Does market demand volatility facilitate collusion?

Kit Pong Wong[*]

School of Economics and Finance, University of Hong Kong, Pokfulam Road, Hong Kong

Abstract

This paper develops a real options model of a price-setting cartel under uncertainty to examine whether market demand volatility facilitates collusion or not. We show that there is a critical level of market demand (the optimal defection trigger) above which firms find it desirable to defect from the cartel. We show further that an increase in the underlying market demand uncertainty has two opposing effects on the optimal defection trigger. First, the increased market demand volatility gives rise to the usual positive effect on option value that lifts up the optimal defection trigger. Second, the increased market demand volatility calls for an upward adjustment of the discount rate and thus creates a negative effect on option value that pushes down the optimal defection trigger. We show that the negative effect dominates (is dominated by) the positive effect when the underlying market demand uncertainty is trivial (significant), thereby rendering a U-shaped pattern of the optimal defection trigger against the market demand volatility.

JEL classification: G13; L13

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1. Introduction

Market demand volatility plays a pivotal role in motivating cartel formations. Prominent examples include the Organization of Petroleum Exporting Countries and the DeBeers controlled international diamond cartel. The former is established with the motive to reduce volatility (Plourde and Watkins, 1998), while the latter is founded so as to prevent volatility (Kretschmer, 1998).

The purpose of this paper is to examine how market demand volatility affects collusive behavior in general, and the timing of cartel defections in particular. To this end, we follow

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[*]Tel.: +852 2859 1044; fax: +852 2548 1152.

E-mail address: kpwong@econ.hku.hk.

Hassan (2006) to develop a real options model of a price-setting cartel under uncertainty (Bagwell and Staiger, 1997; Rotemberg and Saloner, 1986). The model features an industry of symmetric firms that join a cartel and charge the same collusive price in a market where demand fluctuations are governed by a geometric Brownian motion. Unlike Hassan (2006), we follow Sarkar (2000) and Wong (2007) to employ the single-factor intertemporal capital asset pricing model (CAPM) of Merton (1973a) to determine the risk-adjusted rate of return on each firm's profit flows. This approach allows us to distinguish changes in systematic risk from changes in idiosyncratic risk. Since it is reasonable to believe that the underlying market demand uncertainty is unlikely to be purely idiosyncratic in nature, such a distinction is needed for a better understanding of the relationship between market demand volatility and defection timing.

In our model, each firm possesses an option to defect from the cartel, which can be exercised at any time by cutting its price and paying an irreversible cost. The defector benefits from capturing the entire market for a limited time period, after which all other firms retaliate and revert forever to the competitive pricing. The defection option as such resembles a perpetual American call option with an exercise price set equal to the irreversible cost and the underlying asset being the incremental gain from defection. The optimal exercise rule that maximizes the value of the defection option is to defect from the cartel at the first passage time when the market demand reaches a critical level from below, which defines the optimal defection trigger. The incentive to defect from the cartel is thus pro-cyclical. As the market demand grows over time, the cartel becomes increasingly fragile and eventually breaks down with a concomitant price war.

We show that an increase in the underlying market demand uncertainty affects the optimal defection trigger in a non-monotonic manner. There are two opposing forces in effect. First, the increased market demand volatility gives rise to the usual positive effect on option value (Merton, 1973b) that makes waiting beneficial. This positive effect induces each firm in the cartel to raise the optimal defection trigger, as is shown in Hassan (2006). Second, in accord with the CAPM, the increased market demand volatility calls for an upward adjustment of the discount rate and thus creates a negative effect on option value

that makes waiting costly. This negative effect induces each firm in the cartel to lower the optimal defection trigger. When the underlying market demand uncertainty is trivial (significant), we show that the negative effect dominates (is dominated by) the positive effect, thereby rendering a U-shaped pattern of the optimal defection trigger against the market demand volatility. To wit, market demand volatility does not necessarily facilitate collusion, especially for cartels that operate in relatively stable markets.

The rest of this paper is organized as follows. The next section delineates the model and characterizes the optimal defection trigger. Section 3 examines how the optimal defection trigger responds to an increase in the underlying market demand uncertainty. The final section concludes.

2. The model

Consider an industry with N symmetric firms that produce a single homogeneous good in continuous time, where $N \geq 2$ and time is indexed by $t \in [0, \infty)$. At any time t, the firms compete in prices and produce at the same constant marginal cost, $C \geq 0$, under no capacity constraints. The market demand for the good is given by $X(t)D(P)$, where $X(t)$ is the contemporary mass of consumers, $D(P)$ is the identical demand function for each consumer with $D'(P) < 0$, and P is the lowest price charged by the firms. If all the firms set the same price, $P \geq C$, each of them will supply $X(t)D(P)/N$ to the market and earn the profit, $X(t)\Pi(P)/N$, where $\Pi(P) = (P - C)D(P)$. It is well-known that the unique Bertrand-Nash equilibrium in the one-shot pricing-stage game has all the firms setting $P = C$ and thereby earning zero profit.

We introduce uncertainty to the market demand by assuming that $X(t)$ evolves over time according to the following geometric Brownian motion:

$$\mathrm{d}X(t) = \alpha X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}Z(t), \tag{1}$$

where $\mathrm{d}Z(t)$ is the increment of a standard Wiener process, and $\alpha > 0$ and $\sigma > 0$ are the constant drift rate (expected growth rate) and volatility (standard derivation) per unit time, respectively. Eq. (1) implies that the current value of the state variable is known, but the future values are log-normally distributed with a variance that grows linearly with the time horizon. We denote $X_0 > 0$ as the initial mass of consumers at $t = 0$ and $r > 0$ as the constant riskless rate of interest per unit time.

We assume that the stochastic fluctuations of $X(t)$ are spanned by assets traded in complete financial markets, where risk-adjusted rates of return on financial assets are determined by the single-factor intertemporal capital asset pricing model (CAPM) of Merton (1973a). Let $Y(t)$ be the price of an asset or portfolio of assets perfectly correlated with $X(t)$, and denote by $\rho > 0$ as the correlation of $Y(t)$ with the market portfolio. Then, $Y(t)$ evolves over time according to the following geometric Brownian motion:

$$\mathrm{d}Y(t) = (r + \lambda\rho\sigma)Y(t)\mathrm{d}t + \sigma Y(t)\mathrm{d}Z(t), \tag{2}$$

where $\lambda > 0$ is the constant market price of risk per unit time, and $r + \lambda\rho\sigma$ is the risk-adjusted rate of return on $Y(t)$ according to the CAPM. Let $\delta = r + \lambda\rho\sigma - \alpha > 0$ be the return shortfall or convenience yield on $X(t)$.

Since the firms interact among themselves repeatedly in an infinite horizon, the Folk Theorem applies in that the firms can form a cartel to sustain collusive outcomes that strictly dominate the one-shot Bertrand-Nash equilibrium outcome, even when binding contracts cannot be signed. Denote P_m as the unique joint profit-maximizing price, i.e., P_m maximizes $\Pi(P)$. Following Hassan (2006), we consider the simplest mechanism that sustains collusive pricing at P_m from $t = 0$ to some (uncertain) defection time in the future, using trigger strategies as in Friedman (1971). Specifically, at $t = 0$, each firm agrees to maintain the collusive price, P_m, for as long as all others follow suit. If a firm deviates by cutting its price by an arbitrarily small amount below P_m, the defector is able to capture the entire market and earn approximately $X(t)\Pi(P_m)$ for a finite time period, $T > 0$. The other firms then retaliate and revert forever to the one-shot Bertrand-Nash equilibrium, thereby inducing the defector to do the same. T as such measures the retaliation lag. The

defector has to incur an irreversible cost, $I > 0$, at the instant when the defection from the cartel occurs.

At any time t, the market value of a firm if all the firms stay in the cartel is given by

$$V_c[X(t)] = \mathrm{E}\left[\int_t^\infty e^{-(r+\lambda\rho\sigma)(\tau-t)}\frac{X(\tau)\Pi(P_m)}{N}\,\mathrm{d}\tau\Big|X_{(t)}\right] = \frac{X(t)\Pi(P_m)}{\delta N}, \qquad (3)$$

where $\mathrm{E}(\cdot)$ is the expectation operator and $\delta = r + \lambda\rho\sigma - \alpha > 0$. On the other hand, if the firm deviates at that time, then during the period from time t to $t+T$, the defector earns a defection rent equal to the monopoly profit. At time $t+T$, the other firms retaliate and the cartel breaks down, with consequent competitive pricing at the marginal cost thereafter. Hence, the market value of the defector at time t is given by

$$V_d[X(t)] = \mathrm{E}\left[\int_t^T e^{-(r+\lambda\rho\sigma)(\tau-t)}X(\tau)\Pi(P_m)\,\mathrm{d}\tau\Big|X_{(t)}\right] = \frac{(1-e^{-\delta T})X(t)\Pi(P_m)}{\delta}. \quad (4)$$

The incremental gain from defection, in expected present value terms at time t, is given by

$$G[X(t)] = V_d[X(t)] - V_c[X(t)] = \left(1 - e^{-\delta T} - \frac{1}{N}\right)\frac{X(t)\Pi(P_m)}{\delta}, \qquad (5)$$

where the second equality follows from Eqs. (3) and (4). It is evident from Eq. (5) that a necessary condition for defection to occur is that $1 - e^{-\delta T} - 1/N > 0$ or, equivalently,

$$T > \frac{1}{\delta}\ln\left(\frac{N}{N-1}\right). \qquad (6)$$

Condition (6) says that the retaliation lag during which the defector receives the monopoly profit has to be long enough for defection to be potentially attractive, which is assumed to hold throughout the paper. Inspection of Eq. (5) also reveals that the incremental gain from defection increases with the market demand gauged by $X(t)$, thereby rendering pro-cyclical incentives to defect. As the market demand keeps on growing, the cartel becomes increasingly fragile and eventually breaks down with a concomitant price war.

Let $F(X_0)$ be the value of the option to defect from the cartel (hereafter referred to as the defection option), evaluated at $t = 0$. The defection option can be viewed as a perpetual

American call option with an exercise price set equal to the irreversible cost, I, where the underlying asset is the incremental gain from defection, $G[X(t)]$. Thus, we have

$$F(X_0) = \max_{t \geq 0} \ \mathrm{E}\Big\{e^{-(r+\lambda\rho\sigma)t} \max\{G[X(t)] - I, 0\}\Big|X(0) = X_0\Big\}. \tag{7}$$

To solve the optimal stopping time on the right-hand side of Eq. (7) is tantamount to solving the critical mass of consumers, X^*, that triggers the exercise of the defection option. Using the standard contingent-claim approach (see Appendix A), the value of the defection option at time t, $F[X(t)]$, must satisfy the following differential equation:

$$\frac{1}{2}\sigma^2 X(t)^2 F''[X(t)] + (r - \delta)X(t)F'[X(t)] - rF[X(t)] = 0, \tag{8}$$

for all $X(t) \in [0, X^*]$, subject to the following three boundary conditions:

$$F(0) = 0, \tag{9}$$

$$F(X^*) = G(X^*) - I, \tag{10}$$

and

$$F'(X^*) = G'(X^*). \tag{11}$$

The first boundary condition, Eq. (9), simply reflects the fact that zero is an absorbing barrier for the geometric Brownian motion in Eq. (1). The second boundary condition, Eq. (10), is the value-matching condition that ensures the value of the defection option equal to the incremental gain from defection net of the irreversible cost at the instant when the option is exercised. The third boundary condition, Eq. (11), is the smooth-pasting or high-contact condition such that the optimal defection trigger, X^*, is the one that maximizes the value of the defection option.

Eq. (8) is a second-order linear homogeneous differential equation. The general solution to Eq. (8) takes the form of a power function: AX^β, where A is a constant to be determined and β is a solution to the following quadratic equation:

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r = 0. \tag{12}$$

There are two roots for Eq. (12), one positive and the other negative. Eq. (9) implies that the coefficient for the negative β must be zero. Thus, we can ignore the negative solution for β so that

$$\beta = \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{r - \delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{13}$$

Note that Eq. (12) can be written as

$$(\beta - 1)\left(\frac{1}{2}\sigma^2\beta + r\right) = \delta\beta. \tag{14}$$

Since $\delta > 0$, it is evident from Eq. (14) that $\beta > 1$.

Using $F(X) = AX^\beta$ and Eq. (5), we solve Eqs. (10) and (11) to yield

$$X^* = \left(\frac{\beta}{\beta - 1}\right)\left(1 - e^{-\delta T} - \frac{1}{N}\right)^{-1}\frac{\delta I}{\Pi(P_m)}, \tag{15}$$

and $A = [G(X^*) - I]/X^{*\beta}$. Thus, the value of the defection option at $t = 0$ is given by

$$F(X_0) = \begin{cases} [G(X^*) - I](X_0/X^*)^\beta & \text{if } X_0 < X^*, \\ G(X_0) - I & \text{if } X_0 \geq X^*, \end{cases} \tag{16}$$

where $(X_0/X^*)^\beta$ is the stochastic discount factor that accounts for both the timing and the probability of one dollar received at the first instant when the optimal defection trigger, X^*, is reached from below. To ensure some positive value to the defection option, and thus the cartel is sustainable for at least a finite time period, in the sequel we assume that $X_0 < X^*$.

3. Market demand volatility as a collusion facilitating factor

In this section, we study how the market demand volatility affects the optimal defection trigger, X^*. We follow Sarkar (2000) and Wong (2007) to model an increase in the underlying market demand uncertainty as an increase in σ, taking all other parameters, r, λ, ρ, and α, as constants. In other words, the increase in σ contains a systematic risk component

that affects the convenience yield, δ. In accord with the CAPM, we have $\partial\delta/\partial\sigma = \lambda\rho > 0$. In contrast, Hassan (2006) considers another type of increased uncertainty commonly used in the extant literature (see McDonald and Siegel, 1986; Dixit and Pindyck, 1994), wherein the convenience yield is held fixed when σ varies, i.e., $\partial\delta/\partial\sigma = 0$. In his case, the increase in σ has only an idiosyncratic risk component. Since it is reasonable to believe that an increase in the underlying market demand uncertainty is unlikely to be purely idiosyncratic in nature, in this regard our approach offers a better way to model such a change.

Differentiating Eq. (12) with respect to σ yields

$$\frac{\mathrm{d}\beta}{\mathrm{d}\sigma} = \frac{\beta}{\sigma^2(\beta - 1/2) + r - \delta}\left[\frac{\mathrm{d}\delta}{\mathrm{d}\sigma} - \sigma(\beta - 1)\right] = \frac{2\beta^2}{\sigma^2\beta^2 + 2r}\left[\frac{\mathrm{d}\delta}{\mathrm{d}\sigma} - \sigma(\beta - 1)\right], \qquad (17)$$

where the second equality follows from Eq. (12). Using Eq. (14), we can write Eq. (15) as

$$X^* = \left(\frac{1}{2}\sigma^2\beta + r\right)\left(1 - e^{-\delta T} - \frac{1}{N}\right)^{-1}\frac{I}{\Pi(P_m)}. \qquad (18)$$

Differentiating Eq. (18) with respect to σ yields

$$\frac{\mathrm{d}X^*}{\mathrm{d}\sigma} = X^*\left[\left(\frac{1}{2}\sigma^2\beta + r\right)^{-1}\left(\sigma\beta + \frac{1}{2}\sigma^2\frac{\mathrm{d}\beta}{\mathrm{d}\sigma}\right) - \left(1 - e^{-\delta T} - \frac{1}{N}\right)^{-1}e^{-\delta T}T\frac{\mathrm{d}\delta}{\mathrm{d}\sigma}\right]$$

$$= X^*\left\{\frac{2\sigma\beta}{\sigma^2\beta^2 + 2r} + \left[\frac{2\sigma^2\beta^2}{(\sigma^2\beta + 2r)(\sigma^2\beta^2 + 2r)} - \left(1 - e^{-\delta T} - \frac{1}{N}\right)^{-1}e^{-\delta T}T\right]\frac{\mathrm{d}\delta}{\mathrm{d}\sigma}\right\}, \quad (19)$$

where the second equality follows from Eq. (17).

Hassan (2006) considers the case that $\mathrm{d}\delta/\mathrm{d}\sigma = 0$. Inspection of Eq. (19) immediately reveals that $\mathrm{d}X^*/\mathrm{d}\sigma > 0$ in his case. This follows from the fact that an increase in σ enhances the value of the defection option (Merton, 1973b). Each firm as such is induced to raise the optimal threshold mass of consumers, X^*, at which the defection option is exercised, thereby rendering market demand volatility as a collusion facilitating factor.

We consider the alternative case that $\mathrm{d}\delta/\mathrm{d}\sigma = \lambda\rho > 0$. As is shown in the following proposition, X^* is no longer a strictly increasing function of σ in our case.

Proposition 1. *Given that an increase in the underlying market demand uncertainty contains a systematic risk component, i.e., $\mathrm{d}\delta/\mathrm{d}\sigma = \lambda\rho > 0$, if the following condition holds:*

$$\frac{\bar{\sigma} + \lambda\rho + \sqrt{2r}}{\bar{\sigma}\sqrt{2r} + 2r} > \left[1 - e^{-(r+\lambda\rho\bar{\sigma}-\alpha)T} - \frac{1}{N}\right]^{-1} e^{-(r+\lambda\rho\bar{\sigma}-\alpha)T} T\lambda\rho, \tag{20}$$

where $\bar{\sigma} = \sqrt{2\alpha + \lambda^2\rho^2} - \lambda\rho$, there exists a unique point, $\sigma^* \in (0, \bar{\sigma})$, that solves

$$\left.\frac{\mathrm{d}X^*}{\mathrm{d}\sigma}\right|_{\sigma=\sigma^*} = 0, \tag{21}$$

such that the optimal defection trigger, X^*, exhibits a U-shaped pattern against the market demand volatility, σ, with a unique minimum attained at σ^* for all $\sigma \in [0, \bar{\sigma}]$. Otherwise, X^* is strictly decreasing in σ for all $\sigma \in [0, \bar{\sigma})$.

Proof. See Appendix B. □

Condition (20) holds for T reasonably large. To see this, consider the following parameter values taken from Sarkar (2000) and Wong (2007): $\alpha = 1\%$, $r = 4\%$, $\lambda = 0.4$, and $\rho = 0.7$. In this case, we have $\bar{\sigma} = 3.37\%$. If there are $N = 10$ firms in the industry, condition (6) requires that $T > 2.67$ and condition (20) is satisfied for all $T > 15.19$. If there are $N = 100$ firms in the industry, condition (6) requires that $T > 0.25$ and condition (20) is satisfied for all $T > 5.70$.

An immediate implication of Proposition 1 is that market demand volatility does not necessarily facilitate collusion in general, and tends to destroy cartels in relatively stable markets in particular. The intuition of Proposition 1 is as follows. In our case that an increase in the underlying market demand uncertainty contains a systematic risk component, the risk-adjusted rate of return on the incremental gain from defection has to increase according to the CAPM. Since the investment cost remains constant at I, the increase in the discount rate makes defection from the cartel less attractive. Each firm as such is induced to lower the optimal threshold mass of consumers, X^*, at which the defection option is exercised, ceteris paribus. When the market demand volatility is small, it is

evident from Eq. (19) that the positive effect due to the enhanced option value is at best second order. The negative effect due to the increased discount rate, on the other hand, is always first order because the risk-adjusted rate of return is linear in σ in accord with the CAPM. When the market demand volatility becomes greater, condition (20) ensures that the positive effect due to the enhanced option value dominates the negative effect due to the increased discount rate. Such a dominance is driven by the fact that the significance of the positive effect grows with σ, while that of the negative effect declines with σ. This explains why X^* exhibits a U-shaped patten against σ with the unique minimum attained at σ^*.

4. Conclusion

This paper has examined how market demand volatility affects collusive behavior in general, and the timing of cartel defections in particular. We have developed a real options model of a price-setting cartel under uncertainty along the lines of Hassan (2006), with a caveat that the risk-adjusted rate of return on each firm's profit flows is determined by the single-factor intertemporal capital asset pricing model (CAPM) of Merton (1973a). Each firm in the model possesses an option to defect from the cartel, which can be exercised at any time by cutting its price and paying an irreversible cost. The defection option as such resembles a perpetual American call option with an exercise price set equal to the irreversible cost, where the underlying asset is the incremental gain from defection.

We have shown that the optimal exercise rule that maximizes the value of the defection option is to defect from the cartel at the first passage time when the market demand reaches a critical level (the optimal defection trigger) from below, thereby rendering pro-cyclical incentives to defect from the cartel. We have shown further that an increase in the underlying market demand uncertainty creates two opposing effects on the optimal defection trigger: the usual positive effect on option value (Merton, 1973b) and the negative effect on option value due to the upward adjusted discount rate in accord with the CAPM. We have

shown that the negative effect dominates (is dominated by) the positive effect when the market demand volatility is small (large). The relationship between the optimal defection trigger and the market demand volatility thus exhibits a U-shaped pattern, thereby implying that market demand volatility does not necessarily facilitate collusion.

Acknowledgements

Appendix A. Derivation of Eq. (8)

Consider the following dynamic portfolio at time t: (i) Hold the defection option that is worth $F[X(t)]$, and (ii) go short n units of the asset or portfolio of assets that tracks $X(t)$. The value of this portfolio is $F[X(t)] - nY(t)$. The total return from holding the portfolio over a short time interval, $\mathrm{d}t$, is given by

$$\mathrm{d}F[X(t)] - n\mathrm{d}Y(t)$$

$$= F'[X(t)]\mathrm{d}X(t) + \frac{1}{2}F''[X(t)][\mathrm{d}X(t)]^2 - n(r + \lambda\rho\sigma)Y(t)\mathrm{d}t - n\sigma Y(t)\mathrm{d}Z(t)$$

$$= \left\{\alpha X(t)F'[X(t)] + \frac{1}{2}\sigma^2 X(t)^2 F''[X(t)] - n(r + \lambda\rho\sigma)Y(t)\right\}\mathrm{d}t$$

$$+ \{\sigma X(t)F'[X(t)] - n\sigma Y(t)\}\mathrm{d}Z(t), \tag{A.1}$$

where the first equality follows from Ito's Lemma and Eq. (2), and the second equality

follows from Eq. (1) and $[dX(t)]^2 = \sigma^2 X(t)^2 dt$. Substituting $n = X(t)F'[X(t)]/Y(t)$ into Eq. (A.1) yields

$$dF[X(t)] - \frac{X(t)F'[X(t)]}{Y(t)} dY(t) = \left\{ \frac{1}{2}\sigma^2 X(t)^2 F''[X(t)] - \delta X(t)F'[X(t)] \right\} dt, \qquad \text{(A.2)}$$

where $\delta = r + \lambda\rho\sigma - \alpha$. Inspection of Eq. (A.2) reveals that the portfolio is riskless and thus we must have

$$\left\{ \frac{1}{2}\sigma^2 X(t)^2 F''[X(t)] - \delta X(t)F'[X(t)] \right\} dt = r\{F[X(t)] - X(t)F'[X(t)]\} dt, \qquad \text{(A.3)}$$

to rule out arbitrage opportunities. Eliminating dt on both sides of Eq. (A.3) and rearranging terms yields Eq. (8).

Appendix B. Proof of Proposition 1

Using the fact that $d\delta/d\sigma = \lambda\rho > 0$, we can write the expression inside the curly brackets on the right-hand side of Eq. (19) as

$$M(\sigma) = \frac{2\sigma\beta}{\sigma^2\beta^2 + 2r}\left(1 + \frac{\lambda\rho\sigma\beta}{\sigma^2\beta + 2r}\right)$$

$$- \left[1 - e^{-(r+\lambda\rho\sigma-\alpha)T} - \frac{1}{N}\right]^{-1} e^{-(r+\lambda\rho\sigma-\alpha)T} T\lambda\rho. \qquad \text{(A.4)}$$

Differentiating Eq. (A.4) with respect to σ yields

$$M'(\sigma) = \frac{2(2r - \sigma^2\beta^2)}{(\sigma^2\beta^2 + 2r)^2}\left(\beta + \sigma\frac{d\beta}{d\sigma}\right)\left(1 + \frac{\lambda\rho\sigma\beta}{\sigma^2\beta + 2r}\right)$$

$$+ \frac{4r\lambda\rho\sigma\beta}{(\sigma^2\beta + 2r)^2(\sigma^2\beta^2 + 2r)}\left(\beta - \frac{\sigma^2\beta^2}{2r} + \sigma\frac{d\beta}{d\sigma}\right)$$

$$+ \left[1 - e^{-(r+\lambda\rho\sigma-\alpha)T} - \frac{1}{N}\right]^{-2}\left(\frac{N-1}{N}\right) e^{-(r+\lambda\rho\sigma-\alpha)T} T^2\lambda^2\rho^2. \qquad \text{(A.5)}$$

Rearranging terms of Eq. (12) yields

$$2r - \sigma^2\beta^2 = (2\alpha - 2\lambda\rho\sigma - \sigma^2)\beta. \tag{A.6}$$

For all $\sigma \in [0, \bar{\sigma}]$, we have $2\alpha \geq 2\lambda\rho\sigma + \sigma^2$ so that $2\alpha > \sigma^2$ and, from Eq. (A.6), $2r \geq \sigma^2\beta^2$, where the equality holds only at $\sigma = \bar{\sigma}$. Using Eqs. (17) and (A.6), we have

$$\beta + \sigma\frac{d\beta}{d\sigma} = \frac{2r\beta + 2(\lambda\rho + \sigma)\sigma\beta^2 - \sigma^2\beta^3}{\sigma^2\beta^2 + 2r} = \frac{(\sigma^2 + 2\alpha)\beta^2}{\sigma^2\beta^2 + 2r}. \tag{A.7}$$

Substituting Eq. (A.7) into Eq. (A.5) yields

$$M'(\sigma) = \frac{2\beta^2(2r - \sigma^2\beta^2)(\sigma^2 + 2\alpha)(\sigma^2\beta + 2r + \lambda\rho\sigma)}{(\sigma^2\beta + 2r)(\sigma^2\beta^2 + 2r)^3} + \frac{2\lambda\rho\sigma\beta^3(4r\alpha - \sigma^4\beta^2)}{(\sigma^2\beta + 2r)^2(\sigma^2\beta^2 + 2r)^2}$$

$$+ \left[1 - e^{-(r+\lambda\rho\sigma-\alpha)T} - \frac{1}{N}\right]^{-2}\left(\frac{N-1}{N}\right)e^{-(r+\lambda\rho\sigma-\alpha)T}T^2\lambda^2\rho^2. \tag{A.8}$$

For all $\sigma \in [0, \bar{\sigma}]$, we have $2r \geq \sigma^2\beta^2$ and $2\alpha > \sigma^2$ so that $4r\alpha > \sigma^4\beta^2$ and, from Eq. (A.8), $M'(\sigma) > 0$. From Eq. (A.4), we have

$$M(0) = -\left[1 - e^{-(r-\alpha)T} - \frac{1}{N}\right]^{-2}e^{-(r-\alpha)T}T\lambda\rho < 0. \tag{A.9}$$

Condition (20), on the other hand, ensures that $M(\bar{\sigma}) > 0$ since $\bar{\sigma}\beta = \sqrt{2r}$. Hence, we conclude that there exists a unique point, $\sigma^* \in (0, \bar{\sigma})$, such that $M(\sigma^*) = 0$, i.e., σ^* solves Eq. (21). The desired results then follow from the fact that $dX^*/d\sigma < 0$ for all $\sigma \in [0, \sigma^*)$ and $dX^*/d\sigma > 0$ for all $\sigma \in (\sigma^*, \bar{\sigma}]$. If condition (20) is violated, we have $M(\sigma) < 0$ and thus $dX^*/d\sigma < 0$ for all $\sigma \in [0, \bar{\sigma})$.

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