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H_∞ Filter Design for Quantum Stochastic Systems*

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Abstract: This paper is concerned with the H_∞ filtering problem for a class of noncommutative linear stochastic systems, among which quantum technology has been well recognized to be a typical example. A new structure of slack variables is introduced to solve the H_∞ filtering problem which gives a sufficient condition for the existence of desired H_∞ filters for quantum systems in terms of linear matrix inequalities (LMIs). Two examples from quantum optics are given to show the effectiveness of the proposed approach.

Key Words: H_∞ filtering, Quantum optics, Quantum control theory

1 INTRODUCTION

The problem of filter design has been attracted much attention in the field of signal processing and control systems. Much work has been done on the filtering problem for classical control system, e.g., see [1-3]. In recent years, attention has also been drawn to the filtering problem for quantum systems, which has widely been recognized as forming an important basis for the development of various engineering applications of quantum theory with applications including quantum feedback control [4-8], quantum dynamical parameter estimation [9, 10], and quantum information processing [11, 12]. A number of important results on quantum filtering theory have been obtained so far, see [13, 14]. However, earlier results on quantum systems have not focused on the issue of robustness directly. The purpose of this paper is to provide a systematic approach to design H_∞ filters for quantum systems.

In this paper, we present a method for designing H_∞ filters for a class of noncommutative linear stochastic systems, among which quantum technology has been well recognized to be a typical example. We introduce a new structure of slack variables to solve the H_∞ filtering problem. A sufficient condition for existence of the desired H_∞ filters for quantum systems is proposed, which can be expressed in terms of linear matrix inequalities (LMIs). This condition is developed based on a version of the Strict Bounded Real Lemma [15]. Finally, we illustrate the methodology by applying these results to quantum optics, which shows the effectiveness of the proposed approach. A feature worth mentioning is that our method can be used to synthesize a quantum filter or a classical filter for a quantum system. More specifically, the classical filters monitor the quantum system, and then generate an optimal estimate, while the quantum filters are themselves quantum systems.

The systems we consider are defined in a quantum probability space $(\mathscr{A}, \mathbb{P})$ [16], where $\mathscr{A}$ is a von Neumann algebra and $\mathbb{P}$ is a state on this algebra. We write $\mathscr{A} = \mathscr{A}_S \otimes \mathscr{W}$, where $\mathscr{A}_S$ is the CCR (Canonical Commutation Relations) algebra of components of initial plant, and disturbance variables $x, \beta_w(t)$, and $\mathscr{W}$ is the CCR algebra in which the Gaussian noise variables are defined. Also, we write $\mathbb{P} = \rho \otimes \phi$, where ρ is a Gaussian state on $\mathscr{A}_S$, and ϕ is the vacuum state on $\mathscr{W}$. The noncommutative model can be realized in an appropriate Hilbert space framework [16].

2 PROBLEM FORMULATION

Consider the following linear noncommutative stochastic system $\mathcal{S}$:

$$\begin{aligned} \mathcal{S} : \mathrm{d}x(t) &= Ax(t)\,\mathrm{d}t + B\mathrm{d}w(t) + G\mathrm{d}v(t), \\ \mathrm{d}y(t) &= Cx(t)\,\mathrm{d}t + D\mathrm{d}w(t) + H\mathrm{d}v(t), \\ z(t) &= Lx(t) \end{aligned} \tag{1}$$

Here

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^{\mathrm{T}}$$

and $x_1 = q$ is the particle position, and $x_2 = p$ is the momentum. The input $w(t)$ represents a disturbance signal of the form

$$\mathrm{d}w(t) = \beta_w(t)\,\mathrm{d}t + \mathrm{d}\tilde{w}(t)$$

where $\tilde{w}(t)$ is the noise part of $w(t)$ and $\beta_w(t)$ is the self-adjoint finite variation part of $w(t)$. The noise vectors $v(t)$ and $\tilde{w}(t)$ are vectors of noncommutative Wiener processes [17] and have non-zeros Ito tables as follows:

$$\mathrm{d}v(t)\,\mathrm{d}v^{\mathrm{T}}(t) = F_v \mathrm{d}t,\ \mathrm{d}\tilde{w}(t)\,\mathrm{d}\tilde{w}^{\mathrm{T}}(t) = F_{\tilde{w}} \mathrm{d}t, \tag{2}$$

where F_v and $F_{\tilde{w}}$ are non-negative definite Hermitian matrices.

Here we are interested in estimating the signal $z(t)$ by a filter, which is assumed to be noncommutative stochastic system of the form $\mathcal{F}$:

$$\begin{aligned} \mathcal{F} : \mathrm{d}x_F(t) &= A_F x_F(t)\,\mathrm{d}t + B_F \mathrm{d}y(t) + G_F \mathrm{d}v_F(t), \\ z_F(t) &= C_F x_F(t) \end{aligned} \tag{3}$$

where $x_F(t)$ is a vector of self-adjoint filter variables. The noise $v_F(t)$ is a vector of noncommutative Wiener processes (in vacuum states) with non-zero Ito products as in (2) with

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non-negative definite Hermitian matrix F_{v_F}. By extending the state space for the application of von Neumann algebra if necessary, the filter can be defined in the above quantum probability space. Denoting

$$\xi(t) = \left[\begin{array}{cc} x^{\mathrm{T}}(t) & x_F^{\mathrm{T}}(t) \end{array}\right]^{\mathrm{T}} \text{ and } e(t) = z(t) - z_F(t)$$

we define, for a given scalar $\gamma > 0$, the following performance index:

$$J(e(t), \beta_w(t)) \triangleq e^{\mathrm{T}}(t)\, e(t) - \gamma^2 \beta_w^{\mathrm{T}}(t)\, \beta_w(t)$$

Connecting $\mathcal{S}$ and $\mathcal{F}$, we obtain the filtering error system $\mathcal{E}$:

$$\begin{aligned} \mathcal{E} : \mathrm{d}\xi(t) &= \bar{A}\xi(t)\,\mathrm{d}t + \bar{B}\mathrm{d}w(t) + \bar{G}\mathrm{d}\tilde{v}(t), \\ e(t) &= \bar{C}\xi(t) \end{aligned} \tag{4}$$

where $\tilde{v}(t) = \left[\begin{array}{cc} v^{\mathrm{T}}(t) & v_F^{\mathrm{T}}(t) \end{array}\right]^{\mathrm{T}}$ and

$$\begin{aligned} \bar{A} &= \left[\begin{array}{cc} A & 0 \\ B_F C & A_F \end{array}\right], \ \bar{B} = \left[\begin{array}{c} B \\ B_F D \end{array}\right], \\ \bar{G} &= \left[\begin{array}{cc} G & 0 \\ B_F H & G_F \end{array}\right], \bar{C} = \left[\begin{array}{cc} L & -C_F \end{array}\right] \end{aligned} \tag{5}$$

Definition 1 The filtering error system (4) is said to be mean square stable with noise attenuation level γ if there exists a positive operator valued quadratic form $V(\xi) = \xi^{\mathrm{T}} P \xi$, a constant $\epsilon > 0$ and a constant $\lambda > 0$, such that

$$\begin{aligned} &\langle V(\xi(t))\rangle + \int_0^t \Big\langle J(e(s), \beta_w(s)) + \epsilon \Big(\xi^{\mathrm{T}}(s)\, \xi(s) \\ &+ \beta_w^{\mathrm{T}}(s)\, \beta_w(s)\Big)\Big\rangle \mathrm{d}s \leqslant \langle V(\xi(0))\rangle + \lambda t, \ \forall t > 0 \end{aligned} \tag{6}$$

for all Gaussian initial states ρ.

Here we use the short hand notation $\langle \cdot \rangle \equiv \mathbb{P}(\cdot)$.
The problem addressed in this paper is given as follows:
Quantum H_∞ Filtering Problem Given $\gamma > 0$, find a filter of the form (3) that leads to a mean square stable estimation error process $e(t)$ such that (6) is satisfied.

3 MAIN RESULTS

3.1 Strict Bounded Real Lemma

We first introduce a lemma that is proven in [15].

Lemma 1 The following statements are equivalent:

1. The filtering error system (4) is mean square stable with noise attenuation level γ.
2. There exists $Q > 0$ such that
$$\bar{A}^{\mathrm{T}} Q + Q\bar{A} + \bar{C}^{\mathrm{T}}\bar{C} + Q\bar{B}\left(\gamma^2 I\right)^{-1}\bar{B}^{\mathrm{T}} Q < 0 \tag{7}$$

Furthermore, if (7) holds then the required constant λ in Definition 1 can be chosen as

$$\lambda = \mathrm{tr}\left[\left[\begin{array}{c} \bar{B} \\ \bar{G} \end{array}\right] Q \left[\begin{array}{cc} \bar{B} & \bar{G} \end{array}\right] F\right] \tag{8}$$

where the matrix F is defined by the following relation following from (2)

$$F dt = \left[\begin{array}{c} \mathrm{d}\tilde{w}^{\mathrm{T}} \\ \mathrm{d}\tilde{v}^{\mathrm{T}} \end{array}\right]\left[\begin{array}{cc} \mathrm{d}\tilde{w} & \mathrm{d}\tilde{v} \end{array}\right] \tag{9}$$

Using the technique in [18, 19], we can easily obtain the lemma below.

Lemma 2 There exists a $Q > 0$ to the inequality of (7) if and only if there exist matrices (F, M, P) with $P > 0$ such that

$$\left[\begin{array}{cccc} \bar{A}^{\mathrm{T}} F + F^{\mathrm{T}}\bar{A} & P - F^{\mathrm{T}} + \bar{A}^{\mathrm{T}} M & F^{\mathrm{T}}\bar{B} & \bar{C}^{\mathrm{T}} \\ * & -M - M^{\mathrm{T}} & M^{\mathrm{T}}\bar{B} & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{array}\right] < 0 \tag{10}$$

3.2 H_∞ Filter Synthesis

In the sequel, based on Lemma 2, we devote ourselves to the design of quantum H_∞ filters.
Let matrices P, M and F be partitioned as

$$\begin{aligned} P &= \left[\begin{array}{cc} P_1 & P_2 \\ P_2^{\mathrm{T}} & P_3 \end{array}\right], \ M = \left[\begin{array}{cc} M_1 & M_2 \\ W_4 & W_3 \end{array}\right], \\ F &= \left[\begin{array}{cc} F_1 & F_2 \\ \lambda_1 W_4 & \lambda_2 W_3 \end{array}\right] \end{aligned}$$

Without loss of generality, we assume that W_3 and W_4 are invertible. Introduce matrix

$$\phi \triangleq \left[\begin{array}{cc} I & 0 \\ 0 & W_3^{-1} W_4 \end{array}\right] \tag{11}$$

and define

$$\bar{P} = \left[\begin{array}{cc} \bar{P}_1 & \bar{P}_2 \\ * & \bar{P}_3 \end{array}\right] = \phi^{\mathrm{T}} P \phi$$

Performing a congruence transformation to (10) by $\mathrm{diag}\{\phi,\ \phi,\ I,\ I\}$ and taking (5) into account, we obtain

$$\left[\begin{array}{cccc} \Psi_1 + \Psi_1^{\mathrm{T}} & \bar{P} - \Psi_2^{\mathrm{T}} + \Psi_3 & \Psi_4 & \Psi_5 \\ * & \Psi_6 & \Psi_7 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{array}\right] < 0 \tag{12}$$

where

$$\begin{aligned} \Psi_1 &= \left[\begin{array}{c} F_1 A + \lambda_1 W_4^{\mathrm{T}} B_F C \\ W_4^{\mathrm{T}} W_3^{-1} F_2^{\mathrm{T}} A + \lambda_2 W_4^{\mathrm{T}} B_F C \end{array}\right. \\ &\qquad \left.\begin{array}{c} \lambda_1 W_4^{\mathrm{T}} A_F W_3^{-1} W_4 \\ \lambda_2 W_4^{\mathrm{T}} A_F W_3^{-1} W_4 \end{array}\right] \\ \Psi_2 &= \left[\begin{array}{cc} F_1 & F_2 W_3^{-1} W_4 \\ \lambda_1 W_4^{\mathrm{T}} W_3^{-T} W_4 & \lambda_2 W_4^{\mathrm{T}} W_3^{-T} W_4 \end{array}\right] \\ \Psi_3 &= \left[\begin{array}{c} M_1^{\mathrm{T}} A + W_4^{\mathrm{T}} B_F C \\ W_4^{\mathrm{T}} W_3^{-T} M_2^{\mathrm{T}} A + W_4^{\mathrm{T}} B_F C \end{array}\right. \\ &\qquad \left.\begin{array}{c} W_4^{\mathrm{T}} A_F W_3^{-1} W_4 \\ W_4^{\mathrm{T}} A_F W_3^{-1} W_4 \end{array}\right] \\ \Psi_4 &= \left[\begin{array}{c} F_1^{\mathrm{T}} B + \lambda_1 W_4^{\mathrm{T}} B_F D \\ W_4^{\mathrm{T}} W_3^{-T} F_2^{\mathrm{T}} B + \lambda_2 W_4^{\mathrm{T}} B_F D \end{array}\right] \\ \Psi_5 &= \left[\begin{array}{c} L^{\mathrm{T}} \\ -W_4^{\mathrm{T}} W_3^{-T} C_F^{\mathrm{T}} \end{array}\right] \\ \Psi_6 &= \left[\begin{array}{cc} M_1 & M_2 W_3^{-1} W_4 \\ W_4^{\mathrm{T}} W_3^{-T} W_4 & W_4^{\mathrm{T}} W_3^{-T} W_4 \end{array}\right] \\ \Psi_7 &= \left[\begin{array}{c} M_1^{\mathrm{T}} B + W_4^{\mathrm{T}} B_F D \\ W_4^{\mathrm{T}} W_3^{-T} M_2^{\mathrm{T}} B + W_4^{\mathrm{T}} B_F D \end{array}\right] \end{aligned}$$

Define

$$\begin{aligned} X &\triangleq F_1,\ R \triangleq M_1,\ S \triangleq M_2 W_3^{-1} W_4 \\ Y &\triangleq F_2 W_3^{-1} W_4,\ T \triangleq W_4^{\mathrm{T}} W_3^{-1} W_4 \end{aligned}$$

$$\left[\begin{array}{cc} \bar{A}_F & \bar{B}_F \\ \bar{C}_F & 0 \end{array}\right] \triangleq \left[\begin{array}{cc} W_4^{\mathrm{T}} & 0 \\ 0 & I \end{array}\right] \times \left[\begin{array}{cc} A_F & B_F \\ C_F & 0 \end{array}\right] \left[\begin{array}{cc} W_3^{-1} W_4 & 0 \\ 0 & I \end{array}\right]$$

Substituting the above matrices into (12), we obtain

$$\left[\begin{array}{cccccc} \Delta_1 + \Delta_1^{\mathrm{T}} & \Delta_2 + \lambda_1 \bar{A}_F & \bar{P}_1 + \Delta_3^{\mathrm{T}} - X^{\mathrm{T}} & \bar{P}_2 + \Delta_4 - \lambda_1 T & \Delta_5 & L^{\mathrm{T}} \\ * & \lambda_2 \left(\bar{A}_F + A_F^{\mathrm{T}}\right) & \bar{P}_2^{\mathrm{T}} + \bar{A}_F^{\mathrm{T}} - Y^{\mathrm{T}} & \bar{P}_3 + \bar{A}_F^{\mathrm{T}} - \lambda_2 T & \Delta_6 & -\bar{C}_F^{\mathrm{T}} \\ * & * & \Delta_7 & \Delta_8 & \Delta_9 & 0 \\ * & * & * & \Delta_{10} & \Delta_{11} & 0 \\ * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & -\gamma I \end{array}\right] < 0 \quad (13)$$

where

$$\begin{aligned} \Delta_1 &\triangleq X^{\mathrm{T}} A + \lambda_1 \bar{B}_F C,\ \Delta_2 \triangleq A^{\mathrm{T}} Y + \lambda_2 C^{\mathrm{T}} \bar{B}_F^{\mathrm{T}} \\ \Delta_3 &\triangleq R^{\mathrm{T}} A + \bar{B}_F C,\ \Delta_4 \triangleq A^{\mathrm{T}} S + C^{\mathrm{T}} \bar{B}_F^{\mathrm{T}} \\ \Delta_5 &\triangleq X^{\mathrm{T}} B + \lambda_1 \bar{B}_F D,\ \Delta_6 \triangleq Y^{\mathrm{T}} B + \lambda_2 \bar{B}_F D \\ \Delta_7 &\triangleq -R - R^{\mathrm{T}},\ \Delta_8 \triangleq -S - T,\ \Delta_{10} \triangleq -T^{\mathrm{T}} - T \\ \Delta_9 &\triangleq R^{\mathrm{T}} B + \bar{B}_F D,\ \Delta_{11} \triangleq S^{\mathrm{T}} B + \bar{B}_F D \end{aligned}$$

Thus we have the following theorem.

Theorem 3 Given system $\mathcal{S}$ in (1), an admissible filter of the form $\mathcal{F}$ in (3) exists if there exist matrices $\bar{P} = \left[\begin{array}{cc} \bar{P}_1 & \bar{P}_2 \\ * & \bar{P}_3 \end{array}\right] > 0$, X, Y, R, S, T, $\bar{A}_F$, $\bar{B}_F$ and $\bar{C}_F$ satisfying (13). Moreover, under the above conditions, the matrices for an admissible filter in the form of (3) are given by

$$\left[\begin{array}{cc} A_F & B_F \\ C_F & 0 \end{array}\right] = \left[\begin{array}{cc} T^{-1} & 0 \\ 0 & I \end{array}\right] \left[\begin{array}{cc} \bar{A}_F & \bar{B}_F \\ \bar{C}_F & 0 \end{array}\right] \quad (14)$$

Proof Suppose there are matrices $\bar{P} > 0$, X, Y, R, S, T, $\bar{A}_F$, $\bar{B}_F$ and $\bar{C}_F$ satisfying (13). Firstly, the $(4,4)$ block of (13) implies $-T - T^{\mathrm{T}} < 0$, which means that T is nonsingular. Thus, one can always find square and nonsingular matrices W_3 and W_4 satisfying $T = W_4^{\mathrm{T}} W_3^{-1} W_4$. Now define the nonsingular matrix variable ϕ as in (11) and matrices

$$\begin{aligned} P &\triangleq \phi^{-T} \bar{P} \phi^{-1} \\ F &\triangleq \left[\begin{array}{cc} X & Y W_4^{-1} W_3 \\ \lambda_1 W_4 & \lambda_2 W_3 \end{array}\right] \\ G &\triangleq \left[\begin{array}{cc} R & S W_4^{-1} W_3 \\ W_4 & W_3 \end{array}\right] \end{aligned}$$

$$\left[\begin{array}{cc} A_F & B_F \\ C_F & 0 \end{array}\right] \triangleq \left[\begin{array}{cc} W_4^{-T} & 0 \\ 0 & I \end{array}\right] \left[\begin{array}{cc} \bar{A}_F & \bar{B}_F \\ \bar{C}_F & 0 \end{array}\right] \times \left[\begin{array}{cc} W_4^{-1} W_3 & 0 \\ 0 & I \end{array}\right] \quad (15)$$

Note that $P > 0$. By some algebraic matrix manipulations, (13) is equivalent to

$$\left[\begin{array}{cccc} \phi^{\mathrm{T}} \left(\bar{A}^{\mathrm{T}} F + F^{\mathrm{T}} \bar{A}\right) \phi & \phi^{\mathrm{T}} \left(P - F^{\mathrm{T}} + \bar{A}^{\mathrm{T}} G\right) \phi & \phi^{\mathrm{T}} F^{\mathrm{T}} \bar{B} & \phi^{\mathrm{T}} \bar{C}^{\mathrm{T}} \\ * & \phi^{\mathrm{T}} \left(-G - G^{\mathrm{T}}\right) \phi & \phi^{\mathrm{T}} G^{\mathrm{T}} \bar{B} & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & \gamma I \end{array}\right] < 0 \quad (16)$$

Performing a congruence transformation to (16) by $\operatorname{diag}\left\{\phi^{-1}, \phi^{-1}, I, I\right\}$ yields (10). Therefore, we can conclude from Lemma 1 that the filter with a state-space realization (A_F, B_F, C_F) defined in (15) guarantees the filtering error system $\mathcal{E}$ in (4) to be mean square stable with noise attenuation level γ.

Notice that the realization of filter (3) given by $(A_F,\ B_F,\ C_F)$ is algebraically similar to

$$\left(W_4^{-1} W_3 A_F W_3^{-1} W_4,\ W_4^{-1} W_3 B_F,\ C_F W_3^{-1} W_4\right)$$

via a similarity transformation. By substituting the matrices with (15) and by considering the relationship $T = W_4^{\mathrm{T}} W_3^{-1} W_4$, we have

$$(A_F,\ B_F,\ C_F) = \left(T^{-1} \bar{A}_F,\ T^{-1} \bar{B}_F,\ \bar{C}_F\right)$$

Therefore, an admissible filter can be given by (14) and the proof is completed.

Remark 1 By considering Lemma 1 and taking analogy of continuous-time systems, it can be shown that if (A, B, C, D) represents a minimal realization of a stable system with a stable inverse in H_∞, then an exact quantum filter can be obtained with

$$A_F = A - BD^{-1}C,\ B_F = BD^{-1},\ C_F = L.$$

4 ILLUSTRATIVE EXAMPLES

Quantum effects are particularly easy to observe in optical systems, and one of the earliest proposals for quantum information processing uses photons to implement quantum logic, e.g., see [20]. In this section we illustrate the results developed in this paper via two examples of filter design for a simple quantum optical coupled to laser and vacuum optical fields; see [17, 21]. In Example 1, the filter is itself a quantum system which can be implemented in quantum optics, while in Example 2, the filter is a classical system which can be implemented with standard electronics.

Example 1 We consider an optical cavity resonantly coupled to two optical channels v, w as in Fig. 1.

The annihilation operator a for this cavity system (representing a standing wave) evolves in time according to the equations

$$\begin{aligned} \mathrm{d}a &= -\frac{\mathcal{K}}{2} a \mathrm{d}t - \sqrt{k_1} \mathrm{d}V - \sqrt{k_2} \mathrm{d}W \\ \mathrm{d}Y &= \sqrt{k_2} a \mathrm{d}t + \mathrm{d}W \end{aligned} \quad (17)$$

It is required that $\mathcal{K} = k_1 + k_2$. In the quadrature notation of (1), $x_1(t) = q(t) = a(t) + a^{\dagger}(t)$, $x_2(t) =$

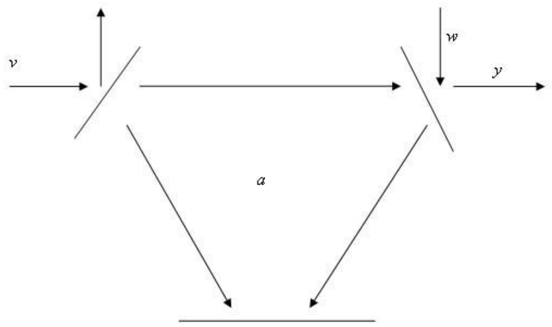

Fig. 1 An optical cavity (plant)

$p(t) = \left(a(t) - a^\dagger(t)\right)/i$, $V(t) = (v_1(t) + iv_2(t))/2$, and $W(t) = (w_1(t) + iw_2(t))/2$. Here, $i = \sqrt{-1}$. The quantum noises v, $\tilde{w}$ have Ito Hermitian matrices F_v, $F_{\tilde{w}}$ equal to

$$F = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

It is not difficult to transform the above equations into a system of the form (1), with the corresponding system matrices given by

$$\begin{aligned} A &= -\frac{\mathcal{K}}{2}I,\ G = \sqrt{k_1}I,\ B = -\sqrt{k_2}I, \\ C &= \sqrt{k_2}I,\ D = I,\ H = 0 \end{aligned}$$

For this system, the boson commutation relation $\left[a, a^\dagger\right] = 1$ holds (here, $[A, B] = AB - BA$).
In our example, we will choose the total cavity decay rate $\mathcal{K} = 2.8$ and the coupling coefficients $k_1 = 2.6$, and $k_2 = 0.2$. Consider $L = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and notice that (A, B, C, D) satisfies the conditions given in Remark 1. Therefore, an exact quantum filter can be found and the filter parameters are given by

$$\begin{aligned} A_F &= \begin{bmatrix} -1.2 & 0 \\ 0 & -1.2 \end{bmatrix},\ C_F = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ B_F &= \begin{bmatrix} -\sqrt{0.2} & 0 \\ 0 & -\sqrt{0.2} \end{bmatrix} \end{aligned} \tag{18}$$

The filter (3), (18) can be implemented with another optical cavity with annihilation operator a_F (with quadratures $x_{F1} = q_F = a_F + a_F^\dagger$, $x_{F2} = p_F = \left(a_F - a_F^\dagger\right)/i$, $x_F = (q_F, p_F)^{\mathrm{T}}$ and a 180° phase shift. This system evolves according to the equations

$$\begin{aligned} \mathrm{d}a_F &= \frac{-\mathcal{K}_F}{2}a_F\mathrm{d}t - \sqrt{k_{F1}}\mathrm{d}V_{F1} \\ &\quad -\sqrt{k_{F2}}\mathrm{d}V_{F2} - \sqrt{k_{F3}}\mathrm{d}Y \\ Z_F &= -\sqrt{k_{F1}}(a_F + a_F^\dagger) \end{aligned}$$

where $k_{F1} = 1$, $k_{F2} = 1.2$, $k_{F3} = 0.2$, and $\mathcal{K}_F = 2.4$. This gives

$$G_F = \begin{bmatrix} -0.4472 & 0 & -1.1 & 0 \\ 0 & -0.4472 & 0 & -1.1 \end{bmatrix}$$

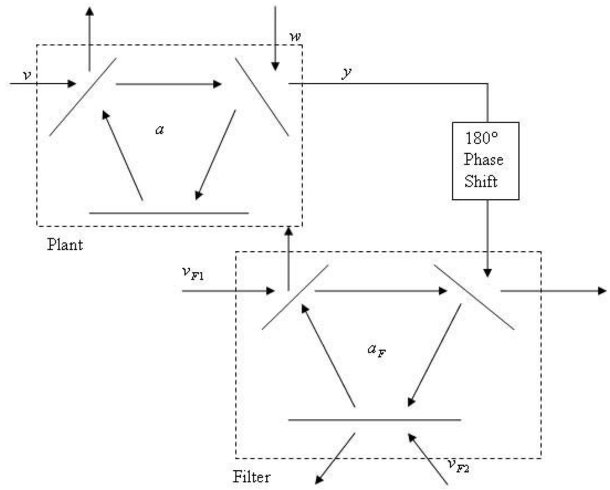

Fig. 2 An optical cavity (filter)

where

$$v_F(t) = \begin{bmatrix} v_{F11}(t) & v_{F12}(t) & v_{F21}(t) & v_{F22}(t) \end{bmatrix}^{\mathrm{T}}$$

and $V_{F1} = (v_{F11} + iv_{F12})/2$, $V_{F2} = (v_{F21} + iv_{F22})/2$. The non-zero Ito products for this noise process are $dV_{F1}(t)\,dV_{F1}^\dagger(t) = dt$ and $dV_{F2}(t)\,dV_{F2}^\dagger(t) = dt$, which defines the Hermitian matrix F_{V_F}. The system is illustrated in Fig. 2.

Example 2 Consider the same optical cavity system as in Fig. 1, but suppose now the information y available to the filter is the real quadrature of the output field (this quadrature can be measured by homodyne photodetection; e.g., see [21]).
The plant with classical output channel is described by the following equations:

$$\begin{aligned} \mathrm{d}a &= -\frac{\mathcal{K}}{2}a\mathrm{d}t - \sqrt{k_1}\mathrm{d}V - \sqrt{k_2}\mathrm{d}W \\ \mathrm{d}Y &= \sqrt{k_2}\left(a + a^\dagger\right)\mathrm{d}t + \mathrm{d}\tilde{w}_1 \end{aligned} \tag{19}$$

In terms of the quadrature from (1), this corresponds to the following choice of system matrices:

$$\begin{aligned} A &= -\frac{\mathcal{K}}{2}I,\ B = -\sqrt{k_2}I,\ G = \sqrt{k_1}I, \\ C &= \sqrt{k_2}\begin{bmatrix} 1 & 0 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

As in the quantum filter case, we will choose the total cavity decay rate $\mathcal{K} = 2.8$, $k_1 = 2.6$, $k_2 = 0.2$. With $L = \begin{bmatrix} 0 & 1 \end{bmatrix}$, it is found that the filtering error system is mean square stable with a disturbance attenuation level $\gamma = 0.3194$ by applying Theorem 3, and the associated filters are given by

$$\begin{aligned} A_F &= \begin{bmatrix} -1.2 & 0 \\ 0 & -1.1755 \end{bmatrix},\ B_F = \begin{bmatrix} 0.4472 \\ 0 \end{bmatrix}, \\ C_F &= \begin{bmatrix} 0 & -0.7103 \end{bmatrix} \end{aligned} \tag{20}$$

We have chosen

$$G_F = 0$$

In this case, the filter (3), (20) is a classical system which can be implemented using standard electronic devices. The system is illustrated in Fig. 3.

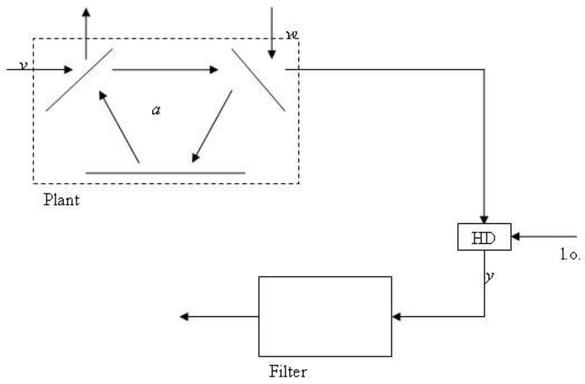

Fig. 3 A classical system (filter $\mathcal{F}$, implemented using standard electronics). The quadrature measurement is achieved by homodyne photodetection (HD).

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