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On Performance Bounds for Timing Estimation under Fading Channels

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***Abstract*—In timing synchronization, the Cramér Rao Bound has been used as performance bounds for timing estimation in AWGN channel. However, the instantaneous CRB for timing estimation in fading channels depends on the channel realizations and may fail to bound the mean square error (MSE) of the estimator because the equivalent signal-to-noise ratio (SNR) is too low. In this paper, we demonstrate that the conventional CRB for timing estimation is no longer valid in fading channels. Furthermore, a new performance bound called Weighted Bayesian CRB (WBCRB) is proposed for the estimation of both single and multiple timing offsets under fading channels. The relationship between the conventional CRB and WBCRB are discussed in details, where numerical results show that the WBCRB is a valid bound for all SNR even under fading channels.**

I. Introduction

Traditionally, Cramér-Rao Bound (CRB) has been widely used as a performance benchmark for timing synchronization [1]-[3]. While CRB is a valid bound for parameters with infinite range, strictly speaking, it is not valid for timing synchronization. This can be seen from the fact that the timing estimation mean square error (MSE) is lower than the CRB at low signal-to-noise ratio (SNR). The reason for this phenomenon is that the derivation of the CRB does not assume any prior information on the parameter, while in fact, the range of the fractional timing offset $[0, 1)$ is informative in that the largest mean square error (MSE) possibly achieved by any timing estimator is on the order of 10^{-1}.

The reason why the conventional CRBs can serve as the performance bounds for synchronization problems in previous studies [1]-[3] is that the conventional CRBs are derived for single user scenario and generally non-fading environments, in which the limitation of the conventional CRBs has not yet begun to show at practical SNR (medium to high SNR) levels. However, for synchronization in fading channels, the conventional CRBs depend on channel realizations and may exceed the maximal possible MSE of the timing parameter estimate even at practical SNR.

In this paper, we apply the concept of the outage probability to the analysis of the conventional CRB and demonstrate that the conventional CRB is no longer valid under fading conditions. Furthermore, the Weighted Bayesian CRB (WBCRB) which incorporates the prior information on the range of the timing offset is derived. WBCRB is more complicated than CRB but it is a valid bound at any SNR even in fading situations.

Notation : The operator diag$(\mathbf{x})$ denotes a diagonal matrix with the elements of $\mathbf{x}$ located on the main diagonal, while $\mathscr{R}(\cdot)$ and $\mathscr{I}(\cdot)$ take the real and imaginary part of the argument respectively. Superscripts $(\cdot)^*$, $(\cdot)^H$ and $(\cdot)^T$ denote the conjugate, conjugate transpose and the transpose operators respectively. Notation $\mathbf{I}$ is the identity matrix and $\mathbb{E}_\theta\{\cdot\}$ assumes the expectation with respect to variable θ. Finally, $||\mathbf{x}||$ represents the L_2 norm of vector $\mathbf{x}$ and $|\cdot|$ takes the modulus of a complex number.

II. System Model

In single carrier systems, the received signal (within $0 \le t \le L_oT$) under flat fading has a complex envelope as follows

$$r(t) = h \cdot \sum_{i=-L_g}^{L_o+L_g-1} d(i)g(t - iT - \epsilon T) + n(t),$$

where h is the complex channel coefficient assumed to be zero mean, circular complex Gaussian random variable with variance σ_h^2. The term $n(t)$ is the zero mean, circular complex additive white Gaussian noise (AWGN) with variance σ_n^2. Notation T is the symbol duration, $d(i)$ is the transmitted training symbol with average symbol energy E; $\epsilon \in [0, 1)$ is the unknown timing offset normalized to the symbol duration and $g(t)$ is the pulse shaping filter. Symbol L_o represents the observation interval while L_g is the approximated effective duration of the tail of $g(t)$ on one side.

Upon reception, the signal is oversampled by a ratio $Q \ge 2$ and thus the sampling interval is $T_s = T/Q$. By stacking L_oQ received samples, the received vector is given by [3]

$$\mathbf{r} = h\,\mathbf{A}_\epsilon \mathbf{d} + \mathbf{n} \tag{1}$$

where

$$\begin{aligned}
\mathbf{r} &\triangleq [r(0), r(T_s), \cdots, r((L_oQ-1)T_s)]^T \\
\mathbf{n} &\triangleq [n(0), n(T_s), \cdots, n((L_oQ-1)T_s)]^T \\
\mathbf{A}_\epsilon &\triangleq [\mathbf{a}_{-L_g}(\epsilon), \cdots, \mathbf{a}_0(\epsilon), \cdots, \mathbf{a}_{L_o+L_g-1}(\epsilon)] \\
\mathbf{a}_i(\epsilon) &\triangleq [g(-iT-\epsilon T), g(-iT+T_s-\epsilon T), \\
&\quad \cdots, g(-iT+(L_oQ-1)T_s-\epsilon T)]^T \\
\mathbf{d} &\triangleq [d(-L_g), \cdots, d(0), \cdots, d(L_o+L_g-1)]^T.
\end{aligned}$$

Note that the phase offsets between the transmitter and receiver are not explicitly specified in the system model (1), because

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it can be incorporated into the unknown channel coefficient while keeping the formulation of the system model unchanged.

III. Cramér-Rao Bound Analysis

With the unknown fading channel h, the CRB is derived as the joint CRB for both the timing parameter ϵ and the channel h. In the following, we first derive the joint CRB for the timing parameter and the channel, and then go on to analyze the effects of fading channel on the CRB for the timing parameter ϵ.

A. Cramér-Rao Bound for Timing Estimation under Fading

The joint CRB for timing parameter ϵ and channel h is obtained by inverting the Fisher Information Matrix (FIM). Let $\boldsymbol{\theta} \triangleq [\epsilon, \mathscr{R}(h), \mathscr{I}(h)]^T$ denote the parameters of interest. Based on the procedure taken in [4], the FIM for ϵ and h is derived as

$$\mathbf{F}(\boldsymbol{\theta}) = \frac{1}{\sigma_n^2} \times \begin{bmatrix} \mathscr{R}(\mathbf{d}^H \mathbf{D}_\epsilon^H \mathbf{D}_\epsilon \mathbf{d}) & \mathscr{R}(h^* \mathbf{d}^H \mathbf{D}_\epsilon^H \mathbf{A}_\epsilon) & \mathscr{I}(h^* \mathbf{d}^H \mathbf{D}_\epsilon^H \mathbf{A}_\epsilon) \\ \mathscr{R}(h \mathbf{A}_\epsilon^H \mathbf{D}_\epsilon \mathbf{d}) & \mathscr{R}(\mathbf{d}^H \mathbf{A}_\epsilon^H \mathbf{A}_\epsilon \mathbf{d}) & \mathscr{I}(\mathbf{d}^H \mathbf{A}_\epsilon^H \mathbf{A}_\epsilon \mathbf{d}) \\ -\mathscr{I}(h \mathbf{A}_\epsilon^H \mathbf{D}_\epsilon \mathbf{d}) & -\mathscr{I}(\mathbf{d}^H \mathbf{A}_\epsilon^H \mathbf{A}_\epsilon \mathbf{d}) & \mathscr{R}(\mathbf{d}^H \mathbf{A}_\epsilon^H \mathbf{A}_\epsilon \mathbf{d}) \end{bmatrix}.$$

Now the CRB for ϵ is obtained by taking the $(1,1)^{\text{th}}$ element in $\mathbf{F}(\boldsymbol{\theta})^{-1}$ and the result is given by

$$\text{CRB}(\epsilon) = \frac{1}{|h|^2} \frac{\sigma_n^2}{\mathbf{d}^H \boldsymbol{\Pi}_\epsilon \mathbf{d}}, \tag{2}$$

where $\boldsymbol{\Pi}_\epsilon = \mathbf{D}_\epsilon^H \mathbf{P}_{\mathbf{d}}^{\perp}(\epsilon) \mathbf{D}_\epsilon$ with $\mathbf{D}_\epsilon = \partial \mathbf{A}_\epsilon / \partial \epsilon$ and

$$\mathbf{P}_{\mathbf{d}}^{\perp}(\epsilon) = \mathbf{I} - \frac{\mathbf{A}_\epsilon \mathbf{d} \mathbf{d}^H \mathbf{A}_\epsilon^H}{\mathbf{d}^H \mathbf{A}_\epsilon^H \mathbf{A}_\epsilon \mathbf{d}}. \tag{3}$$

B. Outage Probability

From (2), it is noted that the CRB depends on the realization of channel h. Therefore, there are some realizations of the channel h that could drive the equivalent SNR to low values and render the Cramér-Rao Bound to exceed the MSE constraint due to the parameter's finite range.

Without loss of generality, the MSE constraint by the finite range is denoted as $\text{MSE}_{\text{max}} = \alpha^2$. For each realization of ϵ, the CRB fails to bound the MSE of the timing parameter if

$$\text{CRB}(\epsilon) > \alpha^2. \tag{4}$$

However, according to (2), the $\text{CRB}(\epsilon)$ also depends on the choice of training $\mathbf{d}$. Therefore, we define the CRB outage condition as

$$\min_{\mathbf{d}} \{\text{CRB}(\epsilon)\} > \alpha^2, \tag{5}$$

meaning that outage occurs when the CRB does not hold, even if the optimal training is used. According to the properties of Rayleigh quotient and $||\mathbf{d}||^2 = (L_o + 2L_g)E$, the minimum of the CRB can be readily obtained as

$$\text{CRB}_{\text{min}}(\epsilon) = \frac{\sigma_n^2}{|h|^2 \lambda_{\text{max}}(\epsilon)(L_o + 2L_g)E},$$

where $\lambda_{\text{max}}(\epsilon)$ represents the largest singular value of the matrix $\boldsymbol{\Pi}_\epsilon$ given ϵ. Hence, an outage is declared when

$$\frac{1}{|h|^2} \frac{1}{\lambda_{\text{max}}(\epsilon)(L_o + 2L_g)\text{SNR}} > \alpha^2, \tag{6}$$

or equivalently

$$|h| < \frac{1}{\alpha} \sqrt{\frac{1}{\lambda_{\text{max}}(\epsilon)(L_o + 2L_g)\text{SNR}}}, \tag{7}$$

where SNR$= E/\sigma_n^2$.

When there is no fading (i.e., AWGN channel), we have $|h| = 1$. Then according to (6), the SNR threshold for the CRB to be applicable under non-fading channels becomes

$$\text{SNR}_{\text{non-fading}} = \frac{1}{\alpha^2 \lambda_{\text{max}}(\epsilon)(L_o + 2L_g)}, \tag{8}$$

which demonstrates that the conventional CRB works at finite SNR (especially when $L_o + 2L_g$ is large) under non-fading channels [1]-[3].

On the other hand, when there is fading, the outage (7) needs to be characterized in a probabilistic way. Since the envelope of the a circular Gaussian complex channel $|h|$ follows a Rayleigh distribution, we can obtain the outage probability according to the cumulative distribution

$$\begin{aligned} &P\left(|h| < \frac{1}{\alpha} \cdot \sqrt{\frac{1}{\lambda_{\text{max}}(\epsilon)(L_o + 2L_g)\text{SNR}}}\right) \\ &= 1 - \exp\left(-\frac{1}{2\alpha^2 \sigma_h^2 \lambda_{\text{max}}(\epsilon)(L_o + 2L_g)\text{SNR}}\right), \end{aligned} \tag{9}$$

where σ_h^2 is the variance of the channel coefficient h. From the outage probability expression (9), it can be observed that

- When the parameter of interest does not have a finite range (i.e., $\alpha^2 \to \infty$), then the outage probability goes to zero, which points out the effect of the range information of the timing parameter and meanwhile verifies that the conventional CRB works for parameters with an infinite range.
- If the SNR$\to \infty$, the probability in (9) also goes to zero. This means that the conventional CRB still works at infinite SNR under fading channels, while at practical SNRs, the outage probability of the CRB is strictly positive (see Section V-B and Fig. 2).

Hence, under fading condition, the estimation performance for timing parameters with finite range cannot be well bounded by the conventional CRB even at practical SNR.

IV. Weighted Bayesian Cramér-Rao Bound

As discussed in Section III, the CRB for timing estimation is not applicable under fading channels due to the fact that the given finite range (i.e., α^2) on the timing estimates serves as an informative prior during estimation [5]. The most common bound that considers the prior information of the parameter of interest is the Bayesian CRB (BCRB). However, as demonstrated in [5], BCRB does not exist for parameters with uniform distribution. In order to derive a valid performance bound for parameters with a given finite range and uniform distribution, the Weighted Bayesian CRB (WBCRB) was introduced and studied in [5].

We hereby derive the WBCRB for timing estimation problems. In the derivation, a more general system model involving multiple timing offsets is considered. This system is of interest in cooperative communication systems [6], [7]. The derived WBCRB can be easily reduced to single timing offset case by setting the number of timing offset to one. Since the conventional CRB is the foundation of the derivation for WBCRB, we will first present the conventional CRB for multiple timing offsets.

A. Conventional CRB for Multiple Timing Offsets

In order to derive the CRB for timing synchronization with multiple timing offsets, the model (1) is extended to a multi-user case with a total number of user K

$$\mathbf{x} = \underbrace{\Big[\mathbf{A}_{\epsilon_1}\mathbf{d}_1,\ \mathbf{A}_{\epsilon_2}\mathbf{d}_2, \cdots, \mathbf{A}_{\epsilon_K}\mathbf{d}_K\Big]}_{\triangleq \boldsymbol{\Omega}_{\boldsymbol{\epsilon}}} \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_K \end{bmatrix}}_{\triangleq \mathbf{h}} + \mathbf{v} \tag{10}$$

where $\mathbf{x} = [x(0), x(T_s), \cdots, x((L_oQ-1)T_s)]^T$ is the received vector and $\mathbf{d}_k = [d_k(-L_g), \cdots, d_k(L_o + L_g - 1)]^T$ is the training sequence from the k^{th} transmitter with average symbol energy E_k, and $\boldsymbol{\epsilon} \triangleq [\epsilon_1, \cdots, \epsilon_K]^T$. The matrix $\mathbf{A}_{\epsilon_k}$ follows the same definition of $\mathbf{A}_{\epsilon}$, and $\mathbf{v}$ contains the discrete noise samples with variance σ_v^2.

In the following, we first derive the joint CRB for timing and channel estimations. Using the general model (10), the FIM for the parameter set $\boldsymbol{\Theta} \triangleq [\boldsymbol{\epsilon}^T, \mathscr{R}(\mathbf{h})^T, \mathscr{I}(\mathbf{h})^T]^T$ is calculated as

$$\mathbf{F}(\boldsymbol{\Theta}) = \frac{1}{\sigma_v^2}\times \begin{bmatrix} \mathscr{R}\{\mathbf{H}^H\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}\mathbf{H}\} & \mathscr{R}\{\mathbf{H}^H\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}\} & \mathscr{I}\{\mathbf{H}^H\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}\} \\ \mathscr{R}\{\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}\mathbf{H}\} & \mathscr{R}\{\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}\} & \mathscr{I}\{\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}\} \\ -\mathscr{I}\{\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}\mathbf{H}\} & -\mathscr{I}\{\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}\} & \mathscr{R}\{\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}\} \end{bmatrix} \tag{11}$$

where $\boldsymbol{\Psi}_{\boldsymbol{\epsilon}} = \big[\mathbf{D}_{\epsilon_1}\mathbf{d}_1,\ \mathbf{D}_{\epsilon_2}\mathbf{d}_2, \cdots, \mathbf{D}_{\epsilon_K}\mathbf{d}_K\big]$ with $\mathbf{D}_{\epsilon_k} = \partial\mathbf{A}_{\epsilon_k}/\partial\epsilon_k$, and $\mathbf{H} = \text{diag}(\mathbf{h})$.

With the FIM in (11), the CRB can be computed through similar mathematical derivations as in [4], and assumes the expression

$$\text{CRB}(\boldsymbol{\epsilon}) = \sigma_v^2 \cdot \left[\mathbf{H}\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}^H\mathbf{P}_{\boldsymbol{\Omega}}^{\perp}(\boldsymbol{\epsilon})\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}\mathbf{H}\right]^{-1}, \tag{12}$$

where $\mathbf{P}_{\boldsymbol{\Omega}}^{\perp}(\boldsymbol{\epsilon}) = \mathbf{I} - \boldsymbol{\Omega}_{\boldsymbol{\epsilon}}(\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Omega}_{\boldsymbol{\epsilon}})^{-1}\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H$.

B. Weighted Bayesian Cramér-Rao Bound

In this subsection, we employ the finite range of the parameters to derive a valid for timing synchronization in the considered system. The WBCRB can be shown to be a valid lower bound at any SNR and is evaluated as [8]

$$\text{WBCRB}(\boldsymbol{\epsilon}) = \mathbb{E}_{\boldsymbol{\epsilon}}\{\mathbf{Q}(\boldsymbol{\epsilon})\}\left\{\mathbb{E}_{\boldsymbol{\epsilon}}\{\mathbf{F}_w(\boldsymbol{\epsilon})\} + \mathbb{E}_{\boldsymbol{\epsilon}}\{\mathbf{P}_w(\boldsymbol{\epsilon})\}\right\}^{-1}\mathbb{E}_{\boldsymbol{\epsilon}}\{\mathbf{Q}(\boldsymbol{\epsilon})\}, \tag{13}$$

where $\mathbf{F}_w(\boldsymbol{\epsilon})$ is the weighted Fisher Information Matrix (FIM) for the timing offset parameters and $\mathbf{P}_w(\boldsymbol{\epsilon})$ is the weighted Prior Information Matrix (PIM), which are defined below in (17) and (16), respectively. The symbol $\mathbf{Q}(\boldsymbol{\epsilon}) \triangleq \text{diag}(q(\epsilon_1), \cdots, q(\epsilon_K))$ represents the weighting matrix, with $q(\epsilon_k)$ being the individual weighting function for the timing offset from the k^{th} transmitter. As suggested by [5], the weighting function $q(\epsilon_k)$ is chosen as

$$q(\epsilon_k) = \begin{cases} \epsilon_k^{\gamma}(1-\epsilon_k)^{\gamma}, & 0 \le \epsilon_k < 1 \\ 0, & \text{otherwise} \end{cases}, \tag{14}$$

where γ is the *weighting index*. The value of γ is chosen to adjust the tightness of the WBCRB, and the optimal value of γ can only be determined numerically [5].

1) Calculation of $\mathbb{E}_{\boldsymbol{\epsilon}}\{\mathbf{Q}(\boldsymbol{\epsilon})\}$ and $\mathbb{E}_{\boldsymbol{\epsilon}}\{\mathbf{P}_w(\boldsymbol{\epsilon})\}$: The evaluation of $\mathbb{E}_{\boldsymbol{\epsilon}}\{\mathbf{Q}(\boldsymbol{\epsilon})\}$ can be obtained easily as an extension of the derivation in [5], and leads to the following result

$$\mathbb{E}_{\boldsymbol{\epsilon}}\{\mathbf{Q}(\boldsymbol{\epsilon})\} = -\beta(\gamma+1, \gamma+1)\,\mathbf{I}, \tag{15}$$

where $\beta(\cdot,\cdot)$ denotes the beta function $\beta(a,b) = \int^1 x^{a-1}(1-x)^{b-1}dx$. Meanwhile, the weighted PIM $\mathbf{P}_w(\boldsymbol{\epsilon})$ is defined in [8] as

$$[\mathbf{P}_w(\boldsymbol{\epsilon})]_{i,j} = \left(q(\epsilon_i)q(\epsilon_j)\frac{\partial \ln q(\epsilon_i)P(\boldsymbol{\epsilon})}{\partial\epsilon_i}\cdot\frac{\partial \ln q(\epsilon_j)P(\boldsymbol{\epsilon})}{\partial\epsilon_j}\right) \tag{16}$$

where $P(\boldsymbol{\epsilon})$ is the prior distribution of timing offsets. With ϵ_k being uniformly distributed in $[0, 1)$ and the weighting function $q(\epsilon_k)$ defined in (14), it can be shown that [5]

$$\mathbb{E}_{\boldsymbol{\epsilon}}\Big\{\big[\mathbf{P}_w(\boldsymbol{\epsilon})\big]_{i,j}\Big\} = \begin{cases} \gamma\cdot\beta(2\gamma-1, 2\gamma+1), & i = j \\ 0, & i \ne j \end{cases}.$$

2) Calculation of $\mathbf{F}_w(\boldsymbol{\epsilon})$: The weighted FIM for the timing offset parameters is defined in [8] as

$$[\mathbf{F}_w(\boldsymbol{\epsilon})]_{i,j} = \left(q(\epsilon_i)q(\epsilon_j)\frac{\partial \ln P(\mathbf{x}|\boldsymbol{\epsilon})}{\partial\epsilon_i}\cdot\frac{\partial \ln P(\mathbf{x}|\boldsymbol{\epsilon})}{\partial\epsilon_j}\right), \tag{17}$$

where the vector $\mathbf{x}$ represents the received samples in (10). Notice that $P(\mathbf{x}|\boldsymbol{\epsilon})$ is the conditional probability distribution of $\mathbf{x}$ given $\boldsymbol{\epsilon}$. However, with the presence of the nuisance parameters $\mathbf{h}$, we only have $P(\mathbf{x}|\boldsymbol{\epsilon}, \mathbf{h})$. In order to eliminate the nuisance parameter $\mathbf{h}$, we employ the method introduced in [1] of using a conditional approach to asymptotically (i.e., K is large) obtain the distribution by substituting the estimate of $\hat{\mathbf{h}} = (\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H\boldsymbol{\Omega}_{\boldsymbol{\epsilon}})^{-1}\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}^H\mathbf{d}$ back into the joint distribution function $P(\mathbf{d}|\boldsymbol{\epsilon}, \mathbf{h})$. Then it can be readily shown [1] that

$$\frac{\partial \ln P(\mathbf{x}|\boldsymbol{\epsilon})}{\partial\epsilon_i}\cdot\frac{\partial \ln P(\mathbf{x}|\boldsymbol{\epsilon})}{\partial\epsilon_j} \cong \frac{1}{\sigma_v^2}\left[\mathbf{H}\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}^H\mathbf{P}_{\boldsymbol{\Omega}}^{\perp}(\boldsymbol{\epsilon})\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}\mathbf{H}\right]_{i,j}.$$

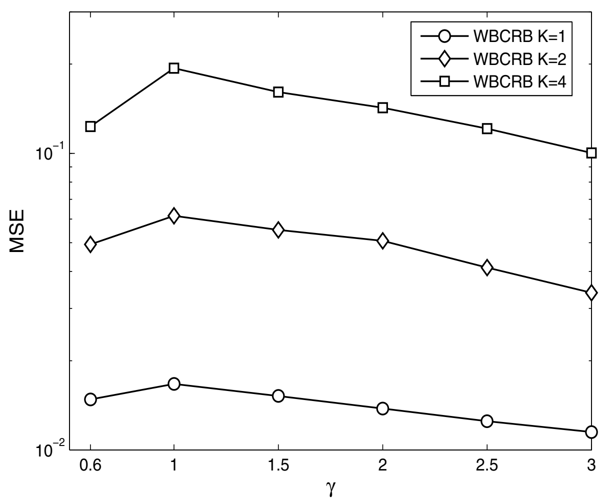

Fig. 1. WBCRB against different values of weighting index γ at SNR$= -10dB$

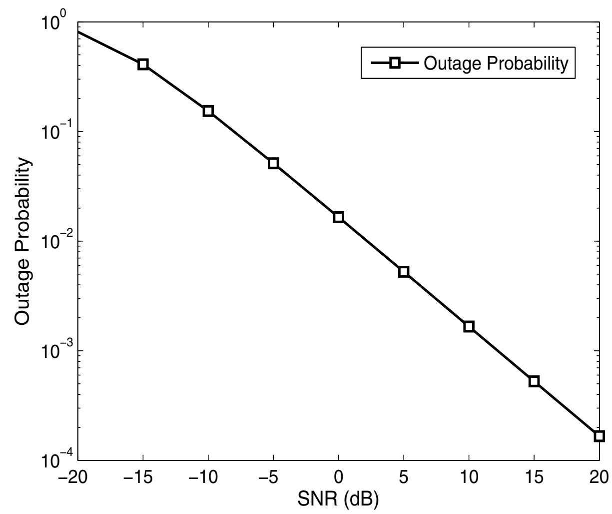

Fig. 2. Outage probability of the conventional CRB for single timing offset estimation ($K = 1$)

Last but not least, even though the expression $\mathbf{F}_w(\boldsymbol{\epsilon})$ is in closed form, the value of $\mathbb{E}_{\boldsymbol{\epsilon}}\Big\{\mathbf{F}_w(\boldsymbol{\epsilon})\Big\}$ can only be obtained numerically because it depends on the pulse shaping filter $g(t)$ in a very complicated and mathematically intractable manner. It is also worth noting that if $\gamma = 0$, the WBCRB has the following relationship with CRB in (12).

$$\text{WBCRB}(\boldsymbol{\epsilon})|_{\gamma=0} = \Big(\mathbb{E}_{\boldsymbol{\epsilon}}\{\text{CRB}^{-1}(\boldsymbol{\epsilon})\}\Big)^{-1}. \tag{18}$$

By setting the number of user to one and replacing the expressions with their counterparts in model (1), the WBCRB for the single offset case is

$$\text{WBCRB}(\epsilon) = \frac{\beta^2(\gamma+1,\gamma+1)}{\mathbb{E}_\epsilon\{F_w(\epsilon)\} + \gamma\cdot\beta(2\gamma-1,2\gamma+1)}$$

where $F_w(\epsilon) = q^2(\epsilon)|h|^2\mathbf{d}^H\mathbf{\Pi}_\epsilon\mathbf{d}/\sigma^2$.

Note that the WBCRB derived in this paper is novel and different from the results in the literature [1], [5], [8] because there exist multiple nuisance parameters $\mathbf{h}$, and also the bound is with respect to multiple timing parameters $\boldsymbol{\epsilon}$. Comparisons between CRB and WBCRB will be provided in Section V-B.

V. Numerical Results

In this section, the proposed WBCRB and the conventional CRB are compared with the simulated MSEs of the maximum likelihood timing estimators

$$\hat{\boldsymbol{\epsilon}} = \arg\min_{\boldsymbol{\epsilon}} \parallel \mathbf{P}^{\perp}_{\mathbf{\Omega}}(\boldsymbol{\epsilon})\mathbf{x} \parallel^2, \tag{19}$$

where the estimation is implemented by alternating projection [9] using Monte Carlo simulations, with each point obtained from 10^4 simulation runs. The timing MSE is defined as $\text{MSE}(\boldsymbol{\epsilon}) = \sum_{k=1}^{K}(\hat{\epsilon}_k - \epsilon_k)^2$, while the CRB and WBCRB are correspondingly calculated as the sum of all the lower bounds for different timing offsets.

The training sequences are all generated as $[\exp(-j\phi_{-L_g}),\cdots,\exp(-j\phi_{L_o+L_g})]$, where ϕ_i is uniformly distributed between $[-\pi,\pi]$. The pulse shaping filter $g(t)$ is assumed as root-raised cosine waveform with roll-off factor 0.22 and normalized energy $\int_{-\infty}^{\infty} g^2(t)dt = 1$. The channel coefficients are modeled as independent identically distributed (i.i.d.) complex Gaussian random variables with zero mean and unit variance.

For simplicity, it is assumed that $E_1 = \cdots = E_K = E$. The SNR is defined as the average transmit signal-to-noise ratio, namely SNR$= E/\sigma_v^2$ for the general model. The length of the observation is $L_o = 32$ and $L_g = 2$ and the timing offset ϵ_k is uniformly generated from $[0, 1)$.

A. Weighting Index for Weighted Bayesian CRB

Fig. 1 shows the WBCRBs for timing as a function of γ for $K = 1$, $K = 2$ and $K = 4$. Although numerous values of γ are evaluated during the simulations, only representative values of $\gamma = [0.6, 1, 1.5, 2, 2.5, 3]$ at SNR$= -10$dB are shown in the graph for a clear presentation. We only show the simulation results at low SNR since at high SNR, WBCRBs with different γ all asymptotically converge to the CRB. From Fig. 1, it is noticed that $\gamma = 1$ gives the tightest bound in all cases, and thus we will use $\gamma = 1$ for the rest of the simulations.

B. Outage Probability of the conventional CRB

In Fig. 2, the outage probability is evaluated to illustrate the inapplicability of the conventional CRB. For timing estimation where ϵ is uniformly distributed between $[0, 1)$, the MSE constraint α^2 can be taken as the variance of the uniform distribution $U[0, 1)$, which is $\alpha^2 = 1/12$. The outage probability (9) is averaged over all the simulated realizations of the timing offset ϵ. As seen from Fig. 2, when the SNR is relatively low, the outage probability approaches 1.

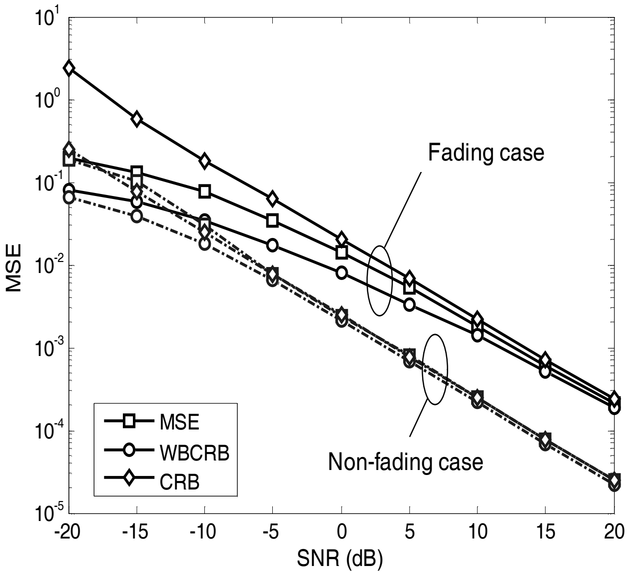

Fig. 3. MSE, CRB and WBCRB of single timing offset estimation in fading and non-fading environment

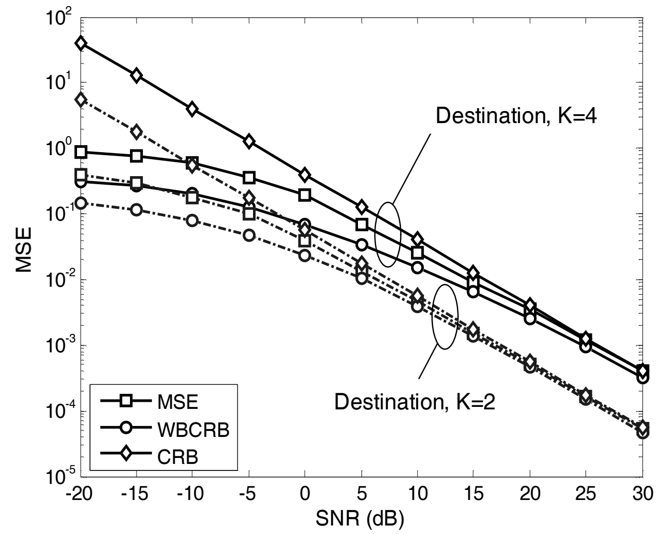

Fig. 4. MSE, CRB and WBCRB for multiple timing estimation at the destination ($K = 2$ and 4) with fading channels

C. WBCRB versus the conventional CRB

In Fig. 3, we present a comparison between CRB, WBCRB and the MSE of the single timing offset estimation ($K = 1$) in both fading and non-fading channels. The non-fading channel is generated as $h = e^{j\phi}$ with unit gain and random phase shift. From Fig. 3, it is clear that the conventional CRB is generally a valid bound in non-fading channels. However, in fading case, the validity of CRB does not hold until the average SNR becomes considerably large (at least > 20dB). On the other hand, by incorporating the prior knowledge on timing, the WBCRB is valid for both fading and non-fading cases. Notice that at low SNR, the WBCRB approaches to the variance of the uniform distribution of timing offsets (i.e., $\mathrm{Var}\{\epsilon\} = 1/12$), while at high SNR the WBCRB asymptotically converges to the CRB.

Furthermore, by viewing Fig. 2 together with Fig. 3, it can be seen that the conventional CRB is above the MSE curve when the outage probability grows above 10^{-3}. In other words, being affected by the outage realization, the CRB fails to bound the MSE of the maximum likelihood estimator.

In Fig. 4, the CRB, the WBCRB and the corresponding timing estimation MSE are plotted as a function of SNR for $K = 2$ and $K = 4$. It can be seen that the CRB and WBCRB coincide in high SNR region while the CRB becomes inapplicable even when the SNR is high (30dB). On the contrary, the proposed WBCRB serves as a valid lower bound for all the SNR. Also, at low SNR, the WBCRB approaches the variance of the uniform distribution of timing offsets, i.e., $\sum_{k=1}^{K} \mathrm{Var}\{\epsilon_k\} = K/12$.

VI. Conclusions

In this paper, with the concept of outage probability of the CRB, it is demonstrated that the conventional CRB for timing estimation is no longer valid under fading conditions. Then a novel Weighted Bayesian CRB has been proposed to incorporate the range constraints of the timing parameters and it has been numerically shown that the WBCRB is a valid lower bound for timing parameters under fading conditions for all SNRs.

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