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AN INTEGRAL INEQUALITY OF AN INTRINSIC MEASURE ON BOUNDED DOMAINS IN $\mathbf{C}^n$

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1. Introduction. Let D be a complete hyperbolic bounded domain in $\mathbf{C}^n$ in the sense of [**7**]. We denote by M_D^E the differential Eisenman-Kobayashi n-measure (defined with respect to the unit ball) on D. Since we may endow D with a global coordinate system, M_D^E can therefore be viewed as a function. The main goal of this paper is to prove the following theorem.

Theorem. *If we assume there exists a neighborhood N of ∂D in $\mathbf{C}^n$ where M_D^E satisfies the growth condition*

$$|M_D^E(z)| \geq \frac{k}{(r(z))^{m+s}},$$

where $k =$ positive constant, $r(z) =$ the euclidean distance from z to ∂D, $z \in N \cap D$, m and s positive numbers, then we can find a neighborhood U of ∂D in $\mathbf{C}^n$ such that for all $z_0 \in U \cup D$, whenever the closed disk $\{z_0 + \rho z_1 : \rho \in \mathbf{C}, |\rho| \leq 1\}$, $z_1 \in \mathbf{C}^n$, lies in $U \cap D$, the inequality

$$\ln |M_D^E(z_0)| \leq \frac{n}{m\pi} \int_0^{2\pi} \ln |M_D^E(z_0 + e^{i\theta} z_1)| \, d\theta$$

for $|M_D^E|$ holds.

Typical examples satisfying conditions of our theorem include analytic polyhedra, strongly pseudoconvex domains and certain domains of holomorphy with smooth real analytic boundary [**1**]. For the case of strongly pseudoconvex domains we have the following corollary.

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Corollary. *Let D be a strongly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. Then there exists a neighborhood U of ∂D in $\mathbf{C}^n$ such that for all $z_0 \in U \cap D$, whenever the closed disk*

$$\{z_0 + \rho z_1; \rho \in \mathbf{C}, |\rho| \le 1\}, \qquad z_1 \in \mathbf{C}^n,$$

lies in $U \cap D$, the inequality

$$\ln |M_D^E(z_0)| \le \frac{1}{\pi} \int_0^{2\pi} \ln |M_D^E(z_0 + e^{i\theta} z_1)| \, d\theta$$

holds.

Our proof rests on the boundary assumption of the intrinsic measure and the classical Hartogs' construction of analytic family of disks [**11**]. It is quite clear from a minor modification of the proof that our integral inequality also holds on an analytically embedded disk (i.e., the image of a holomorphic embedding $f : B \to D$, here $B = \{z \in \mathbf{C} : |z| < 1\}$, which is homeomorphic up to the boundary ∂B with $f(\partial B) \subset D$ and $f(0) = z_0$). One can see this inequality imposes a restriction on M_D^E when the analytic disk is large and sufficiently close to the boundary. This type of inequality can be generalized to other low dimensional intrinsic measures. A sharpened result can also probably be derived along our line. The arrangement of our paper can be summarized as below.

Section 2. Definition of Eisenman-Kobayashi measure. Section 3. A boundary estimate of Eisenman-Kobayashi measure on strongly pseudoconvex domain and proof of the Corollary. Section 4. Proof of our main statement.

2. Definition of Eisenman-Kobayashi measure. For the basic definitions and a survey of this subject, one should consult [**7, 8**]. Since our *Eisenman-Kobayashi n-measures* are defined with respect to the ball in $\mathbf{C}^n$, which is somewhat different from what had been done in [**7, 8**], we shall include our definition here.

Let N be a complex manifold of dimension n. The Eisenman Kobayashi n-measure M_N^E is an (n,n)-form $|M_N^E| \cdot (dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge$

$dz_n \wedge d\bar{z}_n$) on N, such that $|M_N^E|$ is defined for all $x \in N$ as

$$|M_N^E(x)| = \inf\left\{\frac{1}{R^{2n}} : \exists\, f \in \operatorname{Hol}(B_n^R, N) \text{ such that } f(0) = x, \det(Jf(0)) = 1\right\},$$

where B_n^R is the euclidean ball with center 0 and radius R in $\mathbf{C}^n$, $\operatorname{Hol}(B_n^R, N)$ the set of all holomorphic maps from B_n^R to N, and $Jf(0)$ the Jacobian matrix of f at 0.

It is easy to check that for a bounded domain D in $\mathbf{C}^n$, $|M_D^E(x)| \neq 0$ for all $x \in D$. $|M_D^E|$ is in general a semicontinuous function [**12**]. When D is a complete hyperbolic bounded domain, $|M_D^E|$ can be proved to be continuous. All complete hyperbolic domains in $\mathbf{C}^n$ are pseudoconvex [**7**].

3. A boundary estimate of $\mathbf{M_D^E}$ on S.P.C. domains and proof of the corollary. Let D be a strongly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. The following will be proved in this section.

Theorem. *There exist a neighborhood U of ∂D in $\mathbf{C}^n$ and a positive constant c such that for all $z \in U \cap D$,*

$$|M_D^E(z)| \geq \frac{c}{(r(z))^{n+1}},$$

where $r(z)$ is the euclidean distance function from z to the boundary of D.

Note that since ∂D is compact and everywhere strongly pseudoconvex, the above statement can be reduced to the following local problem:

For all $p \in \partial D$, there exists a neighborhood V of p in $\mathbf{C}^n$ and a positive constant k such that for all $z \in D \cap V$,

$$|M_D^E(z)| \geq \frac{k}{(r(z))^{n+1}}.$$

3.1 Localization lemma. Let D be a bounded domain in $\mathbf{C}^n$, D_1 another domain such that $D \cap D_1$ is nonempty.

Definitions. (1) Let z, w belong to D. Then $d(z,w) = \inf\{P(a,b) : \exists\, f \in \mathrm{Hol}\,(B_n, D)$ s.t. $f(a) = z,\ f(b) = w$, where P is the Kobayashi metric on $B_n\}$; here B_n is the unit ball in $\mathbf{C}^n$.

(2) Let z belong to $D \cap D_1$, then $d(z) = \inf\{d(z,w) : w \in D - D_1\}$.

Lemma A. *Let us denote $\hat{D} = D \cap D_1$; then for all $z \in \hat{D}$, we have*

$$|M^E_{\hat{D}}(z)| \leq (\coth d(z))^{2n} \cdot |M^E_D(z)|,$$

where $M^E_{\hat{D}}$ and M^E_D denote the Eisenman-Kobayashi measures on $\hat{D}$ and D, respectively.

Proof. First of all, let us fix $z \in \hat{D}$ and let

$$r = \sup\{r' : \exists f \in \mathrm{Hol}\,(B_n^{r'}, \hat{D}) \text{ s.t. } f(0) = z, \det\,(Jf(0)) = 1\}.$$

Then we choose a number R which is slightly larger than r. From our choice of r it is obvious that there is an $f \in \mathrm{Hol}\,(B_n^R, D)$ such that $f(0) = z$, $\det\,(Jf(0)) = 1$ and it maps a boundary point of B_n^r to a point belonging to $D - \hat{D}$. One can see that if w is such a point belonging to $D - \hat{D}$, then $d(z) \leq d(z,w)$. From our definition of d, we observe that

$$d(z) \leq d(z,w) \leq (1/2) \ln[(1 + r/R)/(1 - r/R)]$$

(distance-decreasing property under holomorphic mappings; consider f to be a holomorphic map from B_n^R to D [**7**]). Hence,

$$1/r \leq \coth d(z) \cdot (1/R),$$
$$(1/r)^{2n} \leq (\coth d(z))^{2n} \cdot (1/R)^{2n}.$$

This inequality is true for all the R's satisfying the properties mentioned above. Considering the definition of $M^E_D(z)$, one can now conclude our desired inequality

$$|M^E_{\hat{D}}(z)| \leq (\coth d(z))^{2n} \cdot |M^E_D(z)|. \qquad \square$$

Lemma B. *Suppose D is a strongly pseudoconvex bounded domain in $\mathbf{C}^n$ and D_1 is a neighborhood of a boundary point of D. Let this* *boundary point be* p, $\hat{D} = D \cap D_1$, *and* $M^E_{\hat{D}}, M^E_D$ *the Eisenman-Kobayashi measures of* $\hat{D}$ *and* D, *respectively. Then we have*

$$\lim_{z \to p} \frac{|M^E_D(z)|}{|M^E_{\hat{D}}(z)|} = 1 \qquad \text{for all } z \in \hat{D}.$$

Proof. We divide our proof into two steps.

(1) Since the inclusion map $\hat{D} \to D$ is holomorphic, by the volume-decreasing property [**7**] we have

$$|M^E_{\hat{D}}(z)| \geq |M^E_D(z)| \quad \text{i.e.,} \ \frac{|M^E_{\hat{D}}(z)|}{|M^E_D(z)|} \geq 1.$$

(2) From Lemma (A) we have

$$|M^E_{\hat{D}}(z)| \leq (\coth d(z))^{2n} \cdot |M^E_D(z)|.$$

Now it is clear from our definitions that

$$d(z, w) \geq d_k(z, w) \qquad \forall z, w \in D,$$

where d_k is the Kobayashi metric on D [**7**]. However, if we set

$$d_k(z, D - \hat{D}) = \inf\{d_k(z, w) : w \in D - \hat{D}\},$$

it is known that

$$\lim_{z \to p} d_k(z, D - \hat{D}) = \infty$$

if D is strongly pseudoconvex (this statement can easily be derived from the result of I. Graham [**5**]). Thus, $d(z)$ will go to infinity as $z \in \hat{D}$ approaches p; consequently, $\coth d(z)$ will tend to 1 as $z \in \hat{D}$ tends to p. At this point we obtain another inequality:

$$\overline{\lim_{z \to p}} \frac{|M^E_{\hat{D}}(z)|}{|M^E_D(z)|} \leq 1.$$

Combining these two inequalities, we thereby complete the proof of our localization lemma. □

3.2 Proof of our estimate. Before embarking on our proof, we first make two remarks here.

Remark 1. An analytic ellipsoid is a strongly pseudoconvex domain A in $\mathbf{C}^n$ which can locally be described as: If $p \in \partial A$ and W is a sufficiently small open neighborhood of p in $\mathbf{C}^n$, then

$$A \cap W = \{z \in \mathbf{C}^n : g(z) = -z_1 - \bar{z}_1 + \sum_{i,j=1}^{n} b_{ij} z_i \bar{z}_j < 0\},$$

where $[b_{ij}]_{i,j=1}^n$ is a hermitian positive definite matrix.

In our expression p is the origin of the coordinates $\{z_1, z_2, \dots, z_n\}$, z_1 is the complex normal of ∂A at p, and $\{z_2, \dots, z_n\}$ is the basis of the maximal complex tangent space $T_p(\partial A)$. It is known that any analytic ellipsoid is biholomorphically equivalent to the unit ball in $\mathbf{C}^n$, and the Eisenman-Kobayashi measure on the unit ball is equal to the volume form of the Bergman metric (with the reservation of the multiple of a constant). The following estimate can thus be obtained from the explicit formula of the Bergman metric on B_n (see [**13**], for example).

There exists a sufficiently small open neighborhood W_1 of ∂A in $\mathbf{C}^n$ and a positive constant c_1 such that

$$|M_A^E(z)| \approx \frac{c_1}{(r(z))^{n+1}} \qquad \forall z \in W_1 \cap A.$$

Furthermore, by the volume-decreasing property again we have the following estimate.

There exists an open neighborhood W_2 of p in $\mathbf{C}^n$ and a positive constant c_1 such that

$$|M_{\hat{W}_2}^E(z)| \geq \frac{c_2}{(r(z))^{n+1}} \qquad \forall z \in \hat{W}_2 = W_2 \cap A.$$

Remark 2. Let D be a strongly pseudoconvex boundary domain in $\mathbf{C}^n$ with smooth boundary and p be a given boundary point of D as before. With a similar coordinate system $\{z_1, z_2, \dots, z_n\}$ as in *Remark* 1, in which we can locally characterize D around p as

$$D \cap U_1 = \{z \in D; G(z) < 0\},$$

where U_1 is a neighborhood of p in $\mathbf{C}^n$, and

$$G(z) = -z_1 - \bar{z}_1 + \sum a_{ij} z_i \bar{z}_j + 2\mathrm{Re}\left\{ \sum_{i \geq 2} \frac{\partial^2 G}{\partial z_i \partial z_j}(p) z_i z_j \right\} + O(|z|^3)$$

with respect to the above coordinate system. Since D is strongly pseudoconvex, (a_{ij}) is a hermitian positive definite matrix. We can apply our localization in Lemma B to make the assertion: For all $\varepsilon > 0$, there exists a neighborhood U_2 of p in $\mathbf{C}^n$ such that

$$\left| \frac{|M^E_{\hat{U}_2}(z)|}{|M^E_D(z)|} - 1 \right| < \varepsilon$$

for all $z \in \hat{U}_2 = U_2 \cap D$.

Proof of our main estimate. To start, we fix a boundary point $p \in \partial D$ and choose the coordinate system $\{z_1, \dots, z_n\}$ as in *Remark* 2. Now we construct an analytic ellipsoid A_s whose defining equation around the point p is given by

$$g_s = -z_1 - \bar{z}_1 + \sum (a_{ij} - s \cdot \delta_{ij}) z_i \bar{z}_j,$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$

and s is a well-chosen positive constant such that A_s satisfies the following properties:

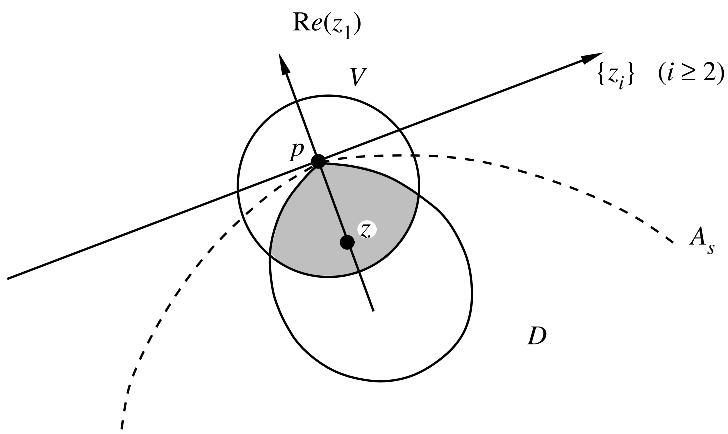
(i) there exists a neighborhood V of p in $\mathbf{C}^n$ such that $V \cap A_s = \{z : g_s(z) < 0\}$;

(ii) with the same V in (i), we have $V \cap A_s \supset V \cap D$.

Graphically, our situation can be illustrated by the picture on the top of the next page.

If we denote by $V_1 = A_s \cap V$ (white area) and $V_2 = D \cap V$ (shaded area), we have by volume decreasing property

$$|M^E_{V_2}(z)| \geq |M^E_{V_1}(z)| \qquad \forall z \in V_2.$$



Moreover, if V is sufficiently small we can use *Remarks* 1 and 2 above to conclude that

$$|M_D^E(z)| \geq \frac{k}{(r(z))^{n+1}} \qquad \forall z \in \mathcal{N}_p$$

where $\mathcal{N}_p = \{\text{the axis Re}\,(z_1)\} \cap V$, k is a suitable constant.

Finally we have to observe that all of our processes described in this section are *uniform* in the following sense. We can choose a sufficiently small neighborhood $T \subset \partial D$ of p in such a way that all the *sizes* of domains of comparison and constants can be unchanged so that all the above arguments will remain valid for all $q \in T$. Then we further refine our $V = \cup_{q \in T}\{\mathcal{N}_q\}$. This completes the whole proof. □

Remark . More precise boundary estimates for both Eisenman-Kobayashi and Carathéodory measures were carried out in [**16**] following the original work of Graham [**4,5**] in the case of metrics. The localization lemma in the case of the Kobayashi metric was first used by Royden [**12**] and Graham [**4,5**]. Some other related results can be found in [**13, 2, 6, 3**].

Proof of the Corollary. Applying our theorem, we let $m = n$ and $s = 1$. ◻

4. Proof of our main statement. Take the neighborhood N as in the assumption of our theorem. Since D is complete hyperbolic in the sense of Kobayashi, consequently $\ln |M_D^E|$ is a continuous function on

$$\{z_0 + \rho z_1 : \rho \in \mathbf{C}, |\rho| \leq 1\} \subset N \cap D.$$

We can always find a real valued function h, defined and continuous on $|\rho| \leq 1$, harmonic in $|\rho| < 1$, and equal to $(1/m) \ln |M_D^E|$ on $|\rho| = 1$, that is,

$$h(\rho) = (1/m) \ln |M_D^E(z_0 + \rho z_1)| \qquad \forall |\rho| = 1.$$

Let h^* be a harmonic conjugate of h, set $g = h + ih^*$. Then g is continuous on $|\rho| \leq 1$ and holomorphic in $|\rho| < 1$. Next, let b be any vector in $\mathbf{C}^n$ with $||b|| = 1$ and λ_0 any real number satisfying $0 < \lambda_0 < 1$. Consider the analytic disk

$$\Sigma_\lambda : \rho \to z_0 + \rho z_1 + \lambda e^{-g(\rho)} b,$$

in $\mathbf{C}^n$ where $|\rho| \leq 1$ and λ fixed, $0 \leq \lambda \leq \lambda_0$.

Claim. $\cup_{0 \leq \lambda \leq \lambda_0} \partial \Sigma_\lambda \subset\subset D$.

Since for all $z \in \partial \Sigma_\lambda$, z is the image of some ρ with $|\rho| = 1$,

$$\begin{aligned} ||z - (z_0 + \rho z_1)|| &= ||\lambda e^{-g(\rho)} b|| \\ &= \lambda e^{-h(\rho)} \leq \lambda_0 e^{-(1/m) \ln |M_D^E(z_0 + \rho z_1)|} \\ &= \lambda_0 \cdot |M_D^E(z_0 + \rho z_1)|^{-1/m}. \end{aligned}$$

By our assumption, since $z_0 + \rho z_1 \in N \cap D$, we have

$$|M_D^E(z_0 + \rho z_1)| \geq \frac{k}{(r(z_0 + \rho z_1))^{m+s}}.$$

Hence,

$$\frac{1}{|M_D^E(z_0 + \rho z_1)|^{1/m}} \leq r(z_0 + \rho z_1) \cdot \left[\frac{(r(z_0 + \rho z_1))^s}{k} \right]^{1/m}.$$

Moreover, we can choose $U \subset N$ to be sufficiently small so that the inequality

$$\left[\frac{(r(z_0+\rho z_1))^s}{k}\right]^{1/m} < 1$$

also holds. Therefore, we obtain

$$||z-(z_0+\rho z_1)|| < \lambda_0 \cdot r(z_0+\rho z_1).$$

However, λ_0 lies between zero and one; this yields

$$||z-(z_0+\rho z_1)|| < r(z_0+\rho z_1),$$

which is independent of z and λ. Hence, the inequality holds for all $z \in \cup\partial\Sigma_\lambda$, that is, $\cup\partial\Sigma_\lambda$ is bounded. Furthermore, for all points $z \in \cup\partial\Sigma_\lambda$ and $w \in \partial D$,

$$||z-w|| \geq ||(z_0+\rho z_1)-w|| - ||z-(z_0+\rho z_1)||,$$

thus

$$r(z) \geq r(z_0+\rho z_1) - ||z-(z_0+\rho z_1)|| > 0,$$

hence

$$\bigcup_{0\leq\lambda\leq\lambda_0} \partial\Sigma_\lambda \subset\subset D.$$

This verifies our claim. Applying Kontinuitätssatz, we therefore obtain

$$z_0+\rho z_1+\lambda e^{-g(\rho)}b \in D \qquad \forall 0\leq\lambda<1, \quad |\rho|\leq 1.$$

Hence,

$$z_0+\lambda e^{-g(0)}be^{i\theta} \in D \qquad \forall 0\leq\lambda<1, \quad 0\leq\theta\leq 2\pi,$$

where b is in arbitrary direction. That is, the ball $B(z_0, ||\lambda e^{-g(0)}b||) \subset D$, for all $0 \leq \lambda < 1$. It implies that

$$B(z_0, ||e^{-g(0)}b||) \subset D;$$

consequently,

$$B(z_0, e^{-h(0)}) \subset D.$$

Hence, the function $f : B_n^{e^{-h(0)}} \to D$ (recall that b is in arbitrary direction) such that $f(z) = z_0 + z$ is well defined and holomorphic. Note that $f(0) = z_0$ and $\det(Jf(0)) = 1$, hence

$$\begin{aligned} |M_D^E(z_0)| &= \inf\bigg\{ \frac{1}{R^{2n}} : \exists f \in \mathrm{Hol}\,(B_n^R, D), \\ &\qquad\qquad \text{such that } f(0) = z_0, \det(Jf(0)) = 1 \bigg\} \\ &\le \frac{1}{e^{-2n\cdot h(0)}} = e^{2n\cdot h(0)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \ln|M_D^E(z_0)| \le 2n\cdot h(0) &= 2n\cdot\frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta})\,d\theta \\ &= \frac{n}{m\pi}\int_0^{2\pi} \ln|M_D^E(z_0 + e^{i\theta}z_1)|\,d\theta, \end{aligned}$$

which is exactly what we want to show. □

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