor of W_{2n}.

Finally, suppose $\widetilde{G} = K_{3,n}^+$ $(n \geq 3)$. Let us call edges of $K_{3,n}$ spoke edges and the others rim edges. Since K^* is not a minor of G, spoke edges in F cannot be incident with any other edge in F, and every rim edge of $\widetilde{G}$ must be incident with every spoke edge in F. If F has no spoke edges, then G is a minor of $K_{3,n+3}$. Thus we may assume that F has at least one spoke edge. It follows that $n = 3$ and F contains no rim edges. Therefore, $|F| = 1, 2$, or 3, and G is a minor of G_2, G_5, or G_6, respectively. ■

4 The Validity of Summing Operations

The purpose of this section is to show that being good is preserved under summing operations.

Let G be a graph and let $Z \subseteq V(G)$. We denote by $E_G(Z)$ (or simply $E(Z)$ when the dependency on G is clear) the set of edges of G that have one end in Z and one end in $V(G) - Z$. Let H be a connected component of G and let (X, Y) be a partition of $V(H)$ such that $X \neq \emptyset \neq Y$. If both $H[X]$ and $H[Y]$ are connected, then the set $E(X) = E(Y)$ is called a *cut* of G. It is well known [9] that cuts of G are precisely circuits of $M^*(G)$.

When $M = M^*(G)$, matrix A in (1.2) is the cut-edge incidence matrix of G. In this situation, the maximization problem in (1.2) will be denoted by $P(G, w)$. It follows from the theorem of Edmonds and Giles [3] that G is good if and only if $P(G, w)$ has a $\frac{1}{2}$-integral optimal solution for all nonnegative integral functions w defined on $E(G)$.

Theorem 4.1 *For $k = 0, 1$, the k-sum of any two good graphs is good.*

Proof. If G is the 0- or 1-sum of two graphs G_1 and G_2, it is not difficult to see that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where A_i $(i = 1, 2)$ is the cut-edge incidence matrix of G_i. Therefore, the result holds obviously. ■

Theorem 4.2 *The 2-sum of any two good graphs is good.*

The remainder of this section consists of a proof of Theorem 4.2. Let G be a graph. We denote by $\mathcal{C}_G$ the set of all cuts of G. For any $\mathcal{C} \subseteq \mathcal{C}_G$ and $e \in E(G)$, let $\mathcal{C}(e) = \{C \in \mathcal{C} : C \ni e\}$. As usual, if z is a function defined on a finite set S and $S_0 \subseteq S$, we denote $z(S_0) = \sum_{s \in S_0} z(s)$.

Lemma 4.1 *Suppose y is an optimal solution of $P(G, w)$. If $C = \{e, f\}$ is a cut of G such that $y(C) < \min\{w(e), w(f)\}$, then $y(\mathcal{C}_G(e)) = w(e)$ and $y(\mathcal{C}_G(f)) = w(f)$.*

Proof. Suppose the lemma is false. By symmetry, we may assume $y(\mathcal{C}_G(e)) \neq w(e)$, which implies $y(\mathcal{C}_G(e)) < w(e)$. If $y(\mathcal{C}_G(f)) \neq w(f)$, then $y(\mathcal{C}_G(f)) < w(f)$, and thus increasing the value of $y(C)$ by a sufficiently small $\epsilon > 0$ would result in a new feasible solution y' of $P(G, w)$, for which $(y')^T\mathbf{1} > y^T\mathbf{1}$. This contradicts the optimality of y, so $y(\mathcal{C}_G(f)) = w(f)$. Since $w(f) > y(C)$, $\mathcal{C}_G(f)$ has a cut $D \neq C$ with $y(D) > 0$. Notice that $D' = (D - \{f\}) \cup \{e\}$ is a cut of G. Then decreasing the value of $y(D)$ by a sufficiently small $\epsilon > 0$, while increasing the values of $y(C)$ and $y(D')$ both by the same ϵ would result in a new feasible solution y' of $P(G, w)$, for which $(y')^T\mathbf{1} > y^T\mathbf{1}$. Again, this contradicts the optimality of y, which proves the lemma. ■

In the rest of this section, let G be a 2-sum of G_1 and G_2. Let a_i, b_i, c_i $(i = 1, 2)$ be defined as in the definition of 2-sum. In addition, let $e_i = a_ic_i$ and $f_i = b_ic_i$ $(i = 1, 2)$. Let w be a nonnegative integral function define on $E(G)$. We aim to show that $P(G, w)$ has a $\frac{1}{2}$-integral optimal solution.

Let $\mathcal{C}_0$ be the set of cuts of G that separate a_1 $(= a_2)$ from b_1 $(= b_2)$. For $i = 1, 2$, let $\mathcal{C}_i$ be the set of cuts of G that are contained in $E(G_i - c_i)$. Clearly, $(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$ is a partition of $\mathcal{C}_G$. For $i = 1, 2$, let $D_i = \{e_i, f_i\}$, $\mathcal{C}_0^i = \{C \cap E(G_i) : C \in \mathcal{C}_0\}$, and $\mathcal{C}_i^0 = \{\{x\} \cup X : x \in \{e_i, f_i\}, X \in \mathcal{C}_0^i\}$. Then it is not difficult to see that $(\mathcal{C}_i, \{D_i\}, \mathcal{C}_i^0)$ is a partition of $\mathcal{C}_{G_i}$.

In our following proof, we will extend the domain of w to the entire $E(G_1) \cup E(G_2)$ by defining $w(e_1) = w(e_2) = \alpha$ and $w(f_1) = w(f_2) = \beta$, for various values of α and β. To simplify our notation, we will write $P(G_i, w)$, instead of $P(G_i, w|_{E(G_i)})$, where $w|_{E(G_i)}$ is the restriction of w to $E(G_i)$.

Lemma 4.2 *Suppose $\alpha \geq \beta$. If y is a feasible solution of $P(G, w)$, then there exist feasible solutions y_1 and y_2 of $P(G_1, w)$ and $P(G_2, w)$, respectively, such that, for $i = 1, 2$, $y_i(\mathcal{C}_i^0) \leq y(\mathcal{C}_0)$, and*

$$y_i^T\mathbf{1} = y(\mathcal{C}_i) + \begin{cases} y(\mathcal{C}_0) + \beta & \text{if} \quad y(\mathcal{C}_0) \leq \alpha - \beta \\ (y(\mathcal{C}_0) + \alpha + \beta)/2 & \text{if} \quad \alpha - \beta \leq y(\mathcal{C}_0) \leq \alpha + \beta \\ \alpha + \beta & \text{if} \quad \alpha + \beta \leq y(\mathcal{C}_0). \end{cases}$$

The two vectors y_1 and y_2 will be called restrictions *of y (with respect to α and β).*

Proof. For any $i \in \{1, 2\}$ and $X \in \mathcal{C}_0^i$, let $y(X)$ be the sum of $y(C)$, over all $C \in \mathcal{C}_0$ with $C \cap E(G_i) = X$. Suppose λ and μ are nonnegative numbers with $\lambda + \mu \leq 1$, $\lambda y(\mathcal{C}_0) \leq \alpha$, and $\mu y(\mathcal{C}_0) \leq \beta$. For $i = 1, 2$, we define y_i as follows: $y_i(C) = y(C)$, $\forall C \in \mathcal{C}_i$; $y_i(\{e_i\} \cup X) = \lambda y(X)$ and $y_i(\{f_i\} \cup X) = \mu y(X)$, $\forall X \in \mathcal{C}_0^i$; and $y_i(D_i) = \min\{\alpha - \lambda y(\mathcal{C}_0), \beta - \mu y(\mathcal{C}_0)\}$. It is routine to verify that $y_i(\mathcal{C}_i^0) = (\lambda + \mu)y(\mathcal{C}_0) \leq y(\mathcal{C}_0)$, $y_i^T\mathbf{1} = y(\mathcal{C}_i) + (\lambda + \mu)y(\mathcal{C}_0) + y_i(\mathcal{D}_i)$, and y_i is a feasible solution of $P(G_i, w)$. Next, we specify (λ, μ) in different cases so that $y_i^T\mathbf{1}$ equals the required value. If $y(\mathcal{C}_0) \leq \alpha - \beta$, let $(\lambda, \mu) = (1, 0)$. Then λ and μ are nonnegative numbers with $\lambda + \mu \leq 1$, $\lambda y(\mathcal{C}_0) \leq \alpha$, and $\mu y(\mathcal{C}_0) \leq \beta$. Moreover, $y_i^T\mathbf{1} = y(\mathcal{C}_i) + y(\mathcal{C}_0) + \min\{\alpha - y(\mathcal{C}_0), \beta\} = y(\mathcal{C}_i) + y(\mathcal{C}_0) + \beta$, as required. In the other two case, if $y(\mathcal{C}_0) = 0$, we take $(\lambda, \mu) = (0, 0)$; if $y(\mathcal{C}_0) > 0$, we take

$$(\lambda, \mu) = (\frac{y(\mathcal{C}_0) + \alpha - \beta}{2y(\mathcal{C}_0)}, \frac{y(\mathcal{C}_0) + \beta - \alpha}{2y(\mathcal{C}_0)}) \quad \text{and} \quad (\frac{\alpha}{y(\mathcal{C}_0)}, \frac{\beta}{y(\mathcal{C}_0)}),$$

respectively. It is straightforward to verify that these choices satisfy the requirements. ■

Lemma 4.3 *For $i = 1, 2$, let y_i be a $\frac{1}{2}$-integral feasible solution of $P(G_i, w)$. Then there exists a $\frac{1}{2}$-integral feasible solution y of $P(G, w)$ such that*

$$y^T\mathbf{1} = y_1^T\mathbf{1} + y_2^T\mathbf{1} - y_1(D_1) - y_2(D_2) - \max\{y_1(\mathcal{C}_1^0), y_2(\mathcal{C}_2^0)\}.$$

Vector y is called a concatenation *of y_1 and y_2.*

Proof. Suppose $i \in \{1, 2\}$. Let $\mathcal{Y}_i$ be the multiset with multiplicity function $2y_i$. That is, $\mathcal{Y}_i$ consists of cuts of G_i such that each cut C of G_i appears in $\mathcal{Y}_i$ exactly $2y_i(C)$ times. Then $|\mathcal{Y}_i| = 2y_i^T\mathbf{1}$. Let $\mathcal{Y}_i'$ and $\mathcal{Y}_i''$ consist of members of $\mathcal{Y}_i$ that belong to $\mathcal{C}_i$ and $\mathcal{C}_i^0$, respectively, and let $\mathcal{Y}_i''' = \mathcal{Y}_i - \mathcal{Y}_i' - \mathcal{Y}_i''$. In addition, let $\mathcal{Y}_i'' = \{C_{i,1}, C_{i,2}, ..., C_{i,k_i}\}$. Clearly, $k_i = 2y_i(\mathcal{C}_i^0)$ and $|\mathcal{Y}_i'''| = 2y_i(D_i)$. Let $k = \min\{k_1, k_2\}$. We define $\mathcal{Y}_0 = \{E(G) \cap (C_{1,j} \cup C_{2,j}) : j = 1, 2, ..., k\}$. It follows that all members of $\mathcal{Y}_0$ are cuts of G. Let $\mathcal{Y} = \mathcal{Y}_1' \cup \mathcal{Y}_2' \cup \mathcal{Y}_0$ and let y' be the multiplicity function of $\mathcal{Y}$. Then it is easy to see that $y = y'/2$ is a $\frac{1}{2}$-integral feasible solution of $P(G, w)$ with $2y^T\mathbf{1} = |\mathcal{Y}| = (2y_1^T\mathbf{1} - k_1 - 2y_1(D_1)) + (2y_2^T\mathbf{1} - k_2 - 2y_1(D_2)) + k = 2y_1^T\mathbf{1} + 2y_2^T\mathbf{1} - 2y_1(D_1) - 2y_1(D_2) - \max\{k_1, k_2\}$, which proves the lemma. ■

In the rest of this section, for any feasible solution y of $P(G, w)$, we denote $q_i = y(\mathcal{C}_i)$ $(i = 0, 1, 2)$. We also denote $p = \lfloor q_0 \rfloor$ and $s = q_0 - p$. For any real number r, we use $[r]$ to denote the smallest $\frac{1}{2}$-integral that is greater than or equal to r.

Lemma 4.4 *Suppose no optimal solution of $P(G, w)$ is $\frac{1}{2}$-integral. Then for any optimal solution y of $P(G, w)$, $[q_1] + [q_2] < q_1 + q_2 + s$.*

Proof. Suppose $[q_1] + [q_2] \geq q_1 + q_2 + s$. Set $\alpha = p$ and $\beta = 0$. Let y_1 and y_2 be the restrictions of y. By Lemma 4.2, $y_i^T\mathbf{1} = q_i + p$, for $i = 1, 2$. Let z_i be a $\frac{1}{2}$-integral optimal solution of $P(G_i, w)$, for $i = 1, 2$. Then $z_i^T\mathbf{1} \geq [p + q_i] = p + [q_i]$. Let z be the concatenation of z_1 and z_2. By Lemma 4.3, $z^T\mathbf{1} \geq z_1^T\mathbf{1} + z_2^T\mathbf{1} - p \geq p + [q_1] + [q_2] \geq q_0 + q_1 + q_2 = y^T\mathbf{1}$, which implies that z is a $\frac{1}{2}$-integral optimal solution of $P(G, w)$, a contradiction. ■

The next lemma is a list of facts that obviously follow from Lemma 4.4.

Lemma 4.5 *Suppose no optimal solution of $P(G, w)$ is $\frac{1}{2}$-integral. Then for any optimal solution y of $P(G, w)$: (i) $s > 0$; (ii) $[q_i] < [q_i + s]$, for $i = 1$ and 2; (iii) $[q_i] < [q_i + \frac{s}{2}]$, for $i = 1$ or 2.*

Lemma 4.6 *G is good if $G_1 = K_4^-$.*

Proof. Let d be the vertex of G_1 other than a_1, b_1, c_1; let $e_1' = a_1d$, $f_1' = b_1d$, and $g = a_1b_1$ be the three edges of G_1 other than e_1, f_1. Clearly, the mapping $d \to c_2$, $e_1' \to e_2$, $f_1' \to f_2$ can be extended into an isomorphism π from $G \backslash g$ to G_2. Moreover, the natural correspondence $\sigma : C \to \pi(C - \{g\})$ is a one-to-one mapping from $\mathcal{C}_G$ to $\mathcal{C}_{G_2}$. For any function z_2 defined on $\mathcal{C}_{G_2}$, let $\varphi : z_2 \to z$, where z is defined on $\mathcal{C}_G$ with $z(C) = z_2(\sigma(C))$, for all $C \in \mathcal{C}_{G_2}$. It is clear that φ is a one-to-one mapping from the set of functions defined on $\mathcal{C}_{G_2}$ to the set of functions defined on $\mathcal{C}_G$. In addition, the equality $z^T\mathbf{1} = z_2^T\mathbf{1}$ always holds.

Suppose G is not good. Then there exists a vector w such that no optimal solution of $P(G,w)$ is $\frac{1}{2}$-integral. We choose such a w with $w(E(G))$ as small as possible.

(1) $w(e_1') \neq 0 \neq w(f_1')$.

Suppose (1) is false. By symmetry, we may assume $w(e_1') = 0$. Let us define w_2 on $E(G_2)$ with $w_2(e_2) = 0$, $w_2(f_2) = \min\{w(f_1'), w(g)\}$, and $w_2(e) = w(e)$, for all other edges e of G_2. For any feasible solution y of $P(G,w)$, let $y_2 = \varphi^{-1}(y)$. Then $y_2(\mathcal{C}_2^0) = y(\mathcal{C}_0)$, which, as $w(e_1') = 0$, is at most $\min\{w(f_1'), w(g)\}$. Therefore, it is easy to see that y_2 is a feasible solution of $P(G_2, w_2)$. Since G_2 is good, $P(G_2, w_2)$ has a $\frac{1}{2}$-integral optimal solution z_2. Let $z = \varphi(z_2)$. Since $w_2(e_2) = 0$, $z_2(\mathcal{C}_2^0) \leq w_2(f_2)$. It follows that $z(\mathcal{C}_0) \leq \min\{w(f_1'), w(g)\}$, and thus z is a $\frac{1}{2}$-integral feasible solution of $P(G,w)$. Consequently, for all feasible solutions y of $P(G,w)$, $z^T\mathbf{1} = z_2^T\mathbf{1} \geq y_2^T\mathbf{1} = y^T\mathbf{1}$, which implies that z is a $\frac{1}{2}$-integral optimal solution of $P(G,w)$. This contradiction proves (1).

In the rest of this proof, let y be an optimal solution of $P(G,w)$. Let $D = \{e_1', f_1'\}$. Then D is the only cut in $\mathcal{C}_1$. Therefore, $q_1 = y(D)$.

(2) $q_1 < 1$.

Suppose $q_1 \geq 1$. Then $\min\{w(e_1'), w(f_1')\} \geq q_1 \geq 1$. Let w' be obtained from w by decreasing the values of $w(e_1')$ and $w(f_1')$ by 1. Let y' be obtained from y by decreasing the values of $y(D)$ by 1. Then y' is a feasible solution of $P(G,w')$. By the minimality of w, $P(G,w')$ has a $\frac{1}{2}$-integral optimal solution z'. Let z be obtained from z' by increasing the value of $z'(D)$ by 1. Then z is a $\frac{1}{2}$-integral feasible solution of $P(G,w)$. Moreover $z^T\mathbf{1} = 1 + (z')^T\mathbf{1} \geq 1 + (y')^T\mathbf{1} = y^T\mathbf{1}$, which means that z is a $\frac{1}{2}$-integral optimal solution of $P(G,w)$. This contradiction proves (2).

(3) $q_0 = w(e_1') + w(f_1') - 2q_1$.

It follows from (1) and (2) that $y(D) < 1 \leq \min\{w(e_1'), w(f_1')\}$. Then we deduce from Lemma 4.1 that $y(\mathcal{C}_0(e_1')) = w(e_1') - q_1$ and $y(\mathcal{C}_0(f_1')) = w(f_1') - q_1$, which implies $q_0 = y(\mathcal{C}_0(e_1')) + y(\mathcal{C}_0(f_1')) = w(e_1') + w(f_1') - 2q_1$. Thus (3) is proved.

(4) $q_1 \neq 1/2$.

By Lemma 4.5 (i), q_0 is not integral. Thus (4) follows from (3) immediately.

(5) $w(g) = w(e_1') + w(f_1') - 1$.

Since y is feasible in $P(G,w)$, $w(g) \geq \lceil y(\mathcal{C}_0) \rceil$, which, by (3), means $w(g) \geq w(e_1') + w(f_1') - \lfloor 2q_1 \rfloor$, and thus, by (2), $w(g) \geq w(e_1') + w(f_1') - 1$. If (5) is false, then $w(g) \geq w(e_1') + w(f_1')$. Let $y_2 = \varphi^{-1}(y)$. It follows that y_2 is a feasible solution of $P(G_2, w_2)$, where $w_2(e) = w(\pi^{-1}(e))$, for all edges e of G_2. Since G_2 is good, $P(G_2, w_2)$ has a $\frac{1}{2}$-integral optimal solution z_2. Let $z = \varphi(z_2)$. Since $w(g) \geq w(e_1') + w(f_1')$, z is feasible in $P(G,w)$. On the other hand, $z^T\mathbf{1} = z_2^T\mathbf{1} \geq y_2^T\mathbf{1} = y^T\mathbf{1}$, so z is a $\frac{1}{2}$-integral optimal solution of $P(G,w)$, a contradiction, which proves (5).

Again, let w_2, y_2, z_2, and z be defined as in the last paragraph. Then the same argument shows that $z^T\mathbf{1} \geq y^T\mathbf{1}$. It follows that z is not a feasible solution of $P(G,w)$, which implies $z(\mathcal{C}_0) > w(g)$. Consequently, $w_2(e_2) + w_2(f_2) - 2z_2(D) \geq z_2(\mathcal{C}_2^0) = z(\mathcal{C}_0) > w(g) = w(e_1') + w(f_1') - 1$, and so $z_2(D) < 1/2$. On the other hand, since y is a feasible solution of $P(G,w)$, we deduce from (3) that $w(g) \geq y(\mathcal{C}_0) = w(e_1') + w(f_1') - 2q_1$, which implies, by (4) and (5), that $q_1 > 1/2$.

Let λ and μ be positive numbers such that $\lambda + \mu = 1$ and $\lambda q_1 + \mu z_2(D) = 1/2$. Let $y' = \lambda y + \mu z_2$. Notice that $y'(\mathcal{C}_0) = \lambda y(\mathcal{C}_0) + \mu z_2(\mathcal{C}_0) \leq \lambda(w(e_1') + w(f_1') - 2q_1) + \mu(w_2(e_2) + w_2(f_2) - 2z_2(D)) = w(g)$, which implies that y' is a feasible solution of $P(G,w)$. From $z_2^T\mathbf{1} = z^T\mathbf{1} \geq y^T\mathbf{1}$ we also know that

y' is an optimal solution of $P(G,w)$. Since (4) holds for an arbitrary optimal solution y of $P(G,w)$, it should also hold for y'. However, $y'(D) = \lambda y(D) + \mu z_2(D) = 1/2$, a contradiction, which proves the lemma. ■

Proof of Theorem 4.2. Suppose the theorem is false. Then there exists w such that no optimal solution of $P(G,w)$ is $\frac{1}{2}$-integral. Let y be an optimal solution of $P(G,w)$. We continue to use the terminology we defined above. We proceed by proving some claims.

(1) $[q_i] = [q_i + s] - \frac{1}{2}$, for $i = 1$ and 2.

Let $i \in \{1,2\}$. By Lemma 4.5 (ii), $[q_i] \le [q_i + s] - \frac{1}{2}$. On the other hand, from $s \le 1$ we deduce that $[q_i + s] \le [q_i] + 1$. Suppose (1) is false. Then we must have $[q_i + s] = [q_i] + 1$ for some $i \in \{1,2\}$, say, for $i = 1$. Set $\alpha = p+1$ and $\beta = 0$. Let y_1, y_2 be the restrictions of y. Let z_i ($i = 1,2$) be a $\frac{1}{2}$-integral optimal solution of $P(G_i, w)$, and let z be the concatenation of z_1, z_2. By Lemma 4.2 and Lemma 4.3, $z^T\mathbf{1} \ge z_1^T\mathbf{1} + z_2^T\mathbf{1} - p - 1 \ge [y_1^T\mathbf{1}] + [y_2^T\mathbf{1}] - p - 1 \ge [q_1 + q_0] + [q_2 + q_0] - p - 1 = [q_1 + s] + [q_2 + q_0] - 1 = [q_1] + [q_2 + q_0] \ge q_1 + q_2 + q_0 = y^T\mathbf{1}$, which means that z is a $\frac{1}{2}$-integral optimal solution of $P(G,w)$. This contradiction proves (1).

(2) Suppose $\alpha = p$ and $\beta = 0$. If z_i ($i \in \{1,2\}$) is a $\frac{1}{2}$-integral feasible solution of $P(G_i, w)$, then $z_i^T\mathbf{1} < p + [q_i + s]$.

Suppose $z_i^T\mathbf{1} \ge p + [q_i + s]$ for some $i \in \{1,2\}$, say, for $i = 1$. Let z be the concatenation of z_1 and z_2, and let y_1 and y_2 be the restrictions of y. Then we deduce from Lemma 4.3 and Lemma 4.2 that, $z^T\mathbf{1} \ge z_1^T\mathbf{1} + z_2^T\mathbf{1} - p \ge [q_1 + s] + [y_2^T\mathbf{1}] \ge [q_1 + s] + [q_2 + p] \ge q_1 + q_2 + q_0 = y^T\mathbf{1}$, which means that z is a $\frac{1}{2}$-integral optimal solution of $P(G,w)$. This is a contradiction and so (2) is proved.

By Lemma 4.5 (iii), we may assume $[q_2] < [q_2 + \frac{s}{2}]$. Therefore, $[q_2] + \frac{1}{2} \le [q_2 + \frac{s}{2}] \le [q_2 + s]$, which, by (1), means that

(3) $[q_2 + \frac{s}{2}] = [q_2 + s]$.

Let G_2' be obtained from G_2 by adding a new edge $g = a_2b_2$. Then there is a natural one-to-one correspondence between cuts of G_2' and cuts of G_2. As in the proof of Lemma 4.6, let $\varphi : z \to z'$ be the natural one-to-one mapping from the set of functions defined on $\mathcal{C}_{G_2}$ to the set of functions defined on $\mathcal{C}_{G_2'}$. Clearly, the equality $(z')^T\mathbf{1} = z^T\mathbf{1}$ always holds.

Set $\alpha = p+1$ and $\beta = 1$. We also extend the domain of w to $E(G_1) \cup E(G_2')$ by setting $w(g) = p+1$. Notice that G_2' is the 2-sum of K_4^- and G_2. By Lemma 4.6, G_2' is good. Let z_2' be a $\frac{1}{2}$-integral optimal solution of $P(G_2', w)$.

(4) $(z_2')^T\mathbf{1} \ge [q_2 + s] + p + 1$.

Let y_2 be the restriction of y. By Lemma 4.2, y_2 is feasible in $P(G_2, w)$ with $y_2(\mathcal{C}_2^0) \le q_0 \le w(g)$. Let $y_2' = \varphi(y_2)$. Then $y_2'(\mathcal{C}_{G_2'}(g)) = y_2(\mathcal{C}_2^0) \le w(g)$, which implies that y_2' is feasible in $P(G_2', w)$. Therefore, $(z_2')^T\mathbf{1} \ge [(y_2')^T\mathbf{1}] = [y_2^T\mathbf{1}] \ge [q_2 + p + 1 + \frac{s}{2}] = [q_2 + s] + p + 1$, where the second inequality follows from Lemma 4.2 and the last equality follows from (3), so (4) is proved.

(5) Let $D = \{e_2, f_2\}$. Then $z_2'(D) = 1/2$.

Since $z_2'(D)$ is $\frac{1}{2}$-integral with $0 \le z_2'(D) \le w(f_2) = \beta = 1$, we must have $z_2'(D) \in \{0, \frac{1}{2}, 1\}$. If $z_2'(D) = 1$, let $z_2 = \varphi^{-1}(z_2'')$, where z_2'' is obtained from z_2' by reducing the value of $z_2'(D)$ by 1. Then it is easy to see that z_2 is a feasible solution of $P(G_2, w_2)$, where $w_2(e_2) = p$, $w_2(f_2) = 0$, and $w_2(e) = w(e)$, for all other edges e of G_2. By (2), we should have $z_2\mathbf{1} < [q_2 + s] + p$. However,

$z_2^T\mathbf{1} = (z_2'')^T\mathbf{1} = (z_2')^T\mathbf{1} - 1$, which, by (4), is at least $[q_2 + s] + p$. This contradiction proves that $z_2'(D) \neq 1$. Therefore, by Lemma 4.1, $z_2'(\mathcal{C}_{G_2'}(e_2)) = p + 1$ and $z_2'(\mathcal{C}_{G_2'}(f_2)) = 1$. Consequently, $2z_2'(D) = (\alpha - z_2'(\mathcal{C}_{G_2'}(e_2)) + (\beta - z_2'(\mathcal{C}_{G_2'}(f_2)) = 1 + w(g) - z_2'(\mathcal{C}_{G_2'}(g)) \geq 1$, which proves (5).

Finally, set $\alpha = p + 1$ and $\beta = 0$. For each $X \in \mathcal{C}_0^2$, let $z_2'(X)$ be the sum of $z_2'(C)$, over all $C \in \mathcal{C}_2^0$ with $C \cap E(G_2) = X$. This time, we define z_2 on $\mathcal{C}_{G_2}$ such that $z_2(C) = 0$, for all $C \in \mathcal{C}_{G_2}(f_2)$; $z_2(C) = z_2'(C)$, for all $C \in \mathcal{C}_2$; and $z_2(C) = z_2'(C - \{e_2\})$, for all $C \in \mathcal{C}_2^0(e_2)$. It is straightforward to verify that z_2 is a $\frac{1}{2}$-integral feasible solution of $P(G_2, w)$. Moreover, by (4) and (1), $z_2^T\mathbf{1} \geq [q_2 + s] + p + \frac{1}{2} \geq q_2 + p + 1$. Let z_1 be a $\frac{1}{2}$-integral optimal solution of $P(G_1, w)$ and let y_1 be the restriction of y. By Lemma 4.2, $z_1^T\mathbf{1} \geq y_1^T\mathbf{1} \geq q_1 + q_0$. Let z be the concatenation of z_1 and z_2. Then $z^T\mathbf{1} \geq z_1^T\mathbf{1} + z_2^T\mathbf{1} - p - 1 \geq q_1 + q_0 + q_2 + p + 1 - p - 1 = q_1 + q_2 + q_0$, and so z is a $\frac{1}{2}$-integral optimal solution of $P(G, w)$. This contradiction completes the proof of Theorem 4.2. ■

5 Truncation

Usually it is very hard to prove directly that a graph is good. To accomplish Lemma 1.1, we introduce a packing property associated with cuts. Let $G = (V, E)$ be a connected graph. For each cut C of G, we denote by (X_C, Y_C) the unique partition of V such that $E(X_C) = E(Y_C) = C$. A cut C is called *big* if $\min\{|X_C|, |Y_C|\} > 1$ and *small* otherwise. Clearly, small cuts are precisely those that can be expressed as $E(\{v\})$, for some $v \in V$. To simplify our notation, $E(\{v\})$ and $E(\{u, v\})$ will be written as $E(v)$ and $E(uv)$, respectively. In the following, we use the word *collection* for multiset, where an element may appear more than once. In contrast, in a *set*, each element may appear at most once.

Let $G = (V, E)$ be a connected graph and let $\mathcal{C}$ be a collection of cuts of G. The multiplicity function of $\mathcal{C}$ will be denoted by $m_{\mathcal{C}}$. For each $e \in E$, set $\mathcal{C}_e = \{C \in \mathcal{C} : C \ni e\}$ and $d_{\mathcal{C}}(e) = |\mathcal{C}_e|$. This notation is slightly different from that in the last section. We make this change since the dependency on G is not emphasized anymore. We call $\mathcal{C}$ *truncatable* if G has a collection $\mathcal{D}$ of cuts, called a *certificate* for the truncatability of $\mathcal{C}$, such that

(1a) $|\mathcal{D}| \geq |\mathcal{C}|/2$, and
(1b) $d_{\mathcal{D}}(e) \leq 2\lceil d_{\mathcal{C}}(e)/4\rceil$, for all $e \in E$.

If, in addition, certificate $\mathcal{D}$ satisfies

(1c) each small cut that appears in $\mathcal{C}$ more than once also appears in $\mathcal{D}$,

then $\mathcal{C}$ is called *strongly truncatable*. We say that G is *truncatable* or *strongly truncatable* if every collection of its cuts is truncatable or strongly truncatable, respectively.

Lemma 5.1 *Every truncatable graph is good.*

Proof. Let $G = (V, E)$ be a truncatable graph. Let A be the cut-edge incidence matrix of G, and let $B = A/2$. Let P_w denote the optimization problem: $\max\{y^T\mathbf{1} \mid y^TB \leq w^T, y \geq \mathbf{0}$, and y is $\frac{1}{2}$-integral$\}$. We aim to show that $Bx \geq \mathbf{1}$, $x \geq \mathbf{0}$ is TDI. This, as proved by Schrijver and Seymour (see Theorem 22.13 of [10]), amounts to that P_w has an integral optimal solution, for all nonnegative integral vectors w.

Let y be an optimal solution of P_w. Then we can regard $2y$ as the multiplicity function of a collection $\mathcal{C}$ of cuts of G. Since G is truncatable, $\mathcal{C}$ has a certificate $\mathcal{D}$. Let z be the multiplicity function of $\mathcal{D}$. For each $e \in E$, let A_e and B_e be the columns of A and B, respectively, that are indexed by e. Then $z^T B_e = z^T A_e/2 = d_{\mathcal{D}}(e)/2 \le \lceil d_{\mathcal{C}}(e)/4 \rceil = \lceil (2y)^T A_e/4 \rceil = \lceil y^T B_e \rceil \le w(e)$, which implies that z is feasible in P_w. On the other hand, $z^T \mathbf{1} = |\mathcal{D}| \ge |\mathcal{C}|/2 = (2y)^T \mathbf{1}/2 = y^T \mathbf{1}$. Therefore, z is an integral optimal solution of P_w, so the lemma is proved. ■

It is not difficult to show that all good graphs are truncatable. We omit its proof since we will not use this claim in proving our theorems. But we do point out the natural consequence of this claim that being good and being truncatable are equivalent. We choose to use the language of truncatability because it simplifies the presentation of our proofs. On the other hand, as we will see later, that there are truncatable graphs, which are not strongly truncatable. We introduce this concept since it will help us to do induction in many cases.

In terms of linear programming, conditions (1a-c) can be strengthened as follows. Let $\hat{\mathcal{C}}$ be the *set* $\{C : C \in \mathcal{C}\}$. Let $A_{\mathcal{C}}$ be the cut-edge incidence matrix of $\hat{\mathcal{C}}$. That is, the $|\hat{\mathcal{C}}|$ rows of $A_{\mathcal{C}}$ are precisely the characteristic vectors of cuts in $\hat{\mathcal{C}}$. Let $w_{\mathcal{C}}$ be defined with $w_{\mathcal{C}}(e) = 2\lceil d_{\mathcal{C}}(e)/4 \rceil$, for all $e \in E$. Let $\ell_{\mathcal{C}} \in \{0,1\}^{\hat{\mathcal{C}}}$ such that $\ell_{\mathcal{C}}(C) = 1$ if and only if C is a small cut with $m_{\mathcal{C}}(C) > 1$.

Lemma 5.2 *Let $\mathcal{C}$ be a collection of cuts of a connected graph G. Then,*
(i) $\mathcal{C}$ is truncatable if $\max\{y^T\mathbf{1} : y^T A_{\mathcal{C}} \le w_{\mathcal{C}},\ y \ge \mathbf{0}\}$ *has an integral optimal solution;*
(ii) $\mathcal{C}$ is strongly truncatable if $\max\{y^T\mathbf{1} : y^T A_{\mathcal{C}} \le w_{\mathcal{C}},\ y \ge \ell_{\mathcal{C}}\}$ *has an integral optimal solution.*

Proof. For each nonnegative integral vector y defined on $\hat{\mathcal{C}}$, let $\mathcal{D}_y$ be the collection of cuts in $\hat{\mathcal{C}}$ such that each $C \in \hat{\mathcal{C}}$ appears in $\mathcal{D}_y$ exactly $y(C)$ times. Clearly, $|\mathcal{D}_y| = y^T\mathbf{1}$. Observe that if y is feasible, in either problem, then $\mathcal{D}_y$ satisfy (1b), and also (1c) in the second case. Moreover, in both problems, the vector $y = \frac{1}{2}\mathbf{1}$ is a feasible solution, which has objective value $|\mathcal{C}|/2$. Therefore, if y is an integral optimal solution, in either case, then $\mathcal{D}_y$ is a certificate. ■

Remark. In both conclusions in Lemma 5.2, having an integral optimal solution is a sufficient condition, but not a necessary condition. This is because, in general, members of a certificate $\mathcal{D}$ do not have to be in $\mathcal{C}$.

For any graph G, let $\overline{G}$ be the *simplification* of G; that is, $\overline{G}$ is the simple spanning subgraph of G such that two vertices are adjacent in $\overline{G}$ if and only if they are adjacent in G. For each cut C of G, it is clear that $\overline{C} = C \cap E(\overline{G})$ is a cut of $\overline{G}$. If $\mathcal{C}$ is a collection of cuts of G, let $\overline{\mathcal{C}} = \{\overline{C} : C \in \mathcal{C}\}$. Then the following lemma follows obviously from (1a-b).

Lemma 5.3 *Let $\mathcal{C}$ be a collection of cuts of a graph G. If $(\overline{G}, \overline{\mathcal{C}})$ is truncatable, then so is $(G, \mathcal{C})$.*

Let $\mathcal{C}$ be a collection of cuts of a connected graph G. Then an edge $e = xy \in E(G)$ is called *contractable* if either $\mathcal{C}_e = \emptyset$, or $\mathcal{C}_e = \{E(x), E(x), E(y), E(y)\}$ and $G - \{x, y\}$ is connected. Next, we prove that, if e is contractable, then the truncatability of $\mathcal{C}$ can be reduced to the truncatability of $\mathcal{C}/e$, which is a collection of cuts of G/e defined as follows. If $\mathcal{C}_e = \emptyset$, let $\mathcal{C}/e = \mathcal{C}$. If $\mathcal{C}_e \ne \emptyset$, let $\mathcal{C}' = (\mathcal{C} - \mathcal{C}_e) \cup \{C, C\}$, where $C = E(xy)$. One can see from our proof below that we could just define $\mathcal{C}/e$ to be $\mathcal{C}'$. However, to smooth the rest of our proof, we make the following adjustment. If $m_{\mathcal{C}}(C) \le 1$, let $\mathcal{C}/e = \mathcal{C}'$; if $m_{\mathcal{C}}(C) \ge 2$, let $\mathcal{C}/e = \mathcal{C}' - \{C, C, C, C\}$. Let $(G, \mathcal{C})/e = (G/e, \mathcal{C}/e)$.

Lemma 5.4 *If $(G, \mathcal{C})$ has a contractable edge e, then $\mathcal{C}/e$ is a collection of cuts of G/e. Moreover,*
(i) if $\mathcal{C}_e = \emptyset$ and $(G, \mathcal{C})/e$ is truncatable, then $(G, \mathcal{C})$ is also truncatable;
(ii) if $(G, \mathcal{C})/e$ is strongly truncatable, then so is $(G, \mathcal{C})$.

Proof. Since contracting edges keeps a connected graph connected, by definition, if a cut C of G does not contain e, then C is a cut of G/e. Notice that all members of $\mathcal{C}/e$ are cuts of G that do not contain e, thus $\mathcal{C}/e$ is a collection of cuts of G/e.

(i) Since $\mathcal{C}_e = \emptyset$, by definition, $\mathcal{C}/e = \mathcal{C}$. Clearly, if $\mathcal{D}$ is a collection of cuts of G/e, then $\mathcal{D}$ is also a collection of cuts of G. To prove (i), we only need to show that if $(\mathcal{C}, \mathcal{D})$ satisfies (1a) or (1b) in G/e, then it also satisfies the corresponding condition in G. Since (1a) depends only on $|\mathcal{C}|$ and $|\mathcal{D}|$, so this part is clear. For each $f \in E(G/e)$, we observe that f is also an edge of G, and the values of $d_{\mathcal{D}}(f)$ and $d_{\mathcal{C}}(f)$ in G are the same as these values in G/e. The only other edge in G is $f = e$, for which we have $(d_{\mathcal{D}}(f), d_{\mathcal{C}}(f)) = (0, 0)$ in G. Therefore, $(\mathcal{C}, \mathcal{D})$ satisfies (1b) in G/e implies $(\mathcal{C}, \mathcal{D})$ satisfies (1b) in G.

(ii) Suppose $(G, \mathcal{C})/e$ is strongly truncatable. Then it has a certificate $\mathcal{D}'$. Let $e = xy$ and $C = E(xy)$. We consider three cases: $\mathcal{C}_e = \emptyset$; $\mathcal{C}_e \neq \emptyset$ and $m_{\mathcal{C}}(C) \geq 2$; and $\mathcal{C}_e \neq \emptyset$ and $m_{\mathcal{C}}(C) \leq 1$. In the first two cases, let $\mathcal{D}$ be $\mathcal{D}'$ and $\mathcal{D}' \cup \{C, E(x), E(y)\}$, respectively. In the last case, notice that C is a small cut of G/e with $m_{\mathcal{C}/e}(C) > 1$, so $C \in \mathcal{D}'$. In this case, let $\mathcal{D} = (\mathcal{D}' - \{C\}) \cup \{E(x), E(y)\}$. In all cases, it is routine to verify that $\mathcal{D}$ satisfies (1a-c). Thus, $(G, \mathcal{C})$ is strongly truncatable. ∎

Let x and y be two vertices of a connected graph G. Let $G + xy$ be obtained from G by adding a new edge $f = xy$. For each cut C of G, let $C + xy = C \cup \{f\}$ if x and y are separated by C, and $C + xy = C$ if otherwise. Then $\mathcal{C} + xy = \{C + xy : C \in \mathcal{C}\}$ is a collection of cuts of $G + xy$.

Lemma 5.5 *Suppose x and y are vertices of a connected graph G.*
(i) If $\mathcal{C} + xy$ is truncatable in $G + xy$, then $\mathcal{C}$ is truncatable in G.
(ii) if $\mathcal{C} + xy$ is strongly truncatable in $G + xy$, then $\mathcal{C}$ is strongly truncatable in G.

Proof. In both cases, let $\mathcal{D}$ be a certificate for $\mathcal{C} + xy$. Notice that every cut D of $G + xy$ has a subset D' such that D' is a cut of G. Let $\mathcal{D}' = \{D' : D \in \mathcal{D}\}$. Then $|\mathcal{D}'| = |\mathcal{D}| \geq |\mathcal{C} + xy|/2 = |\mathcal{C}|/2$. For each $e \in E(G)$, we also have $d_{\mathcal{D}'}(e) \leq d_{\mathcal{D}}(e)$ and $d_{\mathcal{C}}(e) = d_{\mathcal{C}+xy}(e)$, which imply that $\mathcal{D}'$ is a certificate for the truncatability of $\mathcal{C}$. To prove (ii), we observe that if $C \in \mathcal{C}$ is small, then $C + xy \in \mathcal{C} + xy$ is also small, as G is connected. Moreover, $(C + xy)' = C$. Therefore, if $(\mathcal{C} + xy, \mathcal{D})$ satisfies (1c), then so does $(\mathcal{C}, \mathcal{D}')$, hence $\mathcal{D}'$ is a certificate for the strong truncatability of $\mathcal{C}$. ∎

6 Non-truncatability

6.1 A simple observation

Let $\mathcal{C}$ be a collection of cuts of a graph G. An edge e of G is *critical* if $d_{\mathcal{C}}(e) \equiv 0 \pmod 4$.

Lemma 6.1 *Suppose a collection $\mathcal{C}$ of cuts of a graph $G = (V, E)$ is not strongly truncatable. Then the following statements hold.*
(i) $m_{\mathcal{C}}(C)$ is odd for at least one cut $C \in \mathcal{C}$;

(ii) $\mathcal{C}$ has at least two different big cuts, provided that all critical edges form a connected spanning subgraph of G, and $m_{\mathcal{C}}(C) \le 3$, for all $C \in \mathcal{C}$.

Proof. (i) Suppose $m_{\mathcal{C}}(C)$ is even for every cut $C \in \mathcal{C}$. Let $\mathcal{D}$ be a subcollection of $\mathcal{C}$ such that $m_{\mathcal{D}}(C) = m_{\mathcal{C}}(C)/2$, for all $C \in \mathcal{C}$. It is straightforward to verify that $\mathcal{D}$ satisfies (1a-c), so $\mathcal{C}$ is strongly truncatable, contradicting the hypothesis.

(ii) We claim that $\mathcal{C}$ has at least one big cut. Suppose the contrary: all cuts in $\mathcal{C}$ are small. Let $V_i = \{v \in V : m_{\mathcal{C}}(E(v)) = i\}$, for $i = 0, 1, 2, 3$. Then (V_0, V_1, V_2, V_3) is a partition of V. In view of (i), $V_1 \cup V_3 \neq \emptyset$. Next, observe that $V_0 = V_2 = \emptyset$, for otherwise, the hypothesis would guarantee the existence of a path Q from $V_1 \cup V_3$ to $V_0 \cup V_2$, such that all edges on Q are critical. So Q must contain an edge xy between $V_1 \cup V_3$ and $V_0 \cup V_2$. Hence $d_{\mathcal{C}}(xy) = m_{\mathcal{C}}(E(x)) + m_{\mathcal{C}}(E(y)) \not\equiv 0 \pmod 4$, a contradiction. Let $\mathcal{D} = \{E(v) : v \in V\}$ (the collection of all small cuts taken with multiplicity one). Then both $\mathcal{D}$ and $\mathcal{C} - \mathcal{D}$ satisfy (1b) and (1c), which implies that at least one of them is a certificate for the strong truncatability of $\mathcal{C}$, a contradiction. Thus the claim is proved.

Suppose (ii) is false. By the above claim, $\mathcal{C}$ has a big cut C such that all other big cuts in $\mathcal{C}$ are copies of C. For $i = 0, 1, 2, 3$, let $X_i = \{x \in X_C : m_{\mathcal{C}}(E(x)) = i\}$ and $Y_i = \{y \in Y_C : m_{\mathcal{C}}(E(y)) = i\}$. For any two sets $Z, Z' \subseteq V$, let $E(Z, Z')$ denote the set of edges with one end in Z and one end in Z'. Suppose $m_{\mathcal{C}}(C) = 2$. Then the critical edges are precisely those in $E(X_0, Y_2)$, $E(X_2, Y_0)$, $E(X_1, Y_1)$, $E(X_3, Y_3)$, $E(X_1, X_3)$, $E(Y_1, Y_3)$, $E(X_0, X_0)$, and $E(Y_0, Y_0)$, $E(X_2, X_2)$, and $E(Y_2, Y_2)$. Since critical edges form a connected spanning subgraph, using an argument similar to the proof in the preceding paragraph, we can deduce from (i) that $X_i = Y_i = \emptyset$, for $i = 0, 2$. Let $\mathcal{D}_1 = \{C\} \cup \{E(x) : x \in X_C\} \cup \{E(y), E(y) : y \in Y_3\}$ and $\mathcal{D}_2 = \mathcal{C} - \mathcal{D}_1$. It is straightforward to verify that both $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy (1b) and (1c). Since $|\mathcal{C}| = |\mathcal{D}_1| + |\mathcal{D}_2|$, some $\mathcal{D}_i$ must also satisfy (1a), which means $\mathcal{C}$ is strongly truncatable, a contradiction. Therefore, we must have $m_{\mathcal{C}}(C) \in \{1, 3\}$. Again, by analyzing critical edges, we may assume $X_0 = X_2 = Y_1 = Y_3 = \emptyset$. We distinguish among the following four cases.

If $m_{\mathcal{C}}(C) = 1$ and $Y_0 = \emptyset$, let $\mathcal{D}_1 = \{E(x) : x \in V\}$ and $\mathcal{D}_2 = \mathcal{C} - \mathcal{D}_1$;
If $m_{\mathcal{C}}(C) = 1$ and $Y_0 \neq \emptyset$, let $\mathcal{D}_1 = \{D\} \cup \{E(x) : x \in X_C \cup Y_2\}$ and $\mathcal{D}_2 = \mathcal{C} - \mathcal{D}_1 - \{C\}$;
If $m_{\mathcal{C}}(C) = 3$ and $Y_0 = \emptyset$, let $\mathcal{D}_1 = \{C, C\} \cup \{E(x) : x \in V\}$ and $\mathcal{D}_2 = \mathcal{C} - \mathcal{D}_1$;
If $m_{\mathcal{C}}(C) = 3$ and $Y_0 \neq \emptyset$, let $\mathcal{D}_1 = \{C\} \cup \{E(x) : x \in X_C \cup Y_2\}$ and $\mathcal{D}_2 = \{D\} \cup (\mathcal{C} - \mathcal{D}_1 - \{C\})$;

where D is a cut of G with $D \subseteq E(Y_0)$. In all the four cases, it is straightforward to verify that $|\mathcal{C}| = |\mathcal{D}_1| + |\mathcal{D}_2|$, and both $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy (1b) and (1c). Therefore, some $\mathcal{D}_i$ is a certificate for the strong truncatability of $\mathcal{C}$, a contradiction, which proves (ii). ∎

6.2 Basic properties

Suppose a connected graph H is not truncatable. Then a *non-truncatable pair* $(G, \mathcal{C})$ contained in H consists of a non-truncatable graph $G = (V, E)$ and a non-truncatable collection $\mathcal{C}$ of cuts of G, where G is obtained from H by contracting a (possibly empty) set of edges. Throughout this subsection, we assume that $(G, \mathcal{C})$ is chosen such that

(2a) $|E|$ is minimized;
(2b) subject to (2a), $f(\mathcal{C}) = \sum_{e \in E} \lceil d_{\mathcal{C}}(e)/4 \rceil$ is minimized;
(2c) subject to (2a-b), $|\mathcal{C}|$ is maximized;

(2d) subject to (2a-c), $s(\mathcal{C})$, the number of small cuts in $\mathcal{C}$, is maximized;
(2e) subject to (2a-d), $g(\mathcal{C}) = \sum_{C \in \mathcal{C}}(|X_C|^2 + |Y_C|^2)$ is maximized; and
(2f) subject to (2a-e), $|\hat{\mathcal{C}}|$, the number of distinct cuts in $\mathcal{C}$, is minimized.

In the following, we establish some basic properties for $(G, \mathcal{C})$. We say that two cuts C_1, C_2 *cross* if $X_{C_1} \cap X_{C_2}$, $X_{C_1} \cap Y_{C_2}$, $Y_{C_1} \cap X_{C_2}$, and $Y_{C_1} \cap Y_{C_2}$ are all nonempty. If C is a big cut, for which there is no other big cut $D \in \mathcal{C}$ with $X_D \subseteq X_C$, then X_C is called an *end*.

Lemma 6.2 *The non-truncatable pair $(G, \mathcal{C})$ enjoys the following properties:*
(i) Every edge belongs to a cut in $\mathcal{C}$;
(ii) $m_{\mathcal{C}}(C) \leq 3$, for every cut C of G;
(iii) Suppose $\mathcal{C}'$ is a collection of cuts of G with $\lceil d_{\mathcal{C}'}(e)/4 \rceil \leq \lceil d_{\mathcal{C}}(e)/4 \rceil$, for all $e \in E$. Then $(|\mathcal{C}'|, s(\mathcal{C}'), g(\mathcal{C}'), -|\hat{\mathcal{C}}'|)$ is lexicographically less than or equal to $(|\mathcal{C}|, s(\mathcal{C}), g(\mathcal{C}), -|\hat{\mathcal{C}}|)$;
(iv) $\mathcal{C}$ is cross-free. That is, no two cuts in $\mathcal{C}$ cross;
(v) The set of critical edges form a connected spanning subgraph;
(vi) Let $C \in \mathcal{C}$ be a big cut. Then every $v \in V$ is incident with a critical edge not in C;
(vii) If v belongs to an end X_C, then $E(v) \in \mathcal{C}$.

Proof. (i) The conclusion follows obviously from Lemma 5.4(i) and (2a).

(ii) If $m_{\mathcal{C}}(C) \geq 4$, for some C, then we define $\mathcal{C}' = \mathcal{C} - \{C, C, C, C\}$. It is easy to see that $f(\mathcal{C}') < f(\mathcal{C})$. By (2b), $\mathcal{C}'$ is truncatable, and so, has a certificate, say, $\mathcal{D}'$. Then it is straightforward to verify that $\mathcal{D} = \mathcal{D}' \cup \{C, C\}$ is a certificate for the truncatability of $\mathcal{C}$, a contradiction.

(iii) Suppose the conclusion is false. Since $f(\mathcal{C}') \leq f(\mathcal{C})$, we deduce from the choice of $\mathcal{C}$ that $\mathcal{C}'$ is truncatable. Let $\mathcal{D}$ be a certificate for the truncatability of $\mathcal{C}'$. Then $|\mathcal{D}| \geq |\mathcal{C}'|/2 \geq |\mathcal{C}|/2$ and $d_{\mathcal{D}}(e) \leq 2\lceil d_{\mathcal{C}'}(e)/4 \rceil \leq 2\lceil d_{\mathcal{C}}(e)/4 \rceil$, for all $e \in E$, which means that $\mathcal{D}$ is also a certificate for the truncatability of $\mathcal{C}$, a contradiction.

(iv) If two cuts $C_1, C_2 \in \mathcal{C}$ cross, then they both are big cuts. Let $\mathcal{C}' = (\mathcal{C} - \{C_1, C_2\}) \cup \{C_1', C_2'\}$, where C_1', C_2' are cuts with $C_1' \subseteq E(X_{C_1} \cap X_{C_2})$ and $C_2' \subseteq E(X_{C_1} \cup X_{C_2})$. Clearly, $|\mathcal{C}'| = |\mathcal{C}|$ and $s(\mathcal{C}') \geq s(\mathcal{C})$. In addition, it is routine to verify that $g(\mathcal{C}') > g(\mathcal{C})$ and $d_{\mathcal{C}'}(e) \leq d_{\mathcal{C}}(e)$, for all $e \in E$, which contradicts (iii).

(v) If this graph is disconnected or not spanning, then G has a cut C that does not contain any critical edge. Let $\mathcal{C}' = \mathcal{C} \cup \{C\}$. It follows that $|\mathcal{C}'| > |\mathcal{C}|$ and $\lceil d_{\mathcal{C}'}(e)/4 \rceil = \lceil d_{\mathcal{C}}(e)/4 \rceil$, for all $e \in E$, which contradicts (iii) again.

(vi) Suppose the conclusion fails for some $v \in V$. By (v), $G - v$ is connected. It follows that $E(v)$ is a cut of G, and thus $\mathcal{C}' = (\mathcal{C} - \{C\}) \cup \{E(v)\}$ contradicts (iii).

(vii) By (vi), v is incident with a critical edge $e = uv \notin C$. Then, by (i) and (ii), e belongs to at least two different cuts in $\mathcal{C}$. Since X_C is an end, we deduce from (iv) that all these cuts are small. Notice that there are at most two different small cuts that contain e, namely, $E(u)$ and $E(v)$. Therefore, $E(v)$ must belong to $\mathcal{C}$. ■

Lemma 6.3 *Suppose $C \in \mathcal{C}$ such that $G[X_C]$ is a tree. If $Z \subseteq X_C$ and $G[Z]$ is connected, then $E(Z)$ is a cut of G.*

Proof. We first claim that every vertex of G has at least two neighboring vertices. Suppose, on the contrary, that a vertex v of G has at most one neighboring vertex. Since G is obtained from a connected graph by contracting edges, G is connected. Moreover, since G is non-truncatable, G has more than one vertex. Therefore, v has precisely one neighboring vertex, say u. Notice that the only cut D of G that contains $e = uv$ is $D = E(v)$. By Lemma 6.2(i-ii), it follows that $1 \leq d_{\mathcal{C}}(e) = m_{\mathcal{C}}(D) \leq 3$. This contradicts Lemma 6.2(v), and thus the claim is proved.

Since $G[X_C]$ is a tree and $G[Z]$ is connected, for every vertex $x \in X_C - Z$, there exists a leaf y of $G[X_C]$ such that $G[X_C] - Z$ has a (unique) path between x and y. By our claim above, y has at least two neighboring vertices. Since y is a leaf in $G[X_C]$, there is an edge between y and Y_C in G. Consequently, every vertex in $X_C - Z$ is connected through a path in $G - Z$ to Y_C. Therefore, $G - Z$ is connected, as $G[Y_C]$ is, which implies that $E(Z)$ is a cut of G. ■

For each nonnegative integer n, the graph $K_{1,n}$ is called a *star*.

Lemma 6.4 *If $C \in \mathcal{C}$ is a big cut and $G[X_C]$ is a star, then $|X_C| = 2$.*

Proof. Let $x_0 \in X_C$ have degree $t = |X_C| - 1$ in $G[X_C]$ and let $x_1, x_2, ..., x_t$ be the remaining vertices in X_C. Let us consider a big cut $D \in \mathcal{C}$ such that X_D is minimal with $X_D \subseteq X_C$. Then X_D is an end. Since $G[X_C]$ is a star and $G[X_D]$ is connected, x_0 must be contained in X_D. Thus, by Lemma 6.2(vii), $E(x_0) \in \mathcal{C}$. Suppose $|X_C| > 2$. Then $t = |X_C| - 1 \geq 2$. By Lemma 6.3, every $E(x_i)$ $(1 \leq i \leq t)$ is a cut of G. Let $\mathcal{C}' = (\mathcal{C} - \{E(x_0), C\}) \cup \{E(x_1), E(x_2)\}$. It is routine to verify that $|\mathcal{C}'| = |\mathcal{C}|$, $s(\mathcal{C}') > s(\mathcal{C})$ and $d_{\mathcal{C}'}(e) \leq d_{\mathcal{C}}(e)$, for all $e \in E$, which contradicts Lemma 6.2 (iii). ■

Lemma 6.5 *If $C \in \mathcal{C}$ is a big cut and $G[X_C]$ is a path, then $|X_C| = 2$.*

Proof. Suppose there is a counterexample C. We choose such a C with X_C minimal. Let the vertices of X_C be $x_1, x_2, ..., x_t$, ordered as in the path.

Case 1. Suppose $E(x_i) \in \mathcal{C}$, for all $1 < i < t$. We first consider the subcase when either t is odd or $|Y_C| > 2$. Let $Z_0 = \{x_i : i \text{ is even and } i < t\}$ and $Z_1 = \{x_i : i \text{ is odd and } i < t - 1\}$. Let $D = E(x_t)$, if t is odd, and $D = E(\{x_{t-1}, x_t\})$, if t is even. By Lemma 6.3, D and $E(x_i)$ $(1 \leq i \leq t)$ are cuts of G. Let $\mathcal{C}' = (\mathcal{C} - \{C\} - \{E(x) : x \in Z_0\}) \cup \{E(x) : x \in Z_1\} \cup \{D\}$. It is routine to verify that $|\mathcal{C}'| = |\mathcal{C}|$, and $d_{\mathcal{C}'}(e) \leq d_{\mathcal{C}}(e)$, for all $e \in E$. Since $|Z_0| = |Z_1|$, we also have $s(\mathcal{C}') \geq s(\mathcal{C})$. Notice that, if t is odd, then D is a small cut and thus $s(\mathcal{C}') > s(\mathcal{C})$, contradicting Lemma 6.2(iii). On the other hand, if $|Y_C| > 2$, then $\min\{|X_C|, |Y_C|\} > 2 \geq \min\{|X_D|, |Y_D|\}$, which implies $g(\mathcal{C}') > g(\mathcal{C})$, contradicting Lemma 6.2(iii) again.

Next, we consider the subcase when t is even and $|Y_C| = 2$. By Lemma 6.2(ii,v) and Lemma 6.1(ii), $\mathcal{C}$ has another big cut, say D. By Lemma 6.2(iv), we may assume $X_D \subseteq X_C$. Then we deduce from the minimality of X_C that $|X_D| = 2$. So we may further assume $X_D = \{x_k, x_{k+1}\}$, for some k with $1 \leq k < t$. Let $Z_0 = \{x_i : 1 < i < k \text{ and } i \text{ is even, or } k + 1 < i < t \text{ and } i \text{ is odd}\}$ and $Z_1 = \{x_i : 1 \leq i < k \text{ and } i \text{ is odd, or } k + 1 < i \leq t \text{ and } i \text{ is even}\}$. Let $\mathcal{C}_0 = \{E(x) : x \in Z_0\}$ and $\mathcal{C}_1 = \{E(x) : x \in Z_1\}$. If k is even, let $\mathcal{C}' = (\mathcal{C} - \{C, D\} - \mathcal{C}_0) \cup \mathcal{C}_1$. Then $|\mathcal{C}'| = |\mathcal{C}|$, $s(\mathcal{C}') > s(\mathcal{C})$, and $d_{\mathcal{C}'}(e) \leq d_{\mathcal{C}}(e)$, for all $e \in E$, which contradicts Lemma 6.2(iii). It follows that k has to be odd. Since D was chosen arbitrarily, we deduce that every big cut in $\mathcal{C}$ that is different from C can be expressed as $E(x_i x_{i+1})$, for some odd i. It follows that $e = x_1 x_2$ does not belong to any big cut of $\mathcal{C}$. By Lemma 6.2(vi), e is critical, and thus, by Lemma 6.2(i-ii), e belongs to at least two different cuts in $\mathcal{C}$, which implies $E(x_1) \in \mathcal{C}$. Similarly, $E(x_t) \in \mathcal{C}$. Let $m = \min\{m_{\mathcal{C}}(C') : C' \in \{C\} \cup \mathcal{C}_0\}$ and let $\mathcal{C}'$ be obtained from $\mathcal{C}$ by deleting m copies of each member of $\{C\} \cup \mathcal{C}_0$ and adding m

copies of each member of $\{D\} \cup \mathcal{C}_1$. Then $|\mathcal{C}'| = |\mathcal{C}|$, $s(\mathcal{C}') = s(\mathcal{C})$, $g(\mathcal{C}') = g(C)$, and $d_{\mathcal{C}'}(e) \le d_{\mathcal{C}}(e)$, for all $e \in E$. However, since $\{D\} \cup \mathcal{C}_1 \subseteq \mathcal{C}$, we deduce from the choice of m that $|\hat{\mathcal{C}}'| < |\hat{\mathcal{C}}|$, which contradicts Lemma 6.2(iii), and thus Case 1 is settled.

Case 2. Suppose $E(x_i) \notin \mathcal{C}$, for some i with $1 < i < t$. The idea of our proof is similar. Let us consider the set of indices i such that $1 < i < t$ and $E(x_i) \notin \mathcal{C}$. Without loss of generality, we assume that this set consists of $i_1, i_2, ..., i_r$, where $i_1 < i_2 < ... < i_r$ and $r > 0$. Let $i_0 = 1$ and $i_{r+1} = t$. We partition the path $G[X_C]$ into $r+1$ parts, $Q_0, Q_1, ..., Q_r$, where $Q_j = G[\{x_k : i_j \le k \le i_{j+1}\}]$. According to our definition, $E(x) \in \mathcal{C}$, for all the interior vertices x of each Q_j.

Before we proceed, we make two observations. First, every Q_j has at least two edges. Suppose some Q_j has only one edge $e = x_{i_j} x_{i_{j+1}}$. Since $r > 0$, we may assume, by symmetry, that $j \ge 1$. Therefore, $E(x_{i_j}) \notin \mathcal{C}$. Let us consider $\mathcal{C}_e$, the collection $\{D \in \mathcal{C} : D \ni e\}$. By Lemma 6.2(iv), we may assume $X_D \subset X_C$, for all $D \in \mathcal{C}_e$. It follows from Lemma 6.2(vii) and the minimality of X_C that $x_{i_{j+1}} \in X_D$, for all $D \in \mathcal{C}_e$. Moreover, by Lemma 6.2(vii) again, we must have $E(x_{i_{j+1}}) \in \mathcal{C}$, which implies $i_{j+1} = t$. Therefore, all cuts in $\mathcal{C}_e$ are copies of $E(x_{i_{j+1}})$, and thus, by Lemma 6.2(i-ii), $0 < |\mathcal{C}_e| \le 3$. However, since e is the only edge in $G[X_C]$ that is incident with x_t, we deduce from Lemma 6.2(vi) that $|\mathcal{C}_e| \equiv 0 \pmod 4$, a contradiction.

Our second observation is the following. Suppose $x \in X_C$ with $E(x) \notin \mathcal{C}$. If $xy \in E(G[X_C])$ is a critical edge, then y has another neighbor z in X_C such that $E(yz) \in \mathcal{C}$. Since xy is critical, this edge must belong to at least two different cuts in $\mathcal{C}$. It follows that at least one of these cuts, say D, is big, as $E(x) \notin \mathcal{C}$. By Lemma 6.2(iv) and the minimality of X_C, we must have $X_D \subseteq X_C$ and $|X_D| = 2$. Moreover, by Lemma 6.2(vii), $x \notin X_D$. Therefore, X_D consists of two adjacent vertices including y, which proves the second observation.

Let $j \in \{0, 1, ..., r\}$. In the following, we define a set $\mathcal{C}_j$ of cuts in $\mathcal{C}$ and a set $\mathcal{D}_j$ of small cuts of G. If $|V(Q_j)|$ is odd, let $\mathcal{C}_j = \{E(x_i) : i - i_j \text{ is odd and } i_j < i < i_{j+1}\}$ and $\mathcal{D}_j = \{E(x_i) : i - i_j \text{ is even and } i_j \le i \le i_{j+1}\}$. If $|V(Q_j)|$ is even, we chose a vertex $x_k \in V(Q_j)$ such that it has degree one in Q_j but has degree two in $G[X_C]$. Such a vertex x_k must exist since $r > 0$. Without loss of generality, let us assume that $k = i_j$. Let $\mathcal{D}_j = \{E(x_{i_j})\} \cup \{E(x_i) : i - i_j \text{ is odd and } i_j + 3 \le i \le i_{j+1}\}$. To define $\mathcal{C}_j$, we need to consider two cases, depending on if $e_j = x_{i_j} x_{i_j+1}$ is critical. If e_j is critical, by our second observation, $C_j = E(x_{i_j+1} x_{i_j+2})$ is in $\mathcal{C}$. In this case, let $\mathcal{C}_j = \{C_j\} \cup \{E(x_i) : i - i_j \text{ is even and } i_j + 3 < i < i_{j+1}\}$. If e_j is not critical, let $\mathcal{C}_j = \{E(x_i) : i - i_j \text{ is even and } i_j < i < i_{j+1}\}$. By Lemma 6.3, every member in each $\mathcal{D}_j$ is a cut of G. Moreover, from these definitions it is straightforward to verify the following:

- $\mathcal{C}_j \subseteq \mathcal{C}$, for all j;
- every cut in every $\mathcal{D}_j$ is small;
- $\mathcal{C}_j \cap \mathcal{C}_{j'} = \emptyset$, if $j \ne j'$;
- $\mathcal{D}_j \cap \mathcal{D}_{j'} = \emptyset$, if $|j - j'| > 1$;
- $\mathcal{D}_{j-1} \cap \mathcal{D}_j = \{E(x_{i_j})\}$, for $j = 1, 2, ..., r$;
- $|\mathcal{D}_j| = |\mathcal{C}_j| + 1$, for all j.

Let $\mathcal{C}^* = \cup_{j=0}^{r} \mathcal{C}_j$ and $\mathcal{D}^* = \cup_{j=0}^{r} \mathcal{D}_j$, where the union is considered as union of sets, not multisets. In other words, common cuts in $\mathcal{D}_j$ and $\mathcal{D}_{j\pm1}$ are counted only once in $\mathcal{D}^*$. Let $\mathcal{C}' = (\mathcal{C} - \{C\} - \mathcal{C}^*) \cup \mathcal{D}^*$. Then it is routine to verify that $|\mathcal{C}'| = |\mathcal{C}|$, $s(\mathcal{C}') > s(\mathcal{C})$, and, by our first observation, $d_{\mathcal{C}'}(e) \le d_{\mathcal{C}}(e)$, for all $e \in E$, which contradicts Lemma 6.2(iii). ∎

6.3 Compactness and more reductions

A graph G is called *ps-connected* if it is connected and, for every cut C of G, at at least one component of $G\backslash C$ is either a path or a star.

Lemma 6.6 *If G is ps-connected, then so are all its connected minors.*

Proof. Let G' be a connected minor of G with $G' = G\backslash F_1/F_2$ and let C be a cut of G'. Then F_1 can be partitioned into F_1', F_1'' such that $C \cup F_1'$ is a cut of G. Therefore, $G'\backslash C = G\backslash(C\cup F_1')\backslash F_1''/F_2$ and thus components of $G'\backslash C$ are minors of components of $G\backslash(C \cup F_1')$. Since $G\backslash(C \cup F_1')$ has a path- or a star-component, we deduce that $G'\backslash C$ has a path- or a star-component. ■

Let $\mathcal{C}$ be a collection of cuts of a connected graph G. Then $(G, \mathcal{C})$ is *compact* if it satisfies:

(3a) $m_{\mathcal{C}}(C) \leq 3$, for every $C \in \mathcal{C}$;
(3b) the set of critical edges form a connected spanning subgraph of G;
(3c) every big cut in $\mathcal{C}$ has the form $E(xy)$, for some adjacent vertices x and y;
(3d) the set $M = \{xy \in E(G) : E(xy) \in \mathcal{C}$ is a big cut$\}$ of edges is a matching; and
(3e) if $E(xy) \in \mathcal{C}$ is a big cut, then $E(x), E(y) \in \mathcal{C}$ with $m_{\mathcal{C}}(E(x)) + m_{\mathcal{C}}(E(y)) = 4$.

The matching M defined in (3d) will be referred to as the matching of $\mathcal{C}$.

Lemma 6.7 *Let H be a connected and not truncatable graph. If H is ps-connected and $(G, \mathcal{C})$ is chosen subject to (2a-f), then $(G, \mathcal{C})$ is compact.*

Proof. Clearly, (3a) and (3b) follow from (ii) and (v) of Lemma 6.2, respectively. By Lemma 6.6, G is ps-connected and thus (3c) follows from Lemma 6.4 and Lemma 6.5. Consequently, (3d) follows from Lemma 6.2(iv), and (3e) from Lemma 6.2(vi-vii). ■

Lemma 6.8 *Suppose $(G, \mathcal{C})$ has a contractable edge e. If $(G, \mathcal{C})$ is compact, then so is $(G, \mathcal{C})/e$.*

Proof. From its construction we deduce that $\mathcal{C}/e$ satisfies (3a). Since big cuts of $\mathcal{C}/e$ are also big cuts of $\mathcal{C}$, it follows that $\mathcal{C}/e$ satisfies (3c), (3d), and (3e) automatically. To verify (3b), notice that $d_{\mathcal{C}}(f) - d_{\mathcal{C}/e}(f) = 0$, or 4, for all $f \in E(G/e)$. Therefore, if J is the graph formed by the $\mathcal{C}$-critical edges in G, then the graph formed by the $(\mathcal{C}/e)$-critical edges in G/e is exactly J/e, which is connected and spanning, as J is. ■

Lemma 6.9 *Suppose $(G, \mathcal{C})$ is compact but not strongly truncatable. Then the following hold:*
(i) $\mathcal{C}$ has a big cut C with $m_{\mathcal{C}}(C)$ odd;
(ii) $\mathcal{C}$ has at least three different big cuts.

Proof. Let $G = (V, E)$. Suppose (i) is false. Then $m_{\mathcal{C}}(C) = 2$, for all big cuts $C \in \mathcal{C}$. By (3b-e) and Lemma 6.1(i), $m_{\mathcal{C}}(E(x))$ is odd, for all $x \in V$. Let $M = \{x_iy_i : 1 \leq i \leq k\}$ be the matching of $\mathcal{C}$. By (3e), we may assume that $m_{\mathcal{C}}(E(x_i)) = 1$ and $m_{\mathcal{C}}(E(y_i)) = 3$, for all i. Let $Z = V - \{x_i, y_i : 1 \leq i \leq k\}$. Let $\mathcal{D} = \{E(x_iy_i), E(y_i), E(y_i) : 1 \leq i \leq k\} \cup \{E(z) : z \in Z\}$. Then it is routine to verify that both $\mathcal{D}$ and $\mathcal{C} - \mathcal{D}$ satisfy (1b) and (1c), which implies that at least one of them is a certificate for the strong truncatability of $\mathcal{C}$ and this contradiction proves (i).

Next, suppose (ii) is false. By Lemma 6.1(ii), $\mathcal{C}$ has two different big cuts C_1 and C_2 such that all big cuts in $\mathcal{C}$ are copies of one of these two. By (3d), we may assume that V is partitioned into (X_1, X_2, X_3) such that $C_1 = E(X_1)$ and $C_2 = E(X_2)$. For $i = 1, 2, 3$ and $j = 0, 1, 2, 3$, let $X_{ij} = \{x \in X_i : m_{\mathcal{C}}(E(x)) = j\}$. By (3e), $X_{10} = X_{20} = \emptyset$. Let $m_1 = m_{\mathcal{C}}(C_1)$ and $m_2 = m_{\mathcal{C}}(C_2)$.

We claim that $X_{12} = X_{22} = \emptyset$. Suppose, say, $X_{12} \neq \emptyset$. By (3e), $X_{12} = X_1$. Hence the only edge e of $G[X_1]$ is contractable, which implies, by Lemmas 6.8 and 5.4(ii), that $(G, \mathcal{C})/e$ is compact but not strongly truncatable. Consequently, by Lemma 6.1(ii), $\mathcal{C}/e$ should have at least two different big cuts. However, C_2 is the only big cut in $\mathcal{C}/e$, a contradiction, and so the claim is proved.

By (i), at least one of m_1 and m_2 is odd. Then, by (3b) and the above claim, that the other is also odd, and $X_{31} = X_{33} = \emptyset$. By symmetry, we only need to consider the following three cases:
If $(m_1, m_2) = (1, 1)$, let

$\mathcal{D}_1 = \{E(X_1 \cup X_{32})\} \cup \{E(x) : x \in X_1 \cup X_{32}\} \cup \{E(x), E(x) : x \in X_{23}\}$ and
$\mathcal{D}_2 = \{E(X_2 \cup X_{32})\} \cup \{E(x) : x \in X_2 \cup X_{32}\} \cup \{E(x), E(x) : x \in X_{13}\}$.

If $(m_1, m_2) = (1, 3)$, let

$\mathcal{D}_1 = \{C_2, E(X_1 \cup X_{30})\} \cup \{E(x) : x \in X_{32}\} \cup \{E(x), E(x) : x \in X_{13} \cup X_{23}\}$ and
$\mathcal{D}_2 = \{C_2, E(X_1 \cup X_{32})\} \cup \{E(x) : x \in X_1 \cup X_2 \cup X_{32}\}$.

If $(m_1, m_2) = (3, 3)$, let

$\mathcal{D}_1 = \{C_1, C_2, E(X_1 \cup X_{30})\} \cup \{E(x) : x \in X_1 \cup X_{32}\} \cup \{E(x), E(x) : x \in X_{23}\}$ and
$\mathcal{D}_2 = \{C_1, C_2, E(X_2 \cup X_{30})\} \cup \{E(x) : x \in X_2 \cup X_{32}\} \cup \{E(x), E(x) : x \in X_{13}\}$;

where each $E(X_i \cup X_{jk})$ should be interpreted as a cut contained in $E(X_i \cup X_{jk})$. In each case, it is straightforward to verify that $|\mathcal{D}_1| + |\mathcal{D}_2| = |\mathcal{C}|$ and both $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy (1b) and (1c). Therefore, at least one of $\mathcal{D}_1$ and $\mathcal{D}_2$ is a certificate for the strong truncatability of $\mathcal{C}$, a contradiction. ∎

Lemma 6.10 *Suppose $(G, \mathcal{C})$ is compact, not strongly truncatable, and free of contractable edges. Let $M = \{x_i y_i : 1 \leq i \leq k\}$ be the matching of $\mathcal{C}$, and let $Z = V(G) - \{x_i, y_i : 1 \leq i \leq k\}$. Then*
(i) $\{m_{\mathcal{C}}(E(x_i)), m_{\mathcal{C}}(E(y_i))\} = \{1, 3\}$, for $i = 1, 2, ..., k$;
(ii) $m_{\mathcal{C}}(E(\{x_i, y_i\})) \in \{1, 3\}$, for $i = 1, 2, ..., k$;
(iii) $m_{\mathcal{C}}(E(z)) \in \{0, 2\}$, for each $z \in Z$; and
(iv) $m_{\mathcal{C}}(E(z_1)) \neq m_{\mathcal{C}}(E(z_2))$, if z_1, $z_2 \in Z$ and $E(\{z_1, z_2\})$ is a cut of G.

Proof. Conclusion (i) follows from (3e) and the hypothesis that there are no contractable edges. Conclusions (ii) and (iii) follow from (i), Lemma 6.9(i), and (3b). Conclusion (iv) follows from (iii) and the hypothesis that there are no contractable edges. ∎

7 The two infinite families

In this section, we prove that connected minors of $K_{3,n}$ and W_n are truncatable.

Lemma 7.1 *Let $n \geq 3$ be an integer. Then every connected minor of $K_{3,n}$ is truncatable.*

Proof. Suppose a connected minor H of $K_{3,n}$ is not truncatable. We choose a non-truncatable pair $(G, \mathcal{C})$ of H that satisfies (2a-f). Since $K_{3,n}$ is ps-connected, by Lemma 6.6, H is ps-connected. Thus, by Lemma 6.7, $(G, \mathcal{C})$ is compact. As a minor of $K_{3,n}$, G has a set U of vertices such that

$|U| \leq 3$ and $G - U$ is edge-less. On the other hand, since $(G, \mathcal{C})$ is not strongly truncatable either, we deduce from Lemma 6.9(ii) that the matching M of $\mathcal{C}$ has at least three edges. Consequently, M consists of exactly three edges, say u_1v_1, u_2v_2, u_3v_3, where $U = \{u_1, u_2, u_3\}$. By (3e), every $E(u_i)$ and $E(v_i)$ belongs to $\mathcal{C}$. Let $m = \min\{m_{\mathcal{C}}(E(u_i)) : i = 1, 2, 3\}$ and let $\mathcal{C}'$ be obtained from $\mathcal{C}$ by deleting m copies of $E(u_i)$ $(i = 1, 2, 3)$ and also adding m copies of $E(v_i)$ $(i = 1, 2, 3)$. Then $|\mathcal{C}'| = |\mathcal{C}|$, $s(\mathcal{C}') = s(\mathcal{C})$, $g(\mathcal{C}') = g(\mathcal{C})$, and $d_{\mathcal{C}'}(e) \leq d_{\mathcal{C}}(e)$, for all $e \in E(G)$. However, $|\hat{\mathcal{C}}'| < |\hat{\mathcal{C}}|$, contradicting Lemma 6.2(iii). ■

Lemma 7.2 *Let $n \geq 4$ be an integer. Then every connected minor of W_n is truncatable.*

Proof. We apply induction on n. By Lemma 7.1, the assertion holds for $n = 4$. Thus we proceed to the induction step. By Lemma 5.5(i), we only need to consider graphs H obtained from W_n by contracting edges. If at least one edge of W_n is contracted, then $\overline{H}$ is a connected minor of W_{n-1}, so the assertion follows instantly from the induction hypothesis and Lemma 5.3. Hence we only need to justify the case when $H = W_n$. Suppose a non-truncatable pair $(G, \mathcal{C})$ of W_n is chosen, subject to (2a-f). By Lemma 5.3 and the induction hypothesis again, we deduce $G = W_n$.

Let $W_n = (V, E)$. By Lemma 6.7, $(W_n, \mathcal{C})$ is compact. Moreover, all conclusions made in Subsection 6.2 can be applied to $(W_n, \mathcal{C})$. In particular, by Lemma 6.5, the matching M of $\mathcal{C}$ consists of only rim edges. Let Z be the set of vertices that are not incident with any edge in M. For each $x \in V$, we define its rank $r(x)$ to be $m_{\mathcal{C}}(E(x))$ if $x \in Z$, and to be $m_{\mathcal{C}}(E(x)) + m_{\mathcal{C}}(E(xy))$ if $xy \in M$. By (3b), ranks of vertices of G have the same parity, which we call the *parity* of $\mathcal{C}$.

(1) The parity of $\mathcal{C}$ is even.

Suppose the parity is odd. Since $\mathcal{C}$ is not truncatable, it is not strongly truncatable either. Let $(G', \mathcal{C}')$ be obtained from $(W_n, \mathcal{C})$ by repeatedly contracting contractable edges, until no more edge is contractable. By Lemma 5.4(ii), $(G', \mathcal{C}')$ is not strongly truncatable. Moreover, by Lemma 6.8, $(G', \mathcal{C}')$ is compact. Notice that contracting contractable edges preserves the parity of a collection. Therefore, the parity of $\mathcal{C}'$ is odd, contradicting Lemma 6.10(i-iii), which proves (1).

(2) $m_{\mathcal{C}}(E(x))$ is odd, for all $x \in V - Z$.

Suppose $m_{\mathcal{C}}(E(x))$ is even for some $x \in V - Z$. Let $e = xy$ be the matching edge that is incident with x. It follows from (3e) that $m_{\mathcal{C}}(E(x)) = m_{\mathcal{C}}(E(y)) = 2$. Moreover, by (1), $m_{\mathcal{C}}(E(xy)) \geq 2$. Let $G' = \overline{W_n/e}$ and $\mathcal{C}' = \mathcal{C} - \{E(x), E(x), E(y), E(y), E(xy), E(xy)\}$. Clearly, $G' = W_{n-1}$. By our induction hypothesis, $(G', \mathcal{C}')$ is truncatable. Let $\mathcal{D}'$ be a certificate for the truncatability of $\mathcal{C}'$. Let $\mathcal{D} = \mathcal{D}' \cup \{E(x), E(y), E(xy)\}$. Then $|\mathcal{D}| = |\mathcal{D}'| + 3 \geq \frac{1}{2}|\mathcal{C}'| + 3 = \frac{1}{2}|\mathcal{C}|$. Notice that $d_{\mathcal{D}}(f) = d_{\mathcal{D}'}(f) + 2 \leq 2\lceil d_{\mathcal{C}'}(f)/4 \rceil + 2 = 2\lceil (d_{\mathcal{C}'}(f) + 4)/4 \rceil = 2\lceil d_{\mathcal{C}}(f)/4 \rceil$, for all $f \in E(x) \cup E(y)$; and $d_{\mathcal{D}}(f) = d_{\mathcal{D}'}(f) \leq 2\lceil d_{\mathcal{C}'}(f)/4 \rceil = 2\lceil d_{\mathcal{C}}(f)/4 \rceil$, for all $f \in E - E(x) - E(y)$. Therefore, $\mathcal{D}$ is a certificate for the truncatability for $\mathcal{C}$, a contradiction, which proves (2).

It follows from (1) that $xy \in E - M$ is critical if and only if $r(x) \equiv r(y) \pmod 4$. Let u be the hub of W_n and let $v_0, v_1, ..., v_{n-1}$ be the other vertices of W_n, ordered along the cycle $W_n - u$. In the following, subscripts will be taken modulo n.

(3) Every rim edge is critical.

Suppose a rim edge, say v_0v_1, is not critical. By (3e), $v_0v_1 \notin M$, which implies $r(v_0) \not\equiv r(v_1)$ (mod 4). Consequently, either $r(u) \not\equiv r(v_0)$ or $r(u) \not\equiv r(v_1)$ (mod 4). By symmetry, we may assume $r(u) \not\equiv r(v_1) \pmod 4$. Since all edges in M are rim edges, $uv_1 \notin M$ and thus uv_1 is not critical either. By (3b), v_1v_2 has to be critical. Since v_0v_1, uv_1 are not critical, by (3e), they are not contained in M, which implies that either v_1v_2 does not belong to any big cut in $\mathcal{C}$, or $v_2 \notin Z$. In each cases, we deduce from Lemma 6.2(i-ii) or (3e), respectively, that $E(v_2) \in \mathcal{C}$. Let $\mathcal{C}' = (C - \{E(v_2)\}) \cup \{E(v_1), E(v_1v_2)\}$. Then it is easy to see that $d_{\mathcal{C}'}(e) = d_{\mathcal{C}}(e) + 2 = 4\lceil d_{\mathcal{C}}(e)/4\rceil$, for $e = uv_1$ or v_0v_1, and $d_{\mathcal{C}'}(e) = d_{\mathcal{C}}(e)$, for all other edge e of W_n. However, $|\mathcal{C}'| > |\mathcal{C}|$, contradicting Lemma 6.2(iii), which proves (3).

(4) If $v_i \in Z$, then $r(v_i) = m_{\mathcal{C}}(E(v_i)) = 2$.

It follows from the definitions of Z and r that $r(v_i) = m_{\mathcal{C}}(E(v_i))$. Suppose there exists $v_i \in Z$ with $r(v_i) \neq 2$. Then we deduce from (1) and (3a) that $r(v_i) = 0$. Without loss of generality, let $i = 2$. By (3), $r(v_1) \equiv r(v_3) \equiv 0 \pmod 4$, which implies, by Lemma 6.2(i-ii), that v_0v_1, $v_3v_4 \in M$. Consequently, by (3e), $E(v_1)$, $E(v_3) \in \mathcal{C}$. Since $u \in Z$ and $uv_2 \in E$, it is easy to see from Lemma 6.2(i) that $E(u) \in \mathcal{C}$, and thus, by (1), $m_{\mathcal{C}}(E(u)) = 2$, which implies $d_{\mathcal{C}}(uv_2) = 2$. Let $\mathcal{C}' = (\mathcal{C} - \{E(v_1), E(v_3), E(v_0v_1), E(v_3v_4)\}) \cup \{E(v_0), E(v_4), E(v_2), E(v_2)\}$. Then it is routine to check that $|\mathcal{C}'| = |\mathcal{C}|$, $d_{\mathcal{C}'}(uv_2) = 4 = 2\lceil d_{\mathcal{C}}(uv_2)/4\rceil$, and $d_{\mathcal{C}'}(e) \leq d_{\mathcal{C}}(e)$, for all $e \in E - \{uv_2\}$. However, $s(\mathcal{C}') > s(\mathcal{C})$, contradicting Lemma 6.2(iii), which proves (4).

(5) If $r(v_i) \equiv 0 \pmod 4$, then there exists $\varepsilon \in \{1, -1\}$ such that $v_{i-\varepsilon}v_i$, $v_{i+\varepsilon}v_{i+2\varepsilon} \in M$ and $r(v_{i+\varepsilon}) \equiv 0 \pmod 4$

By (4), $v_i \in V - Z$, which means $v_iv_{i-\varepsilon} \in M$, for some $\varepsilon \in \{1, -1\}$. Then, by (3), $r(v_{i+\varepsilon}) \equiv 0 \pmod 4$, and by (4) again, $v_{i+\varepsilon}v_{i+2\varepsilon} \in M$, which proves (5).

For $k = 0, 1$, let $V_k = \{v_i : r(v_i) \equiv 2k \pmod 4\}$. Components of $W_n[V_k]$ are called *$2k$-paths*. Clearly, 0-paths and 2-paths appear on $W_n - u$ alternately. By (3), edges that link a 0-path with a 2-path must belong to M. By (2), every internal vertex of a 2-path must belong to Z. Furthermore, by (5), each 0-path has exactly one edge, which we call a *0-edge*. In the following, we will use this structure to find a certificate $\mathcal{D}$ for the truncatability of $\mathcal{C}$.

Suppose $E(u) \in \mathcal{C}$. We partition V into *blocks* such that each block consists of either a single vertex in Z or vertices of a component of $W_n - Z$. Clearly, each of the second type of blocks can be expressed as $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ such that v_iv_{i+1}, $v_{i+2}v_{i+3} \in M$ and $r(v_{i+1}) \equiv r(v_{i+2}) \equiv 0 \pmod 4$. Let $B_0 = \{u\}$ and let B_1, B_2, ..., B_ℓ be the remaining blocks. For each $i \in \{0, 1, ..., \ell\}$, if $B_i = \{z\}$, let $\mathcal{C}_i = \{E(z), E(z)\}$; if $B_i = \{v_j, v_{j+1}, v_{j+2}, v_{j+3}\}$, let $\mathcal{C}_i$ consist of all cuts in $\mathcal{C}$ of the form $E(x)$ $(x \in B_i)$, $E(v_jv_{j+1})$, or $E(v_{j+2}v_{j+3})$. Then it is easy to see that ($\mathcal{C}_0$, $\mathcal{C}_1$, ..., $\mathcal{C}_\ell$) is a partition of $\mathcal{C}$. Now, for each $i \in \{0, 1, ..., \ell\}$, if $B_i = \{z\}$, let $\mathcal{D}_i = \{E(z)\}$; if $|B_i| = 4$, let $\mathcal{D}_i$ be define as in Figure 7.1, where the numbers indicate $d_{\mathcal{C}_i}(e)$ or $d_{\mathcal{D}_i}(e)$ for the corresponding edge, or the multiplicities for the corresponding cut. Observe that $|\mathcal{D}_i| = |\mathcal{C}_i|/2$. In addition, for each edge e that belongs to a cut in $\mathcal{C}_i$ $(i > 1)$, it is easy to check that $d_{\mathcal{D}_i}(e) = d_{\mathcal{C}_i}(e)/2$, if e is a rim edge, and $d_{\mathcal{D}_i \cup \mathcal{D}_0}(e) \leq 2\lceil d_{\mathcal{C}_i \cup \mathcal{C}_0}(e)/4\rceil$, if e is a spoke edge. Let $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup ... \cup \mathcal{D}_\ell$. It follows that $\mathcal{D}$ is a certificate for the truncatability of $\mathcal{C}$, a contradiction.

Next, suppose $E(u) \notin \mathcal{C}$. This time we partition $V - \{u\}$ into *blocks* such that each block consists of either a vertex in Z or the two ends of a matching edge. Let B_1, B_2, ..., B_ℓ be the blocks, ordered as they appear on the cycle $W_n - u$. Let $i \in \{1, 2, ..., \ell\}$. Let $\mathcal{C}_i$ consist of all cuts in $\mathcal{C}$ of the form $E(x)$ $(x \in B_i)$ or $E(xy)$ $(x, y \in B_i)$. It is clear that $(\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_\ell)$ is a partition of $\mathcal{C}$. We also define a partition $(\mathcal{C}_{i,1}, \mathcal{C}_{i,2})$ of each $\mathcal{C}_i$ (see Figure 7.2). If $B_i = \{z\}$, let $\mathcal{C}_{i,1} = \mathcal{C}_i$. If $B_i = \{x, y\}$, let us assume, without loss of generality, that $r(x) \equiv 0$ and $r(y) \equiv 2 \pmod 4$. If $m_{\mathcal{C}}(E(x)) = 3$, let

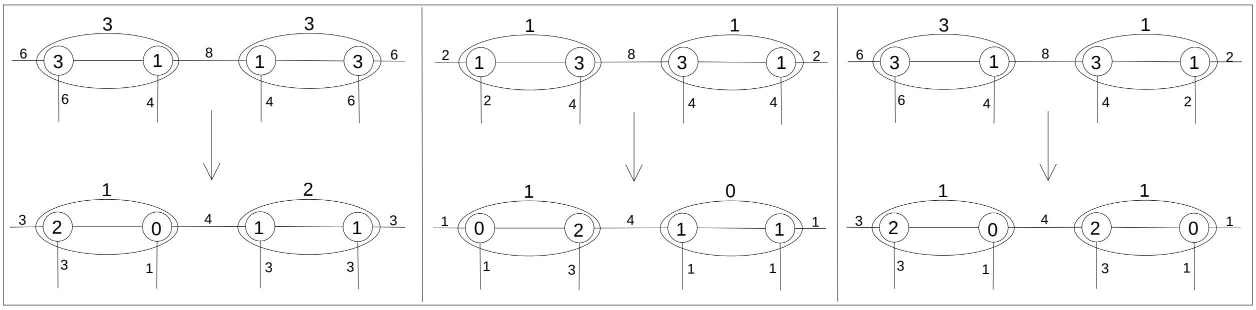

Figure 7.1: The definition of each $\mathcal{C}_i \to \mathcal{D}_i$.

$\mathcal{C}_{i,1} = \{E(x), E(y), E(xy)\}$; if $m_{\mathcal{C}}(E(x)) = 1$, let $\mathcal{C}_{i,1} = \{E(y), E(y), E(xy), E(xy)\}$. Since $\mathcal{C}$ has at least one big cut, it follows that there is at least one 0-edge. Without loss of generality, we assume that the edge between B_1 and B_p is a 0-edge. Let $\mathcal{D}$ be the union of $\mathcal{C}_{i,j_i}$, where $j_i \in \{1, 2\}$ with $j_i + i$ even, over all $i \in \{1, 2, ..., \ell\}$. We verify that $\mathcal{D}$ is a certificate for the truncatability of $\mathcal{C}$.

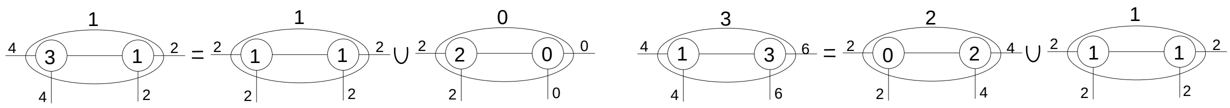

Figure 7.2: $\mathcal{C}_i$ is partitioned into $(\mathcal{C}_{i,1}, \mathcal{C}_{i,2})$.

We will partition $\mathcal{D}$ into groups and consider each group separately. Let Q be a component of $W_n \backslash F_0 - u$, where F_0 is the set of 0-edges. Clearly, Q is a path. Moreover, $V(Q)$ can be expressed as $\{v_q, v_{q+1}, ..., v_{q+p+1}\}$ such that $p \geq 2$, $v_q v_{q+1}$, $v_{q+p} v_{q+p+1} \in M$, $r(v_q) \equiv r(v_{q+p+1}) \equiv 0$ and $r(v_{q+i}) \equiv 2 \pmod 4$, for all $i \in \{1, 2, ..., p\}$. Let B_{t+1}, B_{t+2}, ..., B_{t+p} be the blocks that are contained in $V(Q)$. Let $\mathcal{C}(Q) = \cup_{i=1}^{p} \mathcal{C}_{t+i}$ and $\mathcal{D}(Q) = \mathcal{D} \cap \mathcal{C}(Q)$. It follows from the choice of $\mathcal{D}$ that $|\mathcal{D}(Q)| = |\mathcal{C}(Q)|/2$ (notice that we only need to check this for $p = 2$ or 3, which can be done by inspection). Similarly, for any edge e that is incident with at least one vertex of $V(Q)$, it is routine to verify that $d_{\mathcal{D}(Q)}(e) = d_{\mathcal{C}(Q)}(e)/2$, if e is a rim edge, and $d_{\mathcal{D}(Q)}(e) \leq 2\lceil d_{\mathcal{C}(Q)}(e)/4 \rceil$, if e is a spoke edge. Therefore, $\mathcal{D}$ is a certificate for the truncatability of $\mathcal{C}$, a contradiction. ■

8 Small graphs – Completing the proof of Lemma 1.1

To complete our proof of Lemma 1.1, it remains to consider $G_1, G_2, G_3, G_4, G_5, G_6$, the six graphs shown in Figure 3.2. Let G_2^+ be obtained from G_2 by adding an edge between the two neighbors of the only degree-two vertex. Let G_3^- and G_4^- be the two graphs illustrated in Figure 8.1. Let J_{6a} and J_{6b} be obtained from K_6 by deleting three edges that form a triangle or a star, respectively.

The following statement is easy to verify and hence its proof is omitted.

Lemma 8.1 *Let $G = (V, E)$ be obtained from some G_i by contracting edges. Then*
(i) if $|V| = 9$, then $G = G_6$;
(ii) if $|V| = 8$, then $G = G_3$, $G = G_4$, or G is a subgraph of G_5;

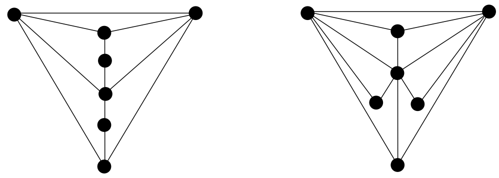

Figure 8.1: G_3^- and G_4^-.

(iii) if $|V| = 7$*, then* $\overline{G}$ *is a subgraph of* G_1, G_2^+, G_3^-*, or* G_4^-*;*
(iv) if $|V| = 6$*, then* $\overline{G}$ *is a subgraph of* J_{6a} *or* J_{6b}*.* ■

In this section, we prove that every compact collection of cuts of a graph listed in Lemma 8.1 must be truncatable. In fact, we shall show that, with only one exception, all these collections are strongly truncatable. Let $M = \{x_iy_i : 1 \le i \le k\}$ be a matching of a graph $G = (V, E)$. Let $Z = V - \{x_i, y_i : 1 \le i \le k\}$. A collection $\mathcal{C}$ of cuts of G is *M-generated* if every big cut in $\mathcal{C}$ has the form $E(x_iy_i)$, every cut $E(x_iy_i)$ is in $\mathcal{C}$, and $m_{\mathcal{C}}$ satisfies the four conclusions in Lemma 6.10. For $i = 1, 2$, let M_i be a matching of a connected graph H_i. We define $(H_1, M_1) \preceq (H_2, M_2)$ if H_1 is isomorphic, under an isomorphism σ, to a spanning subgraph of H_2 such that $\sigma(M_1) = M_2$ and very edge in $E(H_2) - \sigma(E(H_1))$ is incident with at least one edge in M_2. Let $\mathcal{C}_1$ be a collection of cuts of H_1. Let $\mathcal{C}_2$ be the collection $\{E_{H_2}(\sigma(X_C), \sigma(Y_C)) : C \in \mathcal{C}_1\}$ of cuts of H_2.

Lemma 8.2 *If* $\mathcal{C}_1$ *is* M_1*-generated, then* $\mathcal{C}_2$ *is* M_2*-generated.*

Proof. Clearly, we only need to verify that $\mathcal{C}_2$ satisfies the four conclusion in Lemma 6.10. The first three are obvious, since $\mathcal{C}_1$ satisfies them. The last one follows from $\sigma(H_1[Z_1]) = H_2[Z_2]$, where Z_i $(i = 1, 2)$ is the set of vertices of H_i that are not incident with any edge in M_i. ■

In what follows, let $G = (V, E) \in \{J_{6a}, J_{6b}, G_1, G_2^+, G_3^-, G_4^-, G_3, G_4, G_5, G_6\}$, which consists of graphs listed in Lemma 8.2. We assume that $M = \{e_i = x_iy_i : 1 \le i \le k\}$ is a matching of G with $3 \le k \le 4$. Let $Z = V - \{x_i, y_i : 1 \le i \le k\}$. We examine how M can be placed in G.

Notice that, up to isomorphism, there is only one way to choose a perfect matching in J_{6a} and J_{6b}. Let us name the vertices of these two graphs by $\{x_i, y_i : 1 \le i \le 3\}$ such that M is a perfect matching. Then the next lemma follows from the definition of $\preceq$.

Lemma 8.3 *If* $|V| = 6$*, then* $(G, M) \preceq (J_i, M)$*, for some* $i \in \{6a, 6b\}$*.* ■

Lemma 8.4 *Let* J_{7a}*,* J_{7b}*, and* J_{7c} *be defined in Figure 8.2. If* $|V| = 7$*, then* $(G, M) \preceq (J_i, M)$*, for some* $i \in \{7a, 7b, 7c\}$*.*

Proof. Since $|V| = 7$, Z has only one vertex, say z. Let us consider a vertex v_0 of degree two. Let v_1 and v_2 be its only two neighbors. If $z \notin \{v_0, v_1, v_2\}$, then $\{v_0, v_1, v_2\} \subseteq \{x_i, y_i, x_j, y_j\}$, for some distinct $i, j \in \{1, 2, 3\}$, which implies $(G, M) \preceq (J_{7a}, M)$. Therefore, we may assume that $z \in \{v_0, v_1, v_2\}$. Since v_0 can be any vertex of degree two, we conclude that $G \neq G_1$. If $G = G_3^-$ or G_4^-, then z must be the vertex adjacent to both degree-two vertices. In the first case, $(G, M) \preceq (J_{7c}, M)$. The second case won't happen since $G_4^- - z$ has no perfect matching. Finally, we consider $G = G_2^+$. There are three possible choices for z. It is straightforward to see that $(G, M) \preceq (J_{7c}, M)$ if z has degree six, and $(G, M) \preceq (J_{7b}, M)$ in the other two cases. ■

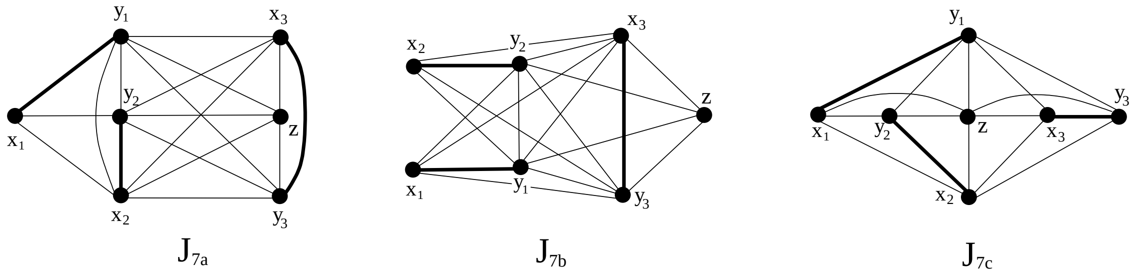

Figure 8.2: Maximal graphs on seven vertices.

Lemma 8.5 *Let J_{8a}, J_{8b}, and J_{8c} be defined in Figure 8.3. (Notice that $J_{8c} = G_3$.) If $|V| = 8$ and $k = 3$, then $(G, M) \preceq (J_{8a}, M)$ or (J_{8b}, M), unless $(G, M) = (J_{8c}, M)$.*

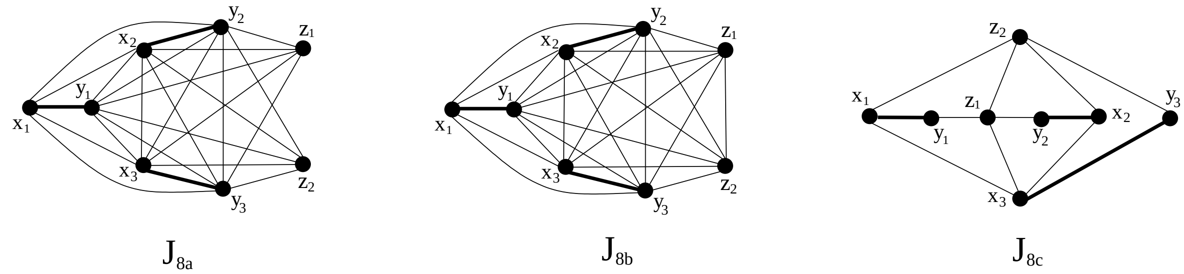

Figure 8.3: Maximal graphs on eight vertices.

Proof. Since $|V| = 8$, we have $|Z| = 2$. Let N_Z be the set of vertices that are adjacent to at least one vertex in Z. If $Z \cup N_Z \neq V$, then it is clear that $(G, M) \preceq (J_i, M)$, for some $i \in \{7a, 7b\}$. Hence we may assume that $Z \cup N_Z = V$. It is routine to verify that $G \neq G_5$. In addition, $G \neq G_4$, because otherwise, Z would consist of two nonadjacent vertices, one with degree two and one with degree four, which implies $G - Z$ has no perfect matching, a contradiction. Therefore, $G = G_3$ and Z consists of two vertices of degree four, including the one that is adjacent to two vertices of degree two. In this case, $G - Z$ has a unique perfect matching, which implies $(G, M) = (J_{7c}, M)$. ■

Lemma 8.6 *Let J_{84} be obtained from from the complete graph on $\{x_i, y_i : i = 1, 2, 3, 4\}$ by deleting four edges x_1x_2, x_2x_3, x_3x_4, and x_4y_1. If $|V| = 8$ and $k = 4$, then $(G, M) \preceq (J_{84}, M)$.*

Proof. Since G_4 has no perfect matching, as deleting the three vertices of degree four results in five isolated vertices, we conclude that $G = G_3$ or G_5.

Suppose $G = G_3$. Let x_2 be the vertex of degree two, for which both of its neighbors have degree four. By symmetry, we may name any of these two neighbors y_2. Notice that, one of the matching edges is between a degree three vertex, say x_1, and a degree two vertex, say y_1. Let x_3 be the other degree two vertex and let x_4 be the other degree four vertex. Then $(G, M) \preceq (J_{84}, M)$.

Next, suppose $G = G_5$. Observe that one of the matching edges is between a degree two vertex, say x_1, and a degree-three vertex, say y_1. Let x_4 be the other vertex of degree-two. Clearly, the other neighbor of x_1 is incident with another matching edge, so may name it x_3. At this point, a simple case analysis shows that the degree-three vertex that is adjacent with y_1 is incident with the last matching edge, so we can name it x_2. Again, we have $(G, M) \preceq (J_{84}, M)$. ■

Lemma 8.7 *Let J_{9a}, J_{9b}, and J_{9c} be obtained from the complete graph on $\{x_i, y_i, z_i : i = 1, 2, 3\}$ by deleting three, two, or one edge within $\{z_1, z_2, z_3\}$, respectively. If $|V| = 9$ and $k = 3$, then $(G, M) \preceq (J_i, M)$, for some $i \in \{9a, 9b, 9c\}$.*

Proof. Since G has no triangle, $G(\{z_1, z_2, z_3\})$ has 0, 1, or 2 edges, which proves the result. ■

Lemma 8.8 *Let J_{94} be defined in Figure 8.4. If $|V| = 9$ and $k = 4$, then $(G, M) \preceq (J_{94}, M)$.*

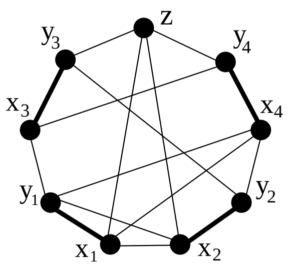

Figure 8.4: Maximal graph J_{94}.

Proof. Let z be the only vertex in Z. If z has degree two, then, up to isomorphism, there is only one way to place the matching. Let x_2, y_4 be the two neighbors of z. Let y_1 be other neighbor of x_2 and let x_3 be other neighbor of y_4. Then it is easy to see that $(G, M) \preceq (J_{94}, M)$. If z has degree three, then there is again only one way to place the matching, up to isomorphic. Let y_4 be the degree-two vertex that is adjacent with z. Let x_2, x_3 be the other two vertices of degree two such that y_2x_4 and y_3z are edges of G. Let x_1 be the other neighbor of z. Then it is easy to see that $(G, M) \preceq (J_{94}, M)$. ■

Lemma 8.9 *Let $J \in \{J_{6a}, J_{6b}, J_{7a}, J_{7b}, J_{7c}, J_{8a}, J_{8b}, J_{84}, J_{9a}, J_{9b}, J_{9c}, J_{94}\}$ and let M be the corresponding matching defined in Lemmas 8.3-8.8. If a collection $\mathcal{C}$ of cuts of J is M-generated, then $\mathcal{C}$ is strongly truncatable.*

Proof. This is the part that we have to use computer. Our program generates all multiplicity functions $m_{\mathcal{C}}$, according to Lemma 6.10, and verifies that the LP in Lemma 5.2(ii) has an integral optimal solution. Therefore, the result follows from Lemma 5.2(ii). ■

Lemma 8.10 *Let M be the matching of J_{8c} defined in Figure 8.3 and let $\mathcal{C}$ be an M-generated collection of cuts of J_{8c}. Then $\mathcal{C}$ is truncatable.*

Proof. Again, we use computer to generates all multiplicity functions $m_{\mathcal{C}}$, according to Lemma 6.10, and verifies that the LP in Lemma 5.2(i) has an integral optimal solution. Therefore, the result follows from Lemma 5.2(i). ■

Remark. $G_3 = J_{8c}$ is not strongly truncatable. To see this, we define an M-generated collection $\mathcal{C}$ with $m_{\mathcal{C}}(E(x_1)) = m_{\mathcal{C}}(E(x_2)) = m_{\mathcal{C}}(E(x_3)) = 1$, $m_{\mathcal{C}}(E(y_1)) = m_{\mathcal{C}}(E(y_2)) = m_{\mathcal{C}}(E(y_3)) = 3$, $m_{\mathcal{C}}(E(x_1y_1)) = m_{\mathcal{C}}(E(x_2y_2)) = 1$, $m_{\mathcal{C}}(E(x_3y_3)) = 3$, $m_{\mathcal{C}}(E(z_1)) = 0$, and $m_{\mathcal{C}}(E(z_2)) = 2$. Then $|\mathcal{C}| = 19$. However, for any collections $\mathcal{D}$ of cuts with $E(z_2) \in \mathcal{D}$, if $d_{\mathcal{D}}(e) \leq 2\lceil d_{\mathcal{C}}(e)/4\rceil$, for all $e \in E(G_3)$, we always have $|\mathcal{D}| < 10$.

Proof of Lemma 1.1. Let H be a graph that contains neither P nor K^* as a minor. We need to show that H is good. It follows from the definition that H is good if and only if its simplification $\overline{H}$ is good, so we may assume that H is simple. Then, by Theorems 3.1, 4.1, and 4.2, we may further assume $H \in \mathcal{G}_0 \cup \mathcal{G}_1$. Finally, since each graph in $\mathcal{G}_0$ is a connected minor of $W_4 \in \mathcal{G}_1$, by Theorem 3.2, we may assume that H is a connected minor of a graph listed in Figure 3.2.

By Lemma 5.1, we only need to show that H is truncatable. If H is a minor of $K_{3,n}$ or W_n, then the result follows from Lemmas 7.1 and 7.2. So H is minor of some G_i $(1 \leq i \leq 5)$. By Lemma 5.5(i), we may assume that H is obtained from G_i by contracting edges. Suppose H is not truncatable. Notice that G_i is ps-connected, by Lemma 6.6, H is also ps-connected. Therefore, by By Lemma 6.7, H contains a non-truncatable pair $(G, \mathcal{C})$ such that $\mathcal{C}$ is compact.

Clearly, $(G, \mathcal{C})$ is not strongly truncatable. Let $(G', \mathcal{C}')$ be obtained from $(G, \mathcal{C})$ by repeatedly contracting contractable edges, until no more edge is contractable. By Lemmas 6.8(ii) and 5.4(ii), $(G', \mathcal{C}')$ satisfies all hypotheses in Lemma 6.10. Let M' be the set of edge $xy \in E(G')$ such that $E(xy)$ is a big cut of G'. By (3d) and Lemma 6.9(ii), M' is a matching of size at least three. In addition, by Lemma 6.10, $\mathcal{C}'$ is M'-generated. Since H is obtained from some G_i by contracting edges, we deduce from Lemmas 8.1 and 8.3-8.8 that either $(G', M') = (J_{8c}, M)$ or $(\overline{G'}, M') \preceq (J, M)$, for some $J \in \{J_{6a}, J_{6b}, J_{7a}, J_{7b}, J_{7c}, J_{8a}, J_{8b}, J_{84}, J_{9a}, J_{9b}, J_{9c}, J_{94}\}$, where M is the corresponding matching defined in Lemmas 8.3-8.8. In the first case, since $J_{8c} = G_3$, which is not a minor of any other G_i, it follows that $(G', \mathcal{C}') = (G, \mathcal{C})$. Therefore, by Lemma 8.10, $\mathcal{C} = \mathcal{C}'$ is truncatable, a contradiction. In the second case, to simplify our notation, let us assume that $\overline{G'}$ *is* a subgraph of J with $M' = M$. Let $\mathcal{D} = \{E_J(X_C, Y_C) : C \in \overline{\mathcal{C}'}\}$. By Lemma 8.2, $\mathcal{D}$ is an M-generated collection of cuts of J. Then, by Lemma 8.9, $\mathcal{D}$ is strongly truncatable, which implies, by Lemma 5.5(ii) and Lemma 5.3, $\mathcal{C}'$ is strongly truncatable, a contradiction. ■

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