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SOME DISCRETE POINCARÉ-TYPE INEQUALITIES

WING-SUM CHEUNG

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ABSTRACT. Some discrete analogue of Poincaré-type integral inequalities involving many functions of many independent variables are established. These in turn can serve as generators of further interesting discrete inequalities.

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1. Introduction. It is well known that differential and integral equations appear in virtually every area of analysis, and among all tools available for the study of quantitative as well as qualitative properties of their solutions, integral inequalities are essential and in fact indispensable. Analogously, as discrete phenomena prevail in nature, difference equations are of fundamental importance in finite element analysis and the discrete analogues of integral inequalities naturally serve the subject as a handy effective tool (cf. [1]).

One of the most inspiring integral inequalities is the Poincaré's inequality. It says that for any bounded region Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ and any continuously differentiable real-valued function f on Ω which vanishes on the boundary $\partial\Omega$ of Ω, one has

$$\lambda_0 \int_\Omega f^2 dx \le \int_\Omega |\nabla f|^2 dx, \tag{1.1}$$

where λ_0 is the smallest eigenvalue of the problem

$$\begin{cases} \triangle f + \lambda f = 0 & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{1.2}$$

Exhibiting an effective estimate of the average of f^2 on Ω by that of $|\nabla f|^2$ on Ω, Poincaré inequality is one of the few most important multi-dimensional integral inequalities. Because of its fundamental importance, a vast stock of investigations and generalizations of it has been established in these years. Such generalizations and improvements of the inequality are in general known as Poincaré-type integral inequalities. A brief account of such inequalities can be found in, say, Beckenbach-Bellman [2], Hardy-Littlewood-Pólya [8], Milovanović-Mitrinović-Rassias [12], Mitrinović [13], and Nirenberg [14]. More recent results include those in Horgan et al. [9, 10, 11], Pachpatte [15, 16], Rassias [17, 18], Cheung [3, 4, 6], and Cheung-Rassias [7]. It is the purpose of this paper to establish some new discrete analogues of Poincaré-type inequalities which improve and generalize some existing results in [5]. The importance of the results here does not confine to their neatness and intrinsic beauty, but also lies on the fact that they can be used in turn to serve as generators of other interesting discrete inequalities.

2. Notations and Preliminaries. In this paper, $m \geq 2$ and $n \geq 1$ will denote two fixed integers. For consistency we will use exclusively the Greek alphabet $\alpha, \beta, \gamma, \ldots$ as indices from 1 to m and the English letters $i, j, k, \ldots$ as indices from 1 to n. Let $\Omega = \prod_{i=1}^{n}[0, b_i] \cap \mathbb{Z}^n \subset \mathbb{R}^n$, where $b_i \in \mathbb{N} \cup \{0\}$ for each i, be a fixed rectangular lattice of integral points. As customary, a general point in Ω will be denoted as $t = (t_1, \ldots, t_n)$. $\mathcal{F}(\Omega)$ will denote the space of all real-valued functions on Ω and $\mathcal{F}_0(\Omega)$ the subspace of $\mathcal{F}(\Omega)$ consisting of all those functions in $\mathcal{F}(\Omega)$ which vanish on the boundary $\partial\Omega$ of Ω. For the sake of convenience, we shall extend the domain of definition of each function in $\mathcal{F}(\Omega)$, hence also those in $\mathcal{F}_0(\Omega)$, trivially to the entire $\mathbb{Z}^n$ and think of $\mathcal{F}(\Omega)$ as the collection of real-valued functions on $\mathbb{Z}^n$ with support in Ω, and $\mathcal{F}_0(\Omega)$ as those with support in $\Omega \setminus \partial\Omega$. Furthermore, for the sake of simplicity, since the indices i, j, k will always be running from 1 to n and α, β, γ from 1 to m, summations and products over i, j, k and α, β, γ will be abbreviated as $\sum_\alpha, \prod_i$, and so forth, unless possible confusion may arise.

For any $f \in \mathcal{F}(\Omega)$, define

$$f_j : \mathbb{Z}^n \longrightarrow \mathbb{R} \tag{2.1}$$

by

$$f_j(t) := \triangle_j f(t) = f(t_1, \ldots, t_j, \ldots, t_n) - f(t_1, \ldots, t_j - 1, \ldots, t_n). \tag{2.2}$$

Observe that if $f \in \mathcal{F}_0(\Omega)$, $f_j \in \mathcal{F}(\Omega)$ for all j. However, in general $f_j \notin \mathcal{F}_0(\Omega)$.

As usual, we define the gradient of f as

$$\nabla f := (f_1, \ldots, f_n) \tag{2.3}$$

and its norm as

$$|\nabla f| := \left(\sum_j |f_j|^2\right)^{1/2}. \tag{2.4}$$

For any $p > 0$ and $f \in \mathcal{F}(\Omega)$, the ℓ_p-norm of f is defined as

$$\|f\|_p := \left(\sum_{t \in \Omega} |f(t)|^p\right)^{1/p}. \tag{2.5}$$

If $f \in \mathcal{F}_0(\Omega)$, $f_j \in \mathcal{F}(\Omega)$ for all j and in this case the ℓ_p-norm of ∇f is defined as

$$\|\nabla f\|_p := \left(\sum_{t \in \Omega} |\nabla f(t)|^p\right)^{1/p} = \left[\sum_{t \in \Omega}\left(\sum_j |f_j|^2\right)^{p/2}\right]^{1/p}. \tag{2.6}$$

3. Discrete Poincaré-type inequalities. Let $B = \max\{b_j : 1 \leq j \leq n\}$.

Theorem 3.1. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$, any real numbers $p_\alpha \geq 2$, $q_\alpha \geq 0$ with $\sum_\alpha q_\alpha / p_\alpha = 1$, and any $C_\alpha > 0$,*

$$\left\|\prod_\alpha (f^\alpha)^{q_\alpha}\right\|_1 \leq \frac{C}{n}\sum_\alpha \frac{q_\alpha}{p_\alpha}\left(\frac{BC_\alpha}{2}\right)^{p_\alpha} \|\nabla f^\alpha\|_{p_\alpha}^{p_\alpha}, \tag{3.1}$$

where $C := \prod_\beta C_\beta^{-q_\beta}$.

Theorem 3.1 generalizes and improves some existing results of discrete Poincaré-type inequalities in the literature [5]. For example, the following consequences are easily derivable from Theorem 3.1.

Corollary 3.2. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$ and any real numbers $p_\alpha \geq 2$, $q_\alpha \geq 0$ with $\sum_\alpha q_\alpha/p_\alpha = 1$,*

$$\left\|\prod_\alpha (f^\alpha)^{q_\alpha}\right\|_1 \leq \frac{1}{n}\sum_\alpha \frac{q_\alpha}{p_\alpha}\left(\frac{B}{2}\right)^{p_\alpha} \|\nabla f^\alpha\|_{p_\alpha}^{p_\alpha}. \tag{3.2}$$

Proof. This follows immediately from Theorem 3.1 by letting $C_\alpha = 1$ for all α. □

Corollary 3.3. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$ and any real numbers $p_\alpha \geq 2$, $q_\alpha > 0$ with $\sum_\alpha q_\alpha/p_\alpha = 1$,*

$$\left\|\prod_\alpha (f^\alpha)^{q_\alpha}\right\|_1 \leq \frac{C}{n}\sum_\alpha \|\nabla f^\alpha\|_{p_\alpha}^{p_\alpha}, \tag{3.3}$$

where

$$C := \prod_\alpha \left[\left(\frac{B}{2}\right)^{q_\alpha}\left(\frac{q_\alpha}{p_\alpha}\right)^{q_\alpha/p_\alpha}\right]. \tag{3.4}$$

Proof. This follows immediately from Theorem 3.1 by letting

$$C_\alpha = \frac{2}{B}\left(\frac{p_\alpha}{q_\alpha}\right)^{1/p_\alpha} \quad \forall \alpha. \tag{3.5}$$

□

Corollary 3.4 [5]. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$ and any real numbers $p_\alpha \geq 2$ with $\sum_\alpha 1/p_\alpha = 1$,*

$$\left\|\prod_\alpha f^\alpha\right\|_1 \leq \frac{1}{n}\sum_\alpha \frac{1}{p_\alpha}\left(\frac{B}{2}\right)^{p_\alpha} \|\nabla f^\alpha\|_{p_\alpha}^{p_\alpha}. \tag{3.6}$$

Proof. It is immediate from Corollary 3.2 by putting $q_\alpha = 1$ for all α. □

Corollary 3.5 [5]. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$ and any real numbers $q_\alpha \geq 0$ with $q := \sum_\alpha q_\alpha \geq 2$,*

$$\left\|\prod_\alpha (f^\alpha)^{q_\alpha}\right\|_1 \leq \frac{1}{n}\left(\frac{B}{2}\right)^{q}\sum_\alpha \frac{q_\alpha}{q} \|\nabla f^\alpha\|_{q}^{q}. \tag{3.7}$$

Proof. It is immediate from Corollary 3.2 by putting $p_\alpha = q$ for all α. □

Corollary 3.6 [5]. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$,*

$$\left\|\prod_\alpha (f^\alpha)\right\|_1 \leq \frac{1}{nm}\left(\frac{B}{2}\right)^{m}\sum_\alpha \|\nabla f^\alpha\|_{m}^{m}. \tag{3.8}$$

Proof. This follows from Corollary 3.4 by setting $p_\alpha = m$ for all α or from Corollary 3.5 by setting $q_\alpha = 1$ for all α. □

To establish Theorem 3.1, we need the following basic lemmas.

Lemma 3.7 [8, 13]. *For any $p_\alpha, q_\alpha, c_\alpha > 0$ with $\sum q_\alpha/p_\alpha = 1$,*

$$\prod_\alpha c_\alpha^{q_\alpha} \leq \sum_\alpha \frac{q_\alpha}{p_\alpha} c_\alpha^{p_\alpha}, \tag{3.9}$$

where the equality holds if and only if $c_1 = \cdots = c_m$.

Lemma 3.8 [8, 13]. *For any* $r_i \geq 0$ *and* $s > 0$,

$$\left(\sum_i r_i\right)^s \leq c(s,n)\sum_i r_i^s, \tag{3.10}$$

where

$$c(s,n) = \begin{cases} n^{s-1} & \text{if } s > 1 \\ 1 & \text{if } 0 \leq s \leq 1. \end{cases} \tag{3.11}$$

Lemmas 3.7 and 3.8 are fundamental inequalities easily derivable from the arithmetic-geometric mean inequality. For their proofs, one is referred to, for example, [8, 13].

Lemma 3.9. *For any* $f \in \mathcal{F}_0(\Omega)$ *and any* $t \in \Omega$,

$$|f(t)| \leq \frac{1}{2n}\sum_i \sum_{u_i=1}^{b_i} |f_i(t_1,\ldots,t_{i-1},u_i,t_{i+1},\ldots,t_n)|. \tag{3.12}$$

Proof. Since $f = 0$ on $\partial\Omega$, for each $i = 1,\ldots,n$, we have

$$\begin{aligned} f(t) &= \sum_{u_i=1}^{t_i} f_i(t_1,\ldots,t_{i-1},u_i,t_{i+1},\ldots,t_n), \\ f(t) &= -\sum_{u_i=t_i+1}^{b_i} f_i(t_1,\ldots,t_{i-1},u_i,t_{i+1},\ldots,t_n). \end{aligned} \tag{3.13}$$

Taking absolute value of each of these equations and adding them up with respect to i, we have

$$2n|f(t)| \leq \sum_i \sum_{u_i=1}^{b_i} |f_i(t_1,\ldots,t_{i-1},u_i,t_{i+1},\ldots,t_n)|, \tag{3.14}$$

hence the lemma is proved. □

Proof of Theorem 3.1. By Lemmas 3.7, 3.8, and 3.9, we have

$$\begin{aligned} \prod_\alpha |f^\alpha(t)|^{q_\alpha} &= \left(\prod_\beta C_\beta^{-q_\beta}\right)\prod_\alpha |C_\alpha f^\alpha(t)|^{q_\alpha} \\ &\leq C\sum_\alpha \frac{q_\alpha}{p_\alpha} C_\alpha^{p_\alpha}|f^\alpha(t)|^{p_\alpha} \\ &\leq C\sum_\alpha \frac{q_\alpha}{p_\alpha} C_\alpha^{p_\alpha}\left[\frac{1}{2n}\sum_i \sum_{u_i=1}^{b_i} |f_i^\alpha(t_1,\ldots,u_i,\ldots,t_n)|\right]^{p_\alpha} \\ &\leq C\sum_\alpha \frac{q_\alpha}{p_\alpha} C_\alpha^{p_\alpha}\left(\frac{1}{2n}\right)^{p_\alpha} c(p_\alpha,n)\cdot\sum_i\left(\sum_{u_i=1}^{b_i} |f_i^\alpha(t_1,\ldots,u_i,\ldots,t_n)|\right)^{p_\alpha} \end{aligned} \tag{3.15}$$

for all $t \in \Omega$. Since $p_\alpha \geq 2$, we have $c(p_\alpha,n) = n^{p_\alpha-1}$ and so

$$\prod_\alpha |f^\alpha(t)|^{q_\alpha} \leq \frac{C}{n}\sum_\alpha \frac{q_\alpha}{p_\alpha}\left(\frac{C_\alpha}{2}\right)^{p_\alpha}\sum_i\left(\sum_{u_i=1}^{b_i} |f_i^\alpha(t_1,\ldots,u_i,\ldots,t_n)|\right)^{p_\alpha}. \tag{3.16}$$

By Hölder's inequality, this gives

$$\begin{aligned}
&\prod_\alpha |f^\alpha(t)|^{q_\alpha} \\
&\le \frac{C}{n}\sum_\alpha \frac{q_\alpha}{p_\alpha}\left(\frac{C_\alpha}{2}\right)^{p_\alpha}\cdot\sum_i\left[\left(\sum_{u_i=1}^{b_i}1\right)^{(p_\alpha-1)/p_\alpha}\cdot\left(\sum_{u_i=1}^{b_i}|f_i^\alpha(t_1,\dots,u_i,\dots,t_n)|^{p_\alpha}\right)^{1/p_\alpha}\right]^{p_\alpha} \\
&= \frac{C}{n}\sum_\alpha \frac{q_\alpha}{p_\alpha}\left(\frac{C_\alpha}{2}\right)^{p_\alpha}\sum_i\left[b_i^{p_\alpha-1}\sum_{u_i=1}^{b_i}|f_i^\alpha(t_1,\dots,u_i,\dots,t_n)|^{p_\alpha}\right] \\
&\le \frac{C}{n}\sum_\alpha \frac{q_\alpha}{p_\alpha}\left(\frac{C_\alpha}{2}\right)^{p_\alpha}B^{p_\alpha-1}\sum_i\sum_{u_i=1}^{b_i}|f_i^\alpha(t_1,\dots,u_i,\dots,t_n)|^{p_\alpha}.
\end{aligned} \tag{3.17}$$

Now by a change of the dummy variables, it is easy to see that

$$\begin{aligned}
\sum_i\sum_{t\in\Omega}\sum_{u_i=1}^{b_i}|f_i^\alpha(t_1,\dots,u_i,\dots,t_n)|^{p_\alpha} &= \sum_i\sum_{u_i=1}^{b_i}\sum_{t\in\Omega}|f_i^\alpha(t_1,\dots,u_i,\dots,t_n)|^{p_\alpha} \\
&= \sum_i\sum_{u_i=1}^{b_i}\sum_{t\in\Omega}|f_i^\alpha(t_1,\dots,t_i,\dots,t_n)|^{p_\alpha} \\
&= \sum_i b_i\sum_{t\in\Omega}|f_i^\alpha(t_1,\dots,t_i,\dots,t_n)|^{p_\alpha} \\
&\le B\sum_i\sum_{t\in\Omega}|f_i^\alpha(t)|^{p_\alpha},
\end{aligned} \tag{3.18}$$

thus we have

$$\begin{aligned}
\sum_{t\in\Omega}\prod_\alpha|f^\alpha(t)|^{q_\alpha} &\le \sum_{t\in\Omega}\frac{C}{n}\sum_\alpha\frac{q_\alpha}{p_\alpha}\left(\frac{C_\alpha}{2}\right)^{p_\alpha}B^{p_\alpha-1}\sum_i\sum_{u_i=1}^{b_i}|f_i^\alpha(t_1,\dots,u_i,\dots,t_n)|^{p_\alpha} \\
&\le \frac{C}{n}\sum_\alpha\frac{q_\alpha}{p_\alpha}\left(\frac{C_\alpha}{2}\right)^{p_\alpha}B^{p_\alpha}\sum_i\sum_{t\in\Omega}|f_i^\alpha(t)|^{p_\alpha} \\
&= \frac{C}{n}\sum_\alpha\frac{q_\alpha}{p_\alpha}\left(\frac{BC_\alpha}{2}\right)^{p_\alpha}\sum_{t\in\Omega}\left[\left(\sum_i|f_i^\alpha(t)|^{p_\alpha}\right)^{2/p_\alpha}\right]^{p_\alpha/2} \\
&\le \frac{C}{n}\sum_\alpha\frac{q_\alpha}{p_\alpha}\left(\frac{BC_\alpha}{2}\right)^{p_\alpha}\sum_{t\in\Omega}\left[c\left(\frac{2}{p_\alpha},n\right)\sum_i(|f_i^\alpha(t)|^{p_\alpha})^{2/p_\alpha}\right]^{p_\alpha/2}
\end{aligned} \tag{3.19}$$

by Lemma 3.8. Since $p_\alpha \ge 2$, $c(2/p_\alpha,n) = 1$ and so

$$\begin{aligned}
\sum_{t\in\Omega}\prod_\alpha|f^\alpha(t)|^{q_\alpha} &\le \frac{C}{n}\sum_\alpha\frac{q_\alpha}{p_\alpha}\left(\frac{BC_\alpha}{2}\right)^{p_\alpha}\sum_{t\in\Omega}\left(\sum_i|f_i^\alpha(t)|^2\right)^{p_\alpha/2} \\
&= \frac{C}{n}\sum_\alpha\frac{q_\alpha}{p_\alpha}\left(\frac{BC_\alpha}{2}\right)^{p_\alpha}\|\nabla f^\alpha\|_{p_\alpha}^{p_\alpha}.
\end{aligned} \tag{3.20}$$

□

Note that from the preceding results, discrete Poincaré-type inequalities involving only one function (the case $m = 1$) can be easily obtained. For instance, we have the following corollary.

Corollary 3.10 [5]. *For any $f \in \mathcal{F}_0(\Omega)$ and any real number $q \geq 2$,*

$$\|f^q\|_1 = \|f\|_q^q \leq \frac{1}{n}\left(\frac{B}{2}\right)^q \|\triangledown f\|_q^q. \tag{3.21}$$

Proof. This follows from Corollary 3.5 by letting $f^\alpha = f$ for all α. □

4. Applications

Theorem 4.1. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$, any real numbers $p_\alpha \geq 2$, $q_\alpha \geq 0$ with $\sum_\alpha q_\alpha/q_\alpha = 1$, and any $C_\alpha > 0$,*

$$\left\| \sum_\beta \left(\prod_{\alpha \neq \beta} |f^\alpha|^{q_\alpha} \right) |\triangledown f^\beta|^{q_\beta} \right\|_1 \leq C K(p,q) \sum_\alpha \frac{q_\alpha}{p_\alpha} C_\alpha^{p_\alpha} \|\triangledown f^\alpha\|_{p_\alpha}^{p_\alpha}, \tag{4.1}$$

where

$$C := \prod_\beta C_\beta^{-q_\beta} \tag{4.2}$$

and

$$K(p,q) = K(p_\alpha, q_\alpha) := \sum_\beta \left(\frac{1}{n}\right)^{1-q_\beta/p_\beta} \left(\frac{B}{2}\right)^{\sum_{\alpha\neq\beta} q_\alpha}. \tag{4.3}$$

Proof. By a generalization of Hölder's inequality for the case of many functions and by Corollary 3.10, we have

$$\begin{aligned}
&\sum_{t\in\Omega} \left[\sum_\beta \left(\prod_{\alpha\neq\beta} |f^\alpha(t)|^{q_\alpha} \right) |\triangledown f^\beta(t)|^{q_\beta} \right] \\
&\quad = \sum_\beta \sum_{t\in\Omega} \left[C \left(\prod_{\alpha\neq\beta} |C_\alpha f^\alpha(t)|^{q_\alpha} \right) |C_\beta \triangledown f^\beta(t)|^{q_\beta} \right] \\
&\quad \leq C \sum_\beta \left\{ \left[\prod_{\alpha\neq\beta} \left(\sum_{t\in\Omega} |C_\alpha f^\alpha(t)|^{p_\alpha} \right)^{q_\alpha/p_\alpha} \right] \left(\sum_{t\in\Omega} |C_\beta \triangledown f^\beta(t)|^{p_\beta} \right)^{q_\beta/p_\beta} \right\} \\
&\quad \leq C \sum_\beta \left\{ \left[\prod_{\alpha\neq\beta} \left(\frac{1}{n} \left(\frac{B}{2}\right)^{p_\alpha} \|C_\alpha \triangledown f^\alpha\|_{p_\alpha}^{p_\alpha} \right)^{q_\alpha/p_\alpha} \right] \left(\|C_\beta \triangledown f^\beta\|_{p_\beta}^{p_\beta} \right)^{q_\beta/p_\beta} \right\} \\
&\quad = C \sum_\beta \left(\frac{1}{n}\right)^{\sum_{\alpha\neq\beta} q_\alpha/p_\alpha} \left(\frac{B}{2}\right)^{\sum_{\alpha\neq\beta} q_\alpha} \prod_\alpha \|C_\alpha \triangledown f^\alpha\|_{p_\alpha}^{q_\alpha} \\
&\quad = C K(p,q) \prod_\alpha \|C_\alpha \triangledown f^\alpha\|_{p_\alpha}^{q_\alpha},
\end{aligned} \tag{4.4}$$

thus by Lemma 3.7, we conclude that

$$\begin{aligned}
\left\| \sum_\beta \left(\prod_{\alpha\neq\beta} |f^\alpha|^{q_\alpha} \right) |\triangledown f^\beta|^{q_\beta} \right\|_1 &\leq C K(p,q) \sum_\alpha \frac{q_\alpha}{p_\alpha} \|C_\alpha \triangledown f^\alpha\|_{p_\alpha}^{p_\alpha} \\
&= C K(p,q) \sum_\alpha \frac{q_\alpha}{p_\alpha} C_\alpha^{p_\alpha} \|\triangledown f\|_{p_\alpha}^{p_\alpha}.
\end{aligned} \tag{4.5}$$

□

Corollary 4.2. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$ and any real numbers $p_\alpha \geq 2$, $q_\alpha \geq 0$ with $\sum_\alpha q_\alpha / p_\alpha = 1$,*

$$\left\|\sum_\beta \left(\prod_{\alpha \neq \beta} |f^\alpha|^{q_\alpha}\right) |\nabla f^\beta|^{q_\beta}\right\|_1 \leq K(p,q) \sum_\alpha \frac{q_\alpha}{p_\alpha} \|\nabla f^\alpha\|_{p_\alpha}^{p_\alpha}, \tag{4.6}$$

where $K(p,q) = K(p_\alpha, q_\alpha)$ is as defined in Theorem 4.1.

Proof. It is immediate from Theorem 4.1 by letting $C_\alpha = 1$ for all α. □

Corollary 4.3. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$ and any real numbers $p_\alpha \geq 2$, $q_\alpha > 0$ with $\sum q_\alpha / p_\alpha = 1$,*

$$\left\|\sum_\beta \left(\prod_{\alpha \neq \beta} |f^\alpha|^{q_\alpha}\right) |\nabla f^\beta|^{q_\beta}\right\|_1 \leq C K(p,q) \sum_\alpha \|\nabla f^\alpha\|_{p_\alpha}^{p_\alpha}, \tag{4.7}$$

where $K(p,q) = K(p_\alpha, q_\alpha)$ is as defined in Theorem 4.1, and

$$C := \prod_\beta \left(\frac{q_\beta}{p_\beta}\right)^{q_\beta / p_\beta}. \tag{4.8}$$

Proof. It is immediate from Theorem 4.1 by putting

$$C_\alpha = \left(\frac{p_\alpha}{q_\alpha}\right)^{1/p_\alpha} \quad \forall \alpha. \tag{4.9}$$

□

Corollary 4.4 [5]. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$ and any real numbers $p_\alpha \geq 2$ with $\sum_\alpha 1/p_\alpha = 1$,*

$$\left\|\sum_\beta \left(\prod_{\alpha \neq \beta} |f^\alpha|^{q_\alpha}\right) |\nabla f^\beta|^{q_\beta}\right\|_1 \leq K(p) \sum_\alpha \frac{1}{p_\alpha} \|\nabla f^\alpha\|_{p_\alpha}^{p_\alpha}, \tag{4.10}$$

where

$$K(p) = K(p_\alpha) := \sum_\beta \left(\frac{1}{n}\right)^{1-1/p_\beta} \left(\frac{B}{2}\right)^{m-1}. \tag{4.11}$$

Proof. It follows immediately from Corollary 4.2 by setting $q_\alpha = 1$ for all α. □

Corollary 4.5. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$ and any real numbers $q_\alpha \geq 0$ with $q := \sum q_\alpha \geq 2$,*

$$\left\|\sum_\beta \left(\prod_{\alpha \neq \beta} |f^\alpha|^{q_\alpha}\right) |\nabla f^\beta|^{q_\beta}\right\|_1 \leq \frac{K(q)}{q} \sum_\alpha q_\alpha \|\nabla f^\alpha\|_q^q, \tag{4.12}$$

where

$$K(q) = K(q_\alpha) := \sum_\beta \left(\frac{1}{n}\right)^{1-q_\beta/q} \left(\frac{B}{2}\right)^{q-q_\beta}. \tag{4.13}$$

Proof. It follows immediately from Corollary 4.2 by setting $p_\alpha = q \geq 2$ for all α. □

Corollary 4.6 [5]. *For any $f^\alpha \in \mathcal{F}_0(\Omega)$,*

$$\left\|\sum_\beta\left(\prod_{\alpha\neq\beta} f^\alpha\right)|\nabla f^\beta|\right\|_1 \leq \left(\frac{B}{2}\right)^{m-1}\left(\frac{1}{n}\right)^{1-1/m}\sum_\alpha \|\nabla f^\alpha\|_m^m. \tag{4.14}$$

Proof. It is immediate from Corollary 4.4 by letting $p_\alpha = m$ for all α. □

Again the above results also give discrete inequalities for the case of one dependent function for free. For instance, we have the following corollary.

Corollary 4.7. *For any $f \in \mathcal{F}_0(\Omega)$ and any real numbers $q_\alpha \geq 0$ with $q := \sum_\alpha q_\alpha \geq 2$,*

$$\left\|\sum_\beta |f|^{q-q_\beta}|\nabla f|^{q_\beta}\right\|_1 \leq K(q)\|\nabla f\|_q^q, \tag{4.15}$$

where $K(q) = K(q_\alpha)$ is defined as in Corollary 4.5. In particular,

$$\||f|^{m-1}|\nabla f|\|_1 \leq \left(\frac{1}{n}\right)^{1-1/m}\left(\frac{B}{2}\right)^{m-1}\|\nabla f\|_m^m. \tag{4.16}$$

Proof. These follow from Corollary 4.5 by letting $f^\alpha = f$ for all α and subsequently $q_\alpha = 1$ for all α. □

Remark 4.8. Further interesting discrete type inequalities can be easily generated from the results in the preceding sections. For instance, by taking $m = 3$ in Corollary 4.6, we have

$$\|fg|\nabla h| + gh|\nabla f| + hf|\nabla g|\|_1 \leq \frac{B^2}{4n^{2/3}}\left[\|\nabla f\|_3^3 + \|\nabla g\|_3^3 + \|\nabla h\|_3^3\right]; \tag{4.17}$$

taking $m = 2$ in Corollary 4.6, we have

$$\|f|\nabla g| + g|\nabla f|\|_1 \leq \frac{B}{\sqrt{2n}}\left[\|\nabla f\|_2^2 + \|\nabla g\|_2^2\right], \tag{4.18}$$

and by putting $f = g = h$ in these inequalities (or using Corollary 4.7 directly), we obtain

$$\|f^2|\nabla f|\|_1 \leq \frac{B^2}{4n^{2/3}}\|\nabla f\|_3^3 \tag{4.19}$$

and

$$\|f|\nabla f|\|_1 \leq \frac{B}{\sqrt{2n}}\|\nabla f\|_2^2. \tag{4.20}$$

On the other hand, using Hölder's inequality, we have

$$\|f|\nabla g|\|_1 \leq \|f\|_2\|\nabla g\|_2, \tag{4.21}$$

and so by Corollary 3.10,

$$\|f|\nabla g|\|_1 \leq \frac{B}{2\sqrt{n}}\|\nabla f\|_2\|\nabla g\|_2\ . \tag{4.22}$$

Similarly, other interesting discrete type inequalities involving the gradient of $\mathcal{F}_0(\Omega)$ functions can be easily obtained. Inequalities of such form are in general of great interest and are important in the study of properties of solutions of difference equations. The importance of our result here also lie in that by choosing different combinations of the parameters $m, n, p_\alpha, q_\alpha, C_\alpha$, etc., one can obtain as many as we wish new discrete type inequalities involving the gradient of $\mathcal{F}_0(\Omega)$ functions. Furthermore, the techniques used here are rather algorithmic and easy to apply. It is expected that discrete inequalities of other types like the Wirtinger type and Sobolev type could also be established by similar techniques.

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WING-SUM CHEUNG: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG, POKFULAM ROAD, HONG KONG

E-mail address: wscheung@hkucc.hku.hk


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