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AN EQUIVALENCE FORM OF THE BRUNN-MINKOWSKI INEQUALITY FOR VOLUME DIFFERENCES

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Abstract. In this paper, we establish an equivalence form of the Brunn-Minkowski inequality for volume differences. As an application, we obtain a general and strengthened form of the dual Kneser-Süss inequality.

1. Introduction

If K and L are convex bodies in $\mathbb{R}^n$, then there is convex body $K\dot{+}L$ such that

$$S(K\dot{+}L,\cdot) = S(K,\cdot) + S(K,\cdot),$$

where $S(K,\cdot)$ denotes the surface area measure of K. This is a Minkowski's existence theorem; see [3] or [9]. The operation $\dot{+}$ is called *Blaschke addition.*

Theorem A (The Kneser-Süss inequality [9])**.** *If K and L are convex bodies in $\mathbb{R}^n$, then*

$$V(K\dot{+}L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \tag{1}$$

with equality if and only if K and L are homothetic.

The *volume differences function* of convex bodies K and L was defined by Leng [5]:

$$Dv(K,D) = V(K) - V(D),\ D \subset K.$$

In [5], Leng established the following Brunn-Minkowski inequality for volume differences.

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Theorem B. *If K, L, and D are convex bodies in $\mathbb{R}^n$, $D \subset K$,* and *$D' \subset L$ is a homothetic copy of D, then*

$$Dv(K+L, D+D')^{1/n} \geq Dv(K,D)^{1/n} + Dv(L,D')^{1/n} \tag{2}$$

with equality if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

If $p \geq 1$ and K and L contain the origin in their interiors, a convex body $K +_p L$ can be defined by

$$h(K +_p L, u)^p = h(K,u)^p + h(L,u)^p$$

for $u \in S^{n-1}$. The operation $+_p$ is called the *p-Minkowski addition.* Firey [2] proved the following inequality.

Theorem C$_1$. *If K and L are convex bodies in $\mathbb{R}^n$ containing the origin in their interiors, $p \geq 1$, and $0 \leq i < n$, then*

$$W_i(K +_p L)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}. \tag{3}$$

Furthermore, when $p > 1$, the equality holds if and only if K and L are dilates of each other.

Firey's ideas were transformed into a remarkable extension of the Brunn-Minkowski theory, called the *Brunn-Minkowski-Firey theory*, by Lutwak [6], [7]. Lutwak found the appropriate p-analog $S_p(K, \cdot)$, $p \geq 1$, of the surface area measure of a convex body K in $\mathbb{R}^n$ containing the origin in its interior. In [6], Lutwak generalized Firey's inequality (3). He also generalized Minkowski's existence theorem, deduced the existence of a convex body $K +_p L$ for which

$$S_p(K \dot{+}_p L, \cdot) = S_p(K, \cdot) + S_p(K, \cdot),$$

(when K and L are origin-symmetric convex bodies), and proved the following result.

Theorem C$_2$ (Lutwak's p-surface area measure inequality)**.** *If K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and $n \neq p \geq 1$, then*

$$V(K \dot{+}_p L)^{(n-p)/n} \geq V(K)^{(n-p)/n} + V(L)^{(n-p)/n}. \tag{4}$$

Furthermore, when $p > 1$, the equality holds if and only if K and L are dilates of each other.

In [8], Lutwak established the following dual Brunn-Minkowsi inequality.

Theorem D. *If K, L* are star bodies in $\mathbb{R}^n$, *then*

$$V(K \check{+} L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n}, \tag{5}$$

with equality if and only if K and L are dilates of each other.

The aim of this paper is to extend Kneser-Süss inequality (Theorem A) to the context of volume differences, which is in turn proved to be equivalent to Leng's result (Theorem B). We then extend Lutwak's p-surface area measure inequality (Theorem C$_2$) to the context of volume differences. Finally, a general dual Brunn-Minkowski inequality which strengthens Lutwak's result (Theorem D) is also given.

2. Definitions and preliminaries

The setting of this paper is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{C}^n$ denote the set of non-empty convex figures (compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathcal{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. We denote by $V(K)$ the n-dimensional volume of a convex body K. Let $h_K : S^{n-1} \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$, i.e., $h_K(u) = Max\{u \cdot x : x \in K\}, u \in S^{n-1}$, where $u \cdot x$ denotes the usual inner product of u and x in $\mathbb{R}^n$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by $\rho(K, u) = \mathrm{Max}\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let φ^n denote the set of star bodies in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $| \cdot |_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$ on S^{n-1}.

2.1. Mixed volume and dual mixed volume

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by

$$V(\sum_{i=1}^r \lambda_i K_i) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1 \cdots i_n}), \tag{6}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V(K_{i_1 \cdots i_n})$, which is called the *mixed volume* of $K_i, \ldots, K_{i_n}$, depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (6). If $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1 \cdots K_n)$ is usually written as $V_i(K, L)$.

From (6), we easily get: If $K, L, M \in \mathcal{K}^n$ and $\alpha, \mu \geq 0$, then

$$V_1(M, \alpha K + \mu L) = \alpha V_1(M, K) + \mu V_1(M, L). \tag{7}$$

Further, from (6) it follows immediately that

$$\lim_{\varepsilon \to 0} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon} = n V_1(K, L). \tag{8}$$

If $K_1, \ldots, K_n \in \varphi^n$, then the dual mixed volume of $K_1, \ldots, K_n$ is written as $\tilde{V}(K_1, \ldots, K_n)$. If $K_1 = \cdots = K_{n-i} = K$, and $K_{n-i+1} = \cdots = K_n = L$, then $\tilde{V}(K_1, \ldots, K_n)$ is written as $\tilde{V}_i(K, L)$. If $L = B$, the dual mixed volume

$\tilde{V}(K,B)$ is written as $\tilde{W}_i(K)$ and is called the *i-th dual Quermassintegral* of K.

2.2. The Blaschke addition and the radial Blaschke addition

If K, L and $\alpha,\mu \geq 0$, then the Theorem of Fenchel-Jessen-Alexandrov tells that there exists a convex body, unique up to translation, which we denote by $\alpha \cdot K \dot{+} \mu \cdot L$, such that

$$S(\alpha \cdot K \dot{+} \mu \cdot L, \cdot) = \alpha S(K,\cdot) + \mu S(L,\cdot).$$

This addition is called *Blaschke addition.*

The following result will be used later: If $K, L, M \in \mathcal{K}^n$ and $\alpha,\mu \geq 0$, then

$$V_1(\alpha K \dot{+} \mu L, M) = \alpha V_1(K,M) + \mu V_1(L,M). \tag{9}$$

As an aside, we note that corresponding to (8) one has for $K, L \in \mathcal{K}^n$,

$$\lim_{\varepsilon \to 0} \frac{V(L \dot{+} \varepsilon K) - V(L)}{\varepsilon} = \frac{n}{n-1} V_1(K,L). \tag{10}$$

See Goikkman [4].

If $K, L \in \varphi^n$ and $\alpha,\mu \geq 0$, then the radial Blaschke linear combination, $\alpha \cdot K \check{+} \mu \cdot L$, is the star body whose radial function is given by

$$\rho(\alpha \cdot K \check{+} \mu \cdot L, \cdot)^{n-1} = \alpha \rho(K,\cdot)^{n-1} + \mu \rho(L,\cdot)^{n-1}. \tag{11}$$

We shall call the addition *radial Blaschke addition.*

3. Lemmas

The following well-known results will be required to prove our main Theorems.

Lemma 1 (Bellman's inequality)**.** *Let* $a = \{a_1,\ldots,a_n\}$ *and* $b = \{b_1,\ldots,b_n\}$ *be two sequences of positive real numbers and* $p > 1$ *such that* $a_1^p - \sum_{i=2}^n a_i^p > 0$ *and* $b_1^p - \sum_{i=2}^n b_i^p > 0$, *then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p} \leq \left((a_1+b_1)^p - \sum_{i=2}^n (a_i+b_i)^p\right)^{1/p} \tag{12}$$

with equality if and only if $a = \upsilon b$ *where* υ *is a constant.*

Lemma 2 (Minkowski's inequality for integrals)**.** *If* $f_j \geq 0 (j = 1,\ldots,m)$, $p > 1$, *then*

$$\left(\int_{S^{n-1}} \left(\sum_{j=1}^m f_j(u)\right)^p dS(u)\right)^{1/p} \leq \sum_{j=1}^m \left(\int_{S^{n-1}} f_j^p(u) dS(u)\right)^{1/p}, \tag{13}$$

with equality if and only if f_j *are effectively proportional.*

This inequality is reversed if $0 < p < 1$ *or* $p < 0$.

Lemma 3. *If* K, L, *and* D *are convex bodies in* $\mathbb{R}^n$, $D \subset K$, and $D' \subset L$ *is a homothetic copy of* D, *then*

$$Dv(K\dot{+}L, D\dot{+}D')^{(n-1)/n} \geq Dv(K, D)^{(n-1)/n} + Dv(L, D')^{(n-1)/n} \tag{14}$$

with equality if and only if K *and* L *are homothetic and* $(V(K), V(D)) = \mu(V(L), V(D'))$, *where* μ *is a constant.*

Proof. We will prove the lemma using the method of Leng [5].

Applying the Knesser-Süss inequality (1), we obtain

$$V(K\dot{+}L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n} \tag{15}$$

with equality if and only if K and L are homothetic, and

$$V(D\dot{+}D')^{(n-1)/n} = V(D)^{(n-1)/n} + V(D')^{(n-1)/n}. \tag{16}$$

From (15) and (16), we obtain

$$\begin{aligned} Dv(K\dot{+}L, D\dot{+}D') \geq {} & [V(K)^{(n-1)/n} + V(L)^{(n-1)/n}]^{n/(n-1)} \\ & - [V(D)^{(n-1)/n} + V(D')^{(n-1)/n}]^{n/(n-1)}. \end{aligned} \tag{17}$$

From (17) and applying inequality (12), we have

$$Dv(K\dot{+}L, D\dot{+}D')^{(n-1)/n} \geq (V(K) - V(D))^{(n-1)/n} + (V(L) - V(D'))^{(n-1)/n},$$

with equality if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant. □

Remark 1. In the special case where D and D' are single points, inequality (14) becomes the classical Kneser-Süss Inequality.

4. Main results

We next observe that Lemma 3 is actually equivalent to Leng's result (Theorem B).

Theorem 1. *If* K, L, *and* D *are convex bodies in* $\mathbb{R}^n$, $D \subset K$, and $D' \subset L$ *is a homothetic copy of* D, *then*

$$\begin{aligned} & Dv(K\dot{+}L, D\dot{+}D')^{(n-1)/n} \geq Dv(K, D)^{(n-1)/n} + Dv(L, D')^{(n-1)/n} \\ & \quad \Leftrightarrow Dv(K + L, D + D')^{1/n} \geq Dv(K, D)^{1/n} + Dv(L, D')^{1/n}, \end{aligned} \tag{18}$$

where the conditions of equality are also equivalent.

Proof. (⇒) Suppose that

$$Dv(K\dot{+}L, D\dot{+}D')^{(n-1)/n} \geq Dv(K, D)^{(n-1)/n} + Dv(L, D')^{(n-1)/n},$$

with equality if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

From (10), we obtain

$$\begin{aligned}
&\frac{n}{n-1}(V_1(K,L)-V_1(D,D'))\\
&= \lim_{\varepsilon\to 0}\frac{Dv(L\dot{+}\varepsilon K, D'\dot{+}\varepsilon D)+Dv(D',L)}{\varepsilon} \qquad (19)\\
&\geq \lim_{\varepsilon\to 0}\frac{(Dv(L,D')^{(n-1)/n}+\varepsilon Dv(K,D)^{(n-1)/n})^{n/(n-1)}+Dv(D',L)}{\varepsilon},
\end{aligned}$$

with equality if and only if K and L are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where μ is a constant.

On the other hand, from (19) and in view of L'Hôpital's rule, we have

$$\begin{aligned}
&V_1(K,L)-V_1(D,D')\\
&\geq \lim_{\varepsilon\to 0}(Dv(L,D')^{(n-1)/n}+\varepsilon Dv(K,D)^{(n-1)/n})^{1/(n-1)}Dv(K,D)^{(n-1)/n} \qquad (20)\\
&= Dv(L,D')^{1/n}Dv(K,D)^{(n-1)/n}.
\end{aligned}$$

Suppose that $M,N\in\mathcal{K}^n$ and $N\subset M$, from (7) and (20), it follows that

$$\begin{aligned}
&V_1(M,K+L)-V_1(N,D+D')\\
&=(V_1(M,K)-V_1(N,D))+(V_1(M,L)-V_1(N,D')) \qquad (21)\\
&\geq (Dv(K,D)^{1/n}+Dv(L,D')^{1/n})Dv(M,N)^{(n-1)/n}.
\end{aligned}$$

If we take $M=K+L$ and $N=D+D'$ in (21), in view of $V(K,\dots,K)=V(K)$, we have

$$Dv(K+L,D+D')^{1/n}\geq Dv(K,D)^{1/n}+Dv(K,D)^{1/n},$$

with equality if and only if K and L are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where μ is a constant.

($\Leftarrow$) Suppose that

$$Dv(K+L,D+D')^{1/n}\geq Dv(K,D)^{1/n}+Dv(L,D')^{1/n},$$

with equality if and only if K and L are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where μ is a constant.

From (8), we have

$$\begin{aligned}
&n(V_1(K,L)-V_1(D,D'))\\
&= \lim_{\varepsilon\to 0}\frac{Dv(K+\varepsilon L,D+\varepsilon D')+Dv(D,K)}{\varepsilon} \qquad (22)\\
&\geq \lim_{\varepsilon\to 0}\frac{(Dv(K,D)^{1/n}+\varepsilon Dv(L,D')^{1/n})^n+Dv(D,K)}{\varepsilon},
\end{aligned}$$

with equality if and only if K and L are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where μ is a constant.

On the other hand, from (22) and in view of L'Hôpital's rule, we have

$$\begin{aligned} & V_1(K,L) - V_1(D,D') \\ & \geq \lim_{\varepsilon\to 0}(Dv(K,D)^{1/n} + \varepsilon Dv(L,D')^{1/n})^{n-1} Dv(L,D')^{1/n} \\ & = Dv(K,D)^{(n-1)/n} Dv(L,D')^{1/n}. \end{aligned} \tag{23}$$

From (9) and (23), for any $M, N \in \mathcal{K}^n$ and $N \subset M$, we have

$$\begin{aligned} & V_1(K\dot{+}L, M) - V_1(D\dot{+}D', N) \\ & = (V_1(K,M) - V_1(D,N)) + (V_1(L,M) - V_1(D',N)) \\ & \geq (Dv(K,D)^{(n-1)/n} + Dv(L,D')^{(n-1)/n}) Dv(M,N)^{1/n}. \end{aligned} \tag{24}$$

If we take $M = K\dot{+}L$ and $N = D\dot{+}D'$ in (24), and in view of $V(K,\ldots,K) = V(K)$, we obtain inequality (14). □

Remark 2. In the special case where D and D' are single points, Theorem 1 gives the following important result.

Corollary 1. *The Knesser-Süss inequality is equivalent to the Bunn-Minkowski inequality, namely, for $K, L \in \mathcal{K}^n$,*

$$\begin{aligned} & V(K\ddot{+}L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n} \\ \Leftrightarrow \quad & V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \end{aligned}$$

with equality if and only if K and L are homothetic.

Similarly, from the Lutwak's p-surface area measure inequality (4) and the Bellman's inequality, we can get the following result which is a general form of (4).

Theorem 2. *If K, L, and D are origin-symmetric convex bodies in $\mathbb{R}^n$, $D \subset K$,* and *$D' \subset L$ is a homothetic copy of D, then for $n \neq p \geq 1$,*

$$Dv(K\dot{+}_p L, D\dot{+}_p D')^{(n-p)/n} \geq Dv(K,D)^{(n-p)/n} + Dv(L,D')^{(n-p)/n}. \tag{25}$$

Furthermore, when $p > 1$, the equality holds if and only if K and L are dilates of each other and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

Remark 3. Note that the Knesser-Süss inequality (14) for volume differences corresponds to the case $p = 1$ in (25). On the other hand, if D and D' are single points, (25) reduces to the classical Knesser-Süss inequality.

Finally, the following is a general and strengthened form of Lutwak's dual Brunn-Minkowski inequality.

Theorem 3. *If $K, L \in \varphi^n$, $\alpha \in [0,1]$, then for $i < 1$,*

$$\begin{aligned} & \tilde{W}_i(K\check{+}L)^{(n-1)/(n-i)} \\ & \leq \tilde{W}_i(\alpha K\check{+}(1-\alpha)L)^{(n-1)/(n-i)} + \tilde{W}_i((1-\alpha)K\check{+}\alpha L)^{(n-1)/(n-i)} \\ & \leq \tilde{W}_i(K)^{(n-1)/(n-i)} + \tilde{W}_i(L)^{(n-1)/(n-i)}, \end{aligned} \tag{26}$$

with equality if and only if K and L are dilates of each other.

These inequalities are reversed for $i > n$ or $1 < i < n$.

Proof. Noting that $\tilde{W}_i(K) = \int_{S^{n-1}} \rho(K)^{n-i} dS(u)$, and from (11), (13), we have for $i < 1$,

$$\begin{aligned}
&\tilde{W}_i(K \check{+} L)^{(n-1)/(n-i)}\\
&= \left(\frac{1}{n}\int_{S^{n-1}} \rho(K \check{+} L, u)^{n-i} dS(u)\right)^{(n-1)/(n-i)}\\
&= \left(\frac{1}{n}\int_{S^{n-1}} \left(\rho(K,u)^{n-1} + \rho(L,u)^{n-1}\right)^{(n-i)/(n-1)} dS(u)\right)^{(n-1)/(n-i)}\\
&\leq \left(\frac{1}{n}\int_{S^{n-1}} \left(\alpha\rho(K,u)^{n-1} + (1-\alpha)\rho(L,u)^{n-1}\right)^{(n-i)/(n-1)} dS(u)\right)^{(n-1)/(n-i)}\\
&\quad + \left(\frac{1}{n}\int_{S^{n-1}} \left((1-\alpha)\rho(K,u)^{n-1} + \alpha\rho(L,u)^{n-1}\right)^{(n-i)/(n-1)} dS(u)\right)^{(n-1)/(n-i)}\\
&= \left(\frac{1}{n}\int_{S^{n-1}} \left(\rho(\alpha \cdot K \check{+} (1-\alpha)\cdot L, u)\right)^{n-i} dS(u)\right)^{(n-1)/(n-i)}\\
&\quad + \left(\frac{1}{n}\int_{S^{n-1}} \left(\rho((1-\alpha) \cdot K \check{+} \alpha\cdot L, u)\right)^{n-i} dS(u)\right)^{(n-1)/(n-i)}\\
&= \tilde{W}_i(\alpha \cdot K \check{+} (1-\alpha)\cdot L)^{(n-1)/(n-i)} + \tilde{W}_i((1-\alpha) \cdot K \check{+} \alpha\cdot L)^{(n-1)/(n-i)}.
\end{aligned}$$

On the other hand, for $i < 1$,

$$\begin{aligned}
&\tilde{W}_i(\alpha \cdot K \check{+} (1-\alpha)\cdot L)^{(n-1)/(n-i)}\\
&= \left(\frac{1}{n}\int_{S^{n-1}} \rho(\alpha \cdot K \check{+} (1-\alpha) L)^{n-i} dS(u)\right)^{(n-1)/(n-i)}\\
&\leq \alpha\left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i} dS(u)\right)^{(n-1)/(n-i)}\\
&\quad + (1-\alpha)\left(\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n-i} dS(u)\right)^{(n-1)/(n-i)}\\
&= \alpha\tilde{W}_i(K)^{(n-1)/(n-i)} + (1-\alpha)\tilde{W}_i(L)^{(n-1)/(n-i)}.
\end{aligned}$$

Similarly, we get

$$\tilde{W}_i((1-\alpha)\cdot K \check{+} \alpha\cdot L)^{(n-1)/(n-i)} \leq (1-\alpha)\tilde{W}_i(K)^{(n-1)/(n-i)} + \alpha\tilde{W}_i(L)^{(n-1)/(n-i)}.$$

Hence,

$$\begin{aligned}
&\tilde{W}_i(\alpha \cdot K \check{+} (1-\alpha)\cdot L)^{(n-1)/(n-i)} + \tilde{W}_i((1-\alpha) \cdot K \check{+} \alpha\cdot L)^{(n-1)/(n-i)}\\
\leq\ & W_i(K)^{(n-1)/(n-i)} + W_i(L)^{(n-1)/(n-i)},
\end{aligned}$$

with equality if and only if K and L are dilates of each other.

The cases of $i > n$ and $1 < i < n$ are obtained analogously. □

Remark 4. Taking $i = 0$, inequality (26) becomes the following strengthened form of the dual Knesser-Süss inequality.

Corollary 2. *If $K, L \in \varphi^n$, $\alpha \in [0,1]$, then*

(27)
$$\begin{aligned} V(K \breve{+} L)^{(n-1)/n} &\le V(\alpha K \breve{+} (1-\alpha)L)^{(n-1)/n} + V((1-\alpha)K \breve{+} \alpha L)^{(n-1)/n} \\ &\le V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \end{aligned}$$

with equality if and only if K and L are dilates of each other.

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