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A Matrix Transformation Approach to $\mathcal{H}_\infty$ Control via Static Output Feedback for Input Delay Systems

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***Abstract*—This paper addresses the static output feedback (SOF) $\mathcal{H}_\infty$ control for continuous-time linear systems with an unknown input delay from a novel perspective. New equivalent characterizations on the stability and $\mathcal{H}_\infty$ performance of the closed-loop system are established in terms of nonlinear matrix inequalities with free parametrization matrices. These delay-dependent characterizations possess a special monotonic structure, which leads to linearized iterative computation. The effectiveness and merits of the proposed approach are shown through numerical examples.**

I. Introduction

It is well known that even a simple linear system with a single delay imposes difficulties and restrictions on the design of a stabilization controller. The stabilization problem of linear systems with an unknown delay in the input signal is still under great attention as shown in [2], [13], [16] (and the references therein). For this type of systems, many stabilization methods have been developed via state feedback controllers. An easy way to deal with this problem is to reduce it to an ordinary delay-free system by the Artstein model reduction method in [1], [10]. However, the complete transformation in Artstein model reduction is valid only for a fully known system, and to implement a stabilizing controller with distributed system state seems to be much difficult. The bounded state feedback controllers, designed both in [5] and [11], that globally stabilize an oscillator system involve a saturation function also require an explicit knowledge of the size of the input delay. It is unfortunate that full access to the state vector is not always possible, while a controller based on available output measurements has to be used in such cases.

Another difficulty is that the conditions for the delay-dependent output feedback stabilization and controller design cannot be expressed in terms of strict linear matrix inequalities (LMIs). Geromel *et al.* in [9] modified the cone complementarity linearization (CCL) algorithm to solve inversely coupled Lyapunov inequality problems under certain additional assumptions, and Moon *et al.* [12] devised an LMI-based iterative algorithm to solve the problem of designing a delay-dependent state feedback stabilization controller. Recently, Gao *et al.* tried to employ this method to deal with the output feedback stabilization of discrete-time systems with a time-varying state delay in [8]. However, as far as we know, there are no effective methods to resolve the coupling of controller gain with not only the state matrix but also the input matrix while the delay term appearing in the input signal, unless introducing extra restrictions on the Lyapunov matrix (for example ([6], [7]) referring to state feedback control of linear systems with state delays).

Related to the background mentioned above, a natural question to ask is how to design a static output feedback controller to stabilize an unknown input-delayed system. It is also expected that the controller can guarantee certain performance requirements. This paper discusses in detail the output feedback stabilization problem for linear input-delayed systems using a new approach in the state space. In Section 2, a new characterization of static output feedback (SOF) stabilization is established in terms of matrix inequalities. The advantage of such a characterization is twofold. First, the decoupling of the input and the gain-output matrix enables us to parameterize the controller matrix by a free matrix parameter. Second, the separation of the Lyapunov matrix and the controller matrix avoids imposing any constraint on the Lyapunov matrix when the controller matrix is parametrized. Besides, on that basis, no coupling terms of controller gain and the redundant matrices introduced in the free-weighting matrix method appear, due to the use of Finsler's Lemma. Based on the new characterization for SOF control, $\mathcal{H}_\infty$ performance analysis is obtained in Section 3. To obtain the controller gain matrices, an iterative algorithm is given in Section 4 to solve the nonlinear matrix inequalities. The effectiveness and merits of the proposed approach are illustrated in Section 5 through numerical examples.

Notation: Throughout this paper, let $\mathbb{R}$ be the set of real numbers; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{m\times n}$ is the set of $m\times n$ matrices for which all entries belong to $\mathbb{R}$. The space of functions in $\mathbb{R}^q$ that are square Lebesgue integrable over $[0\ \infty)$ is denoted by $\mathcal{L}_2^q[0,\infty)$ with the norm $\|\cdot\|_{L_2}$.

For real symmetric matrices X and Y, the notation $X\geq Y$ (respectively, $X>Y$) means that the matrix $X-Y$ is positive semi-definite (respectively, positive definite). 0 is a null matrix and I is the identity matrix with an appropriate dimension in a matrix inequality. The superscript "T" represents the transpose of the matrix and the asterisk "$*$" in a matrix stands the term

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which is induced by symmetry. $\mathrm{col}\{\cdot\}$ denotes a matrix column with blocks given by the matrices in $\{\cdot\}$. A block diagonal matrix with diagonal blocks A_1, A_2, ..., A_r will be denoted by $\mathrm{diag}\{A_1, A_2, \ldots, A_r\}$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

II. New Closed-loop Stability Characterization

Consider the following linear time-invariant system with delayed and non-delayed inputs,

$$(\Sigma_0): \quad \begin{aligned} \dot{x}(t) &= Ax(t) + B_0 u(t) + B_1 u(t-d) \\ y(t) &= Cx(t) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state with the initial function $\phi(t)$ when $t \in [-d, 0]$, and $y(t)$ is the measurement output. Here, A, B_0, B_1, C are the system state, the control input and the measured output matrices, respectively, and $d > 0$ is an unknown constant input delay. An SOF controller under consideration is of the form

$$(C_1): \quad u(t) = Ky(t)$$

where K is the controller gain to be designed. When SOF controller (C_1) is applied to (Σ_0), the closed-loop system is given by

$$(\Sigma_{0c1}): \quad \dot{x}(t) = A_c x(t) + B_1 KCx(t-d)$$

where $A_c = A + B_0 KC$. We provide a theorem which will be used in the sequel. Construct a Lyapunov-Krasovskii functional with matrices $P_1 > 0$, $R > 0$, and $Q > 0$,

$$\begin{aligned} V(x_t) &= x^T(t) P_1 x(t) + \int_{-d}^{0}\int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ &\quad + \int_{t-d}^{t} x^T(s) Q x(s)\,\mathrm{d}s \end{aligned} \tag{1}$$

where x_t is defined by $x_t(\theta) = x(t+\theta)$ with $-d \le \theta \le 0$.

Theorem 1: Closed-loop system (Σ_{0c1}) is asymptotically stable if there exist matrices $P > 0$, $R > 0$, $Q > 0$ and a scalar $\mu \in \mathbb{R}$ satisfying

$$\begin{bmatrix} \mathbf{P}^T\mathbf{A} + \mathbf{A}^T\mathbf{P} + \mathbf{T}_1 + d\mathbf{A}^T\mathbf{R}\mathbf{A} + \mu\begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} & \mathbf{P}\mathbf{B}_1 + d\mathbf{A}^T\mathbf{R}\mathbf{B}_1 - \mu\begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} & -\mu\mathbf{I} \\ * & \mathbf{T}_2 + d\mathbf{B}_1^T\mathbf{R}\mathbf{B}_1 + \mu\begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} & \mu\mathbf{I} \\ * & * & \mu I - d^{-1}R \end{bmatrix} < 0 \tag{2}$$

where $\mathbf{P} = \begin{bmatrix} P_1 & 0 \\ -P_2KC & P_2 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} A & B_0 \\ KC & -I \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} 0 & B_1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} R & 0 \\ 0 & \frac{1}{4d}P_2 \end{bmatrix}$, $\mathbf{I} = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\mathbf{T}_1 = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$, and $\mathbf{T}_2 = \begin{bmatrix} -Q - C^TK^TP_2KC & C^TK^TP_2 \\ * & -P_2 \end{bmatrix}$.

Proof: The derivative of $V(x_t)$ in (1) along the solution of system (Σ_{0c1}) with respect to t is given by

$$\begin{aligned} &\dot{V}(x_t) \\ =\ & x^T(t)[P_1A_c + A_c^TP_1]x(t) + 2x^T(t)P_1B_1KCx(t-d) \\ & + d\dot{x}^T(t)R\dot{x}(t) - \int_{t-d}^{t}\dot{x}^T(s)R\dot{x}(s)\,\mathrm{d}s \\ & + x^T(t)Qx(t) - x^T(t-d)Qx(t-d) \\ \le\ & x^T(t)[P_1A_c + A_c^TP_1]x(t) \\ & + 2x^T(t)P_1B_1KCx(t-d) + d\dot{x}^T(t)R\dot{x}(t) \\ & - d^{-1}\left[\left(\int_{t-d}^{t}\dot{x}(s)\,\mathrm{d}s\right)^T R \left(\int_{t-d}^{t}\dot{x}(s)\,\mathrm{d}s\right)\right] \\ & + x^T(t)Qx(t) - x^T(t-d)Qx(t-d) \end{aligned} \tag{3}$$

Note that, Jensen's integral inequality [4] has been used to obtain (3). Thus, $\dot{V}(x_t) < 0$ if

$$\xi^T(t)\begin{bmatrix} \Omega_1 + d\Omega_2^TR\Omega_2 & 0 \\ 0 & -d^{-1}R \end{bmatrix}\xi(t) < 0 \tag{4}$$

where $\xi(t) = \mathrm{col}\{x(t), x(t-d), \int_{t-d}^{t}\dot{x}(s)\,\mathrm{d}s\}$, and

$$\begin{aligned} \Omega_1 &= \begin{bmatrix} P_1A_c + A_c^TP_1 + Q & P_1B_1KC \\ * & -Q \end{bmatrix} \\ \Omega_2 &= \begin{bmatrix} A_c & B_1KC \end{bmatrix} \end{aligned}$$

By the Newton-Leibniz formula we have $x(t) - x(t-d) - \int_{t-d}^{t}\dot{x}(s)\,\mathrm{d}s = 0$, that is, $\begin{bmatrix} I & -I & -I \end{bmatrix}\xi(t) = 0$. By Finsler's Lemma [14], this equality and (4) hold together if and only if there exists a scalar $\mu \in \mathbb{R}$ such that

$$\begin{bmatrix} \Omega_1 + d\Omega_2^TR\Omega_2 + \mu J^TJ & -\mu J^T \\ * & \mu I - d^{-1}R \end{bmatrix} < 0 \tag{5}$$

where $J = \begin{bmatrix} I & -I \end{bmatrix}$. This implies that system (Σ_{0c1}) is asymptotically stable. Then comes the proof of the equivalence of (2) and (5) by defining $\mathfrak{B} = \begin{bmatrix} B_0 & B_1 \end{bmatrix}$ and two nonsingular matrices $G_1 = \mathrm{diag}\{\bar{S}, \bar{S}\}$ with $\bar{S} = \begin{bmatrix} I & 0 \\ KC & I \end{bmatrix}$, and

$$G_2 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

Pre- and post-multiplying (2) with $\mathrm{diag}\{G_2^TG_1^T, I\}$ and $\mathrm{diag}\{G_1G_2, I\}$, respectively, by algebraic manipulation, we have

$$
\begin{aligned}
& G_2^T \begin{bmatrix} \bar{S}^T(\mathbf{P}^T\mathbf{A}+\mathbf{A}^T\mathbf{P}+\mathbf{T}_1)\bar{S} & \bar{S}^T\mathbf{P}\mathbf{B}_1\bar{S} \\ * & \bar{S}^T\mathbf{T}_2\bar{S} \end{bmatrix} G_2 \\
= \; & G_2^T \left[\begin{array}{cc|cc} P_1A_c + A_c^TP_1 + Q & P_1B_0 & P_1B_1KC & P_1B_1 \\ * & -2P_2 & 0 & 0 \\ \hline \multicolumn{2}{c|}{*} & -Q & 0 \\ & & * & -P_2 \end{array}\right] G_2 \\
= \; & \left[\begin{array}{c|cc} \Omega_1 & \multicolumn{2}{c}{\begin{bmatrix} \mathfrak{B}^TP_1 & 0 \end{bmatrix}^T} \\ \hline * & -2P_2 & 0 \\ & 0 & -P_2 \end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
& G_2^T \begin{bmatrix} \bar{S}^T\mathbf{A}^T \\ \bar{S}^T\mathbf{B}_1^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & \frac{1}{4d}P_2 \end{bmatrix} \begin{bmatrix} \mathbf{A}\bar{S} & \mathbf{B}_1\bar{S} \end{bmatrix} G_2 \\
= \; & G_2^T \left[\begin{array}{cc} A_c^T & 0 \\ B_0^T & -I \\ \hline C^TK^TB_1^T & 0 \\ B_1^T & 0 \end{array}\right] \begin{bmatrix} R & 0 \\ 0 & \frac{1}{4d}P_2 \end{bmatrix} \\
& \left[\begin{array}{cc|cc} A_c & B_0 & B_1KC & B_1 \\ 0 & -I & 0 & 0 \end{array}\right] G_2 \\
= \; & \begin{bmatrix} \Omega_2^T & 0 \\ \mathfrak{B}^T & \begin{bmatrix} -I \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & \frac{1}{4d}P_2 \end{bmatrix} \\
& \times \begin{bmatrix} \Omega_2 & \mathfrak{B} \\ 0 & \begin{bmatrix} -I \\ 0 \end{bmatrix}^T \end{bmatrix} \\
= \; & \begin{bmatrix} \Omega_2^TR\Omega_2 & \Omega_2^TR\mathfrak{B} \\ * & \mathfrak{B}^TR\mathfrak{B} + \begin{bmatrix} \frac{1}{4d}P_2 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}
\end{aligned}
$$

and $\begin{bmatrix} \mathbf{I}^T & -\mathbf{I}^T \end{bmatrix} G_1G_2 = \begin{bmatrix} J & 0 & 0 \end{bmatrix}$. That is to say, inequality (2) is equivalent to

$$
\left[\begin{array}{cc|c} \Omega_1 + d\Omega_2^TR\Omega_2 + \mu J^TJ & \begin{bmatrix} \mathfrak{B}^TP_1 & 0 \end{bmatrix}^T + d\Omega_2^TR\mathfrak{B} & -\mu J^T \\ * & d\mathfrak{B}^TR\mathfrak{B} + \begin{bmatrix} -\frac{7}{4}P_2 & 0 \\ 0 & -P_2 \end{bmatrix} & 0 \\ \hline * & * & \mu I - d^{-1}R \end{array}\right] < 0
$$

which, with congruent transformation, is equivalent to

$$
\left[\begin{array}{cc|c} \Omega_1 + d\Omega_2^TR\Omega_2 + \mu J^TJ & -\mu J^T & \begin{bmatrix} \mathfrak{B}^TP_1 & 0 \end{bmatrix}^T + d\Omega_2^TR\mathfrak{B} \\ * & \mu I - d^{-1}R & 0 \\ \hline * & * & d\mathfrak{B}^TR\mathfrak{B} + \begin{bmatrix} -\frac{7}{4}P_2 & 0 \\ 0 & -P_2 \end{bmatrix} \end{array}\right] < 0
$$

Notice that only the lower diagonal block matrix in the left hand side of the above inequality is dependent on the matrix $P_2 > 0$. There exists a sufficiently large $P_2 > 0$ such that (2) is equivalent to (5). That completes the proof. □

Remark 1: The advantage of Theorem 1 lies in not only the separation of B_0, B_1 and KC, but also in the separation of the Lyapunov matrix $P_1 > 0$ and the controller matrix K. This feature enables us to parametrize K by a free matrix $P_2 > 0$, independent of the Lyapunov matrix P_1 used for checking stability or performances directly. Therefore, less conservative results will be obtained since no additional constraints have been introduced to deal with the nonconvex terms of the Lyapunov matrix and the controller matrix when it is parametrized.

III. $\mathcal{H}_\infty$ Control

Consider the following continuous-time systems with input delay:

$$
\begin{aligned}
(\Sigma_1): \quad \dot{x}(t) &= Ax(t) + B_0u(t) + B_1u(t-d) + E\omega(t) \\
z(t) &= C_zx(t) + Du(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $z(t) \in \mathbb{R}^l$ and $\omega(t) \in \mathbb{R}^q$ are the system state, exogenous input, measured output, regulated output and disturbance input, respectively, and A, B_0, B_1, E, C_z, D, C are constant system matrices with appropriate dimensions. When $t \in [-d, 0]$, $x(t) = \phi(t)$, where $d > 0$ is a constant lumped delay. A detailed discussion on how to apply the proposed output feedback characterization on $\mathcal{H}_\infty$ performance control will be conducted for the following closed-loop system (Σ_{1c1}), which is derived from (Σ_1) via the SOF controller (C_1):

$$
\begin{aligned}
(\Sigma_{1c1}): \quad \dot{x}(t) &= A_cx(t) + B_1KCx(t-d) + E\omega(t) \\
z(t) &= C_cx(t)
\end{aligned}
$$

where $A_c = A + B_0KC$, and $C_c = C_z + DKC$. Define $\varsigma(t) = \mathrm{col}\left\{x(t), x(t-d), \omega(t), \int_{t-d}^{t} \dot{x}(s)\,\mathrm{d}s\right\}$.

For the closed-loop system (Σ_{1c1}) and a prescribed scalar $\gamma_\infty > 0$, we define the performance index

$$
\mathfrak{J}(\omega) = \int_0^\infty (\gamma_\infty^{-1}z^T(s)z(s) - \gamma_\infty\omega^T(s)\omega(s))\mathrm{d}s
$$

for all nonzero $\omega \in \mathcal{L}_2^q[0, \infty)$.

Theorem 2: Consider the closed-loop system (Σ_{1c1}). For a prescribed scalar $\gamma_\infty > 0$, the cost function $\mathfrak{J}(\omega) < 0$ for all nonzero $\omega \in \mathcal{L}_2^q[0, \infty)$, if there exist $P_1 > 0$, $P_2 > 0$, $R > 0$, $Q > 0$, and a scalar $\mu \in \mathbb{R}$ satisfying

$$
\mathbf{\Gamma}_\infty^S = \begin{bmatrix} \mathbf{\Lambda}_{1\infty} + d\mathbf{\Lambda}_2^T\mathbf{R}\mathbf{\Lambda}_2 + \mu\mathbf{J}^T\mathbf{J} & -\mu\mathbf{J}^T \\ * & \mu I - d^{-1}R \end{bmatrix} < 0 \tag{6}
$$

where

$$
\begin{aligned}
\mathbf{\Lambda}_{1\infty} &= \begin{bmatrix} \begin{matrix} \mathbf{P}^T\mathbf{A} + \mathbf{A}^T\mathbf{P} \\ +\mathbf{T}_1 + \gamma_\infty^{-1}\mathbf{C}_c^T\mathbf{C}_c \end{matrix} & \mathbf{P}^T\mathbf{B}_1 & \mathbf{P}^T\mathbf{E} \\ * & \mathbf{T}_2 & 0 \\ * & * & -\gamma_\infty I \end{bmatrix} \\
\mathbf{\Lambda}_2 &= \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{E} \end{bmatrix}, \ \mathbf{J} = \begin{bmatrix} \mathbf{I} & -\mathbf{I} & 0 \end{bmatrix}
\end{aligned}
$$

with $\mathbf{C}_c = [\begin{array}{cc} C_z & D \end{array}]$, $\mathbf{E} = [\begin{array}{cc} E^T & 0 \end{array}]^T$, and $\mathbf{P}$, $\mathbf{A}$, $\mathbf{B}_1$, $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{I}$ and $\mathbf{R}$ are defined as in Theorem 1.

Proof: Using the arguments in the previous subsection, we apply the Lyapunov-Krasovskii functional $V(x_t)$ defined in (1) and require $\dot{V}(x_t) + \gamma_\infty^{-1} z^T(t)z(t) - \gamma_\infty \omega^T(t)\omega(t) \leq \varsigma^T(t)\Gamma_\infty^S \varsigma(t) < 0$ for all $\varsigma(t)$ such that $[\begin{array}{cccc} I & -I & 0 & -I \end{array}]\varsigma(t) = 0$, where

$$
\begin{aligned}
\Gamma_\infty^S &\triangleq \begin{bmatrix} \Lambda_{1\infty} + d\Lambda_2^T R \Lambda_2 + \mu J^T J & -\mu J^T \\ * & \mu I - d^{-1}R \end{bmatrix} \\
\Lambda_{1\infty} &= \begin{bmatrix} \begin{array}{c} P_1 A_c + A_c^T P_1 \\ +Q + \gamma_\infty^{-1} C_c^T C_c \end{array} & P_1 B_1 K C & P_1 E \\ * & -Q & 0 \\ * & * & -\gamma_\infty I \end{bmatrix} \\
\Lambda_2 &= \begin{bmatrix} A_c & B_1 K C & E \end{bmatrix}, \; J = \begin{bmatrix} I & -I & 0 \end{bmatrix}.
\end{aligned}
$$

We will establish that $\mathbf{\Gamma}_\infty^S < 0$ is equivalent to $\Gamma_\infty^S < 0$ by defining two nonsingular matrices $G_1 = \mathrm{diag}\{\bar{S}, \bar{S}, I\}$ with $\bar{S} = \begin{bmatrix} I & 0 \\ KC & I \end{bmatrix}$, and

$$
G_2 = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 \end{bmatrix}
$$

A similar process as that in the proof of Theorem 1, by pre- and post-multiplying (6) with $\mathrm{diag}\{G_2^T G_1^T, I\}$ and $\mathrm{diag}\{G_1 G_2, I\}$, respectively,

$\mathbf{\Gamma}_\infty^S < 0$ is changed to $\begin{bmatrix} \Gamma_\infty^S & \Upsilon_1 \\ * & \Upsilon_2 \end{bmatrix} < 0$ where

$$
\begin{aligned}
\Upsilon_1 &= \begin{bmatrix} P_1 B_0 + \gamma_\infty^{-1} C_c^T D & P_1 B_1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad + d\Lambda_2^T R \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\
\Upsilon_2 &= d \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} R \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} -\frac{7}{4}P_2 + \gamma_\infty^{-1} D^T D & 0 \\ 0 & -P_2 \end{bmatrix}
\end{aligned}
$$

As only the lower diagonal block matrix Υ_2 is dependent on the matrix $P_2 > 0$, there exists a sufficiently large matrix $P_2 > 0$ such that $\mathbf{\Gamma}_\infty^S < 0$ is equivalent to $\Gamma_\infty^S < 0$. That completes the proof. □

IV. Controller Parametrization and Computation

In this section, we are now in a position to establish a new sufficient condition for SOF $\mathcal{H}_\infty$ control of system (Σ_1) and to compute controller gains via an effective algorithm.

Theorem 3: Closed-loop system (Σ_{1c1}) is asymptotically stable, if there exist matrices $P_1 > 0$, $P_2 > 0$, $R > 0$, $Q > 0$, L, N and a scalar $\mu \in \mathbb{R}$ such that

$$
\Phi \triangleq \begin{bmatrix} \Phi_1 + \mu \mathbf{J}^T \mathbf{J} & -\mu \mathbf{J}^T & \Phi_2^T \\ * & \mu I - d^{-1}R & 0 \\ * & * & \Phi_3 \end{bmatrix} < 0 \qquad (7)
$$

where

$$
\begin{aligned}
\Phi_1 &= \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ * & \Phi_{22} & 0 \\ * & * & -\gamma_\infty I \end{bmatrix} \\
\Phi_{11} &= \left[\begin{array}{cc} \begin{array}{c} P_1 A + A^T P_1 + Q \\ +2M + \gamma_\infty^{-1} C_z^T C_z \end{array} & \begin{array}{c} P_1 B_0 + 2C^T L^T \\ +\gamma_\infty^{-1} C_z^T D \end{array} \\ \hline * & -2P_2 + \gamma_\infty^{-1} D^T D \end{array}\right] \\
\Phi_{12} &= \begin{bmatrix} 0 & P_1 B_1 \\ 0 & 0 \end{bmatrix}, \; \Phi_{13} = \begin{bmatrix} P_1 E \\ 0 \end{bmatrix} \\
\Phi_{22} &= \begin{bmatrix} -Q + M & C^T L^T \\ * & -P_2 \end{bmatrix} \\
\Phi_2 &= \begin{bmatrix} RA & RB_0 & 0 & RB_1 & RE \\ LC & -P_2 & 0 & 0 & 0 \end{bmatrix} \\
\Phi_3 &= \mathrm{diag}\left\{-d^{-1}R, -4P_2\right\} \\
M &= -N^T LC - C^T L^T N + N^T P_2 N
\end{aligned}
$$

and $\mathbf{J}$ is defined in Theorem 2. Under such a condition, the matrices of an SOF $\mathcal{H}_\infty$ controller (C_1) can be parameterized as $K = P_2^{-1} L$.

Proof: Expanding inequality (6) yields that

$$
\begin{bmatrix} \bar{\Phi}_1 + d\mathbf{\Lambda}_2^T \mathbf{R} \mathbf{\Lambda}_2 + \mu \mathbf{J}^T \mathbf{J} & -\mu \mathbf{J}^T \\ * & \mu I - d^{-1}R \end{bmatrix} < 0
$$

where

$$
\bar{\Phi}_1 = \begin{bmatrix} \bar{\Phi}_{11} & \Phi_{12} & \Phi_{13} \\ * & T_2 & 0 \\ * & * & -\gamma_\infty I \end{bmatrix}
$$

$$
\bar{\Phi}_{11} = \begin{bmatrix} \begin{array}{c} P_1 A + A^T P_1 \\ +Q + \gamma_\infty^{-1} C_z^T C_z \\ -2C^T K^T P_2 K C \end{array} & \begin{array}{c} P_1 B_0 + \gamma_\infty^{-1} C_z^T D \\ +2C^T K^T P_2 \end{array} \\ * & -2P_2 + \gamma_\infty^{-1} D^T D \end{bmatrix}
$$

Define

$$
\bar{\Phi}_2 = \begin{bmatrix} RA & RB_0 & 0 & RB_1 & RE \\ P_2 KC & -P_2 & 0 & 0 & 0 \end{bmatrix}
$$

it follows

$$
\begin{aligned}
\mathbf{\Lambda}_2^T \mathbf{R} \mathbf{\Lambda}_2 &= \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{E} \end{bmatrix}^T \mathbf{R} \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{E} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A}^T \\ \mathbf{B}_1^T \\ \mathbf{E}^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} R^{-1} & 0 \\ 0 & \frac{1}{4d} P_2^{-1} \end{bmatrix} \\
&\quad \times \begin{bmatrix} R & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{E} \end{bmatrix} \\
&= \bar{\Phi}_2^T \begin{bmatrix} R^{-1} & 0 \\ 0 & \frac{1}{4d} P_2^{-1} \end{bmatrix} \bar{\Phi}_2
\end{aligned}
$$

Thus, (6) is equivalent to

$$
\begin{bmatrix} \bar{\Phi}_1 + \mu \mathbf{J}^T \mathbf{J} & -\mu \mathbf{J}^T & \bar{\Phi}_2^T \\ * & \mu I - d^{-1}R & 0 \\ * & * & \Phi_3 \end{bmatrix} < 0 \qquad (8)
$$

Now comes the validation that (8) is equivalent to (7).

(Sufficiency) It follows that $P_2 > 0$, which implies that $K = P_2^{-1}L$ is meaningful, and $L = P_2K$. Substituting it to (7) and noting, for any real matrix N with appropriate dimension,

$$(N - KC)^T P_2 (N - KC) \geq 0$$

we have (8) holds with the following property of all the terms $-C^T K^T P_2 KC$ just in the diagonal blocks.

$$-C^T K^T P_2 KC \leq -N^T LC - C^T L^T N + N^T P_2 N = M$$

(Necessity) Assume inequality (8) holds. Then, by setting $N = KC$, we obtain

$$\begin{aligned} & -C^T K^T P_2 KC \\ = \;& -C^T K^T P_2 KC + (N - KC)^T P_2 (N - KC) \\ = \;& -N^T P_2 KC - C^T K^T P_2 N + N^T P_2 N \end{aligned}$$

Substituting it into (8), and denoting $L = P_2K$, (7) is obtained. This completes the proof. □

Remark 2: It is worth pointing out that the parametrization of the controller matrices by our approach is flexible. Indeed, P_2 can be set to be any sufficiently large positive definite matrix, and thus more synthesis problems such as simultaneous stabilization, structural controller synthesis can be treated readily in this framework.

To facilitate exposition, $\Phi(N, d) \equiv \Phi$ in (7) is taken in the sequel. When d and N are fixed, (7) becomes a strict LMI problem to search an optimal performance index γ_∞, which can be verified easily by conventional LMI solver. The remaining problem is how to select the matrix N. It can be seen from the proof of Theorem 2 that the left hand side of (7), $\Phi(N, d)$ achieves its minimum when $N = P_2^{-1}LC$, which can be used to construct an iteration rule. We summarize briefly our analysis on N in the following proposition.

When $P_1 > 0$, $P_2 > 0$, $Q > 0$, L, d and μ are fixed, the following relationship holds for any real matrix N,

$$\Phi(P_2^{-1}LC, d) \leq \Phi(N, d)$$

It follows that the scalar $\varepsilon \in \mathbb{R}$ satisfying $\Phi(N, d) < \varepsilon I$ achieves its global minimum only if $N = P_2^{-1}LC = KC$. Therefore, the following iteration algorithm is constructed to solve the condition of Theorem 2.

Algorithm SOF-HC (SOF $\mathcal{H}_\infty$ Control):

- Step 1. Set $m = 1$, and $\varepsilon_0^* > 0$, $c > 0$ be two prescribed initial values. Select an initial matrix N_1 and a delay upper bound $\bar{d}$ such that the closed-loop system (Σ_{1c1}), when KC is substituted by N_1, is $\mathcal{H}_\infty$ stable with a γ_∞.
- Step 2. For the fixed N_m, solve the following convex optimization problem with respect to L_m, μ_m, $P_{1m} > 0$, $P_{2m} > 0$, $Q_m > 0$:

$$\begin{aligned} & \min \quad \varepsilon_m \\ \text{s.t.} \quad & \Phi(N_m, \bar{d}) < \varepsilon_m I \\ & \varepsilon_m > -c \end{aligned} \tag{9}$$

Denote ε_m^* as the minimized value of ε_m satisfying (9). If $\varepsilon_m^* \leq 0$, the system (Σ_1) is stabilizable via the SOF controller (C_1). The gain matrix K of (C_1) can be obtained as $K = P_{2m}^{-1}L_m$, STOP, else, go to Step 3.

- Step 3. If $|\varepsilon_m^* - \varepsilon_{m-1}^*| \leq \delta$, a prescribed tolerance, then go to Step 4, else update N_{m+1} as

$$N_{m+1} = (P_{2m})^{-1} L_m C$$

and set $m = m + 1$, then go to Step 2.

Remark 3: It follows from that the sequence ε_m^* is monotonic decreasing with respect to m for a fixed d and has a lower bound c. Therefore, the convergence of the iteration is guaranteed which leads to stabilization of system (Σ_1) under a performance level γ_∞.

Remark 4: The initial matrix N_1 can be considered as a state feedback stabilizing controller matrix, which can be found by existing approaches for stabilizability analysis. If no such matrices are found, we will conclude immediately that the system is not stabilizable via (C_1). Like many other iteration algorithms, the sequence of iterates depends on the selection of initial values, and an appropriate selection will improve the solvability. Here, we attempt to utilize a state feedback controller as the initial value N_1 which satisfies $A + B_0N$ or $A + (B_0 + B_1)N$ being Hurwitz stable for system (Σ_{1c1}). There is some conservatism since it is only a approximate solution obtained from a delay-independent stability condition. If it fails, Zhang *et al.* in [17] gives a method to obtain a new state- and input-delay-dependent state feedback controller to ensure the stability of the closed-loop system.

The initial performance level γ_∞ in Step 1 of Algorithm SOF-HC can be chosen as an appropriate value for system (Σ_1) stabilized by a state feedback controller. If this fails and no negative ε_m^* can be found, increase it to $\gamma_\infty = \gamma_\infty + k$ for some $k > 0$ until (9) is feasible, else STOP (i.e. the system may not be $\mathcal{H}_\infty$ stabilizable via the SOF controller (C_1)).

V. Numerical Example

This section presents two numerical examples to illustrate the design approach for SOF controllers described in this paper.

Example 1

An application example presented here is the system proposed in [15], where a 'T-shape' inverted pendulum is controlled through a simulated TCP network. The pendulum dynamics is 4^{th} order, nonminimum phase, open-loop unstable and with coupled nonlinearities. Its linearized model is given in [3] with the parameters as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -21.54 & 0 & 14.96 & 0 \\ 0 & 0 & 0 & 1 \\ 65.28 & 0 & -15.59 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 8.10 \\ 0 \\ -10.31 \end{bmatrix}$$

The input delay τ is constant and the corresponding (Σ_0) has a particular form with $B_0 = 0$ and $C = I$. Suppose the disturbance input matrix $E = [\ 0.1 \quad 0 \quad 0.1 \quad 0\ ]^T$, and $C_z = [\ 1 \quad 1 \quad 1 \quad 1\ ]^T$, $D = -1.0340$.

We now apply the proposed approach to find an SOF controller to stabilize this pendulum. An initial matrix $N_1 = [-11.1384 \quad -2.1632 \quad -3.3258 \quad -0.9217]$ is chosen for (C_1) which is obtained directly by solving a system pair (A, B_1) state stabilization conditions $AX + B_1Y + (AX + B_1Y)^T < 0$ and $X > 0$, with $N_1 = YX^{-1}$. A desired SOF $\mathcal{H}_\infty$ controller is obtained as $u(t) = [\ -11.1342 \quad -2.1651 \quad -3.3257 \quad -0.9226\ ]y(t)$ to satisfy a performance level $\gamma_\infty = 4.6268$ after 3 iterations, corresponding to a delay interval $(0, 0.07]$.

Example 2

A second order input-delayed system in [2] is considered with the following parameters:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Other system matrices in (Σ_1) are selected as $E = [-0.230 \quad 0.190]^T$, $C_z = [0.1462 \quad -0.2361]$, $D = -0.034$ and $C = [1 \quad -1]$ to evaluate the $\mathcal{H}_\infty$ performance. Choose an initial matrix $N_1 = [-0.6149 \quad -1.3964]$ to make $(A+B_1N_1)$ stable for SOF controller (C_1), the numerical results are listed in Table I.

Furthermore, consider the same model with a different output matrix $C = [1 \quad 0]$. It shows clearly in this example that neither (A, B_0) is stabilizable, nor (A, C) is detectable. With the same initial matrix N_1, the desired SOF controller and the corresponding system performance are given in Table I by applying Algorithm SOF-HC again.

TABLE I
NUMERICAL RESULTS FOR EXAMPLE 2 WITH DELAY INTERVAL $(0, 1.7]$

	γ_∞	Controller gain K	Iteration No.
$C = [\ 1 \quad -1\ ]$	0.3101	-0.8621	2
$C = [\ 1 \quad 0\ ]$	0.6041	-0.7739	2

VI. CONCLUSION

This paper has studied the SOF $\mathcal{H}_\infty$ control problem for input-delayed systems from a new perspective. Input-delay-dependent $\mathcal{H}_\infty$ criteria via a SOF controller is derived from a new equivalent characterization on the stabilizability of the system in terms of matrix inequalities by introducing a slack positive definite matrix, and an iterative algorithm is developed to solve the condition. Although the proposed approach is not guaranteed to find a solution even it exists, it is very effective since there is no need to introduce additional constraints to linearize the product term of Lyapunov matrix and controller gain during the parametrization.

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