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Research Article

On Opial-Type Integral Inequalities

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We establish some new Opial-type inequalities involving functions of two and many independent variables. Our results in special cases yield some of the recent results on Opial's inequality and also provide new estimates on inequalities of this type.

1. Introduction

In the year 1960, Opial [1] established the following integral inequality.

Theorem 1.1. *Suppose $f \in C^1[0,h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0,h)$. Then the integral inequality holds*

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{4}\int_0^h (f'(x))^2 dx, \tag{1.1}$$

where this constant $h/4$ is best possible.

Opial's inequality and its generalizations, extensions, and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [2–6]. The inequality (1.1) has received considerable attention and a large number of papers dealing with new proofs, extensions, generalizations, variants, and discrete analogs of Opial's inequality have appeared in some literature [7–22]. For an extensive survey on these inequalities, see [2, 6]. The main purpose of the present paper is to establish some new Opial-type inequalities involving functions of two and many independent variables. Our results in special cases yield some of the recent results on Opial's inequality and provide some new estimates on such types of inequalities.

2. Main results

Our main results are given in the following theorems.

THEOREM 2.1. *Let $u_i(s,t)$, $v_i(s,t)$, $i=1,\ldots,n$, be real-valued absolutely continuous functions defined on $[a,b]\times[c,d]$ and $a,b,c,d\in[0,\infty)$ with $u_i(s,c)=u_i(a,t)=u_i(a,c)=0$, $v_i(s,c)=v_i(a,t)=v_i(a,c)=0$, $i=1,\ldots,n$. Let F, G be real-valued nonnegative continuous and nondecreasing functions on $[0,\infty)^n$ with $F(0,\ldots,0)=0$, $G(0,\ldots,0)=0$ such that all their partial derivatives $\partial^2F/\partial|u_i|^2$, $\partial F/\partial|u_i|$, $\partial^2G/\partial|v_i|^2$, $\partial G/\partial|v_i|$, $i=1,\ldots,n$ are nonnegative continuous and nondecreasing functions on $[0,\infty)^n$. Let $\partial|u_i|/\partial s$, $\partial|u_i|/\partial t$, $\partial^2|u_i|/\partial s\partial t$, $\partial|v_i|/\partial s$, $\partial|v_i|/\partial t$, $\partial^2|v_i|/\partial s\partial t$, $i=1,\ldots,n$, be nonnegative continuous and nondecreasing functions on $[a,b]\times[c,d]$. Then*

$$\begin{aligned}
\int_a^b\int_c^d\Bigg[&F(|u_1(s,t)|,\ldots,|u_n(s,t)|)\sum_{i=1}^n\left(\frac{\partial^2G}{\partial|v_i|^2}\cdot\frac{\partial|v_i|}{\partial t}\cdot\frac{\partial|v_i|}{\partial s}+\frac{\partial G}{\partial|v_i|}\cdot\frac{\partial|v_i|}{\partial s\partial t}\right)\\
&+G(|v_1(s,t)|,\ldots,|v_n(s,t)|)\sum_{i=1}^n\left(\frac{\partial^2F}{\partial|u_i|^2}\cdot\frac{\partial|u_i|}{\partial t}\cdot\frac{\partial|u_i|}{\partial s}+\frac{\partial F}{\partial|u_i|}\cdot\frac{\partial|u_i|}{\partial s\partial t}\right)\\
&+S(s,t)\Bigg]ds\,dt\\
\le F&\left(\int_a^b\int_c^d\left|\frac{\partial^2u_1}{\partial s\partial t}\right|ds\,dt,\ldots,\int_a^b\int_c^d\left|\frac{\partial^2u_n}{\partial s\partial t}\right|ds\,dt\right)\\
\cdot G&\left(\int_a^b\int_c^d\left|\frac{\partial^2v_1}{\partial s\partial t}\right|ds\,dt,\ldots,\int_a^b\int_c^d\left|\frac{\partial^2v_n}{\partial s\partial t}\right|ds\,dt\right),
\end{aligned}\tag{2.1}$$

where

$$S(s,t)=\sum_{i=1}^n\frac{\partial F}{\partial|u_i|}\frac{\partial|u_i|}{\partial s}\cdot\sum_{i=1}^n\frac{\partial G}{\partial|v_i|}\frac{\partial|v_i|}{\partial t}+\sum_{i=1}^n\frac{\partial F}{\partial|u_i|}\frac{\partial|u_i|}{\partial t}\cdot\sum_{i=1}^n\frac{\partial G}{\partial|v_i|}\frac{\partial|v_i|}{\partial s}.\tag{2.2}$$

Proof. From the hypotheses on $u_i(s,t)$, $v_i(s,t)$, $i=1,\ldots,n$, we have

$$\begin{aligned}
|u_i(s,t)|&\le\int_a^s\int_c^t\left|\frac{\partial^2u_i}{\partial\sigma\partial\tau}(\sigma,\tau)\right|d\sigma d\tau,\\
|v_i(s,t)|&\le\int_a^s\int_c^t\left|\frac{\partial^2v_i}{\partial\sigma\partial\tau}(\sigma,\tau)\right|d\sigma d\tau,
\end{aligned}\tag{2.3}$$

for $s\in[a,b]$, $t\in[c,d]$.

From (2.3) and in view of the hypotheses on all partial derivatives, and by letting

$$U_i(s,t)=\int_a^s\int_c^t\left|\frac{\partial^2 u_i}{\partial\sigma\partial\tau}(\sigma,\tau)\right|d\sigma d\tau,$$
$$V_i(s,t)=\int_a^s\int_c^t\left|\frac{\partial^2 v_i}{\partial\sigma\partial\tau}(\sigma,\tau)\right|d\sigma d\tau, \tag{2.4}$$

we obtain

$$\begin{aligned}
&\int_a^b\int_c^d\Bigg[F(|u_1(s,t)|,\dots,|u_n(s,t)|)\sum_{i=1}^n\left(\frac{\partial^2 G}{\partial|v_i|^2}\cdot\frac{\partial|v_i|}{\partial t}\cdot\frac{\partial|v_i|}{\partial s}+\frac{\partial G}{\partial|v_i|}\cdot\frac{\partial^2|v_i|}{\partial s\partial t}\right)\\
&\quad+G(|v_1(s,t)|,\dots,|v_n(s,t)|)\sum_{i=1}^n\left(\frac{\partial^2 F}{\partial|u_i|^2}\cdot\frac{\partial|u_i|}{\partial t}\cdot\frac{\partial|u_i|}{\partial s}+\frac{\partial F}{\partial|u_i|}\cdot\frac{\partial^2|u_i|}{\partial s\partial t}\right)\\
&\quad+\sum_{i=1}^n\frac{\partial F}{\partial|u_i|}\cdot\frac{\partial|u_i|}{\partial s}\cdot\sum_{i=1}^n\frac{\partial G}{\partial|v_i|}\cdot\frac{\partial|v_i|}{\partial t}\\
&\quad+\sum_{i=1}^n\frac{\partial F}{\partial|u_i|}\cdot\frac{\partial|u_i|}{\partial t}\cdot\sum_{i=1}^n\frac{\partial G}{\partial|v_i|}\cdot\frac{\partial|v_i|}{\partial s}\Bigg]ds\,dt\\
&\le\int_a^b\int_c^d\Bigg[F(U_1(s,t),\dots,U_n(s,t))\cdot\sum_{i=1}^n\left(\frac{\partial^2 G}{\partial V_i^2}\cdot\frac{\partial V_i}{\partial t}\cdot\frac{\partial V_i}{\partial s}+\frac{\partial G}{\partial V_i}\cdot\frac{\partial^2 V_i}{\partial s\partial t}\right)\\
&\qquad+G(V_1(s,t),\dots,V_n(s,t))\cdot\sum_{i=1}^n\left(\frac{\partial^2 F}{\partial U_i^2}\cdot\frac{\partial U_i}{\partial t}\cdot\frac{\partial U_i}{\partial s}+\frac{\partial F}{\partial U_i}\cdot\frac{\partial^2 U_i}{\partial s\partial t}\right)\\
&\qquad+\sum_{i=1}^n\frac{\partial F}{\partial U_i}\frac{\partial U_i}{\partial s}\cdot\sum_{i=1}^n\frac{\partial G}{\partial V_i}\frac{\partial V_i}{\partial t}+\sum_{i=1}^n\frac{\partial F}{\partial U_i}\frac{\partial U_i}{\partial t}\cdot\sum_{i=1}^n\frac{\partial G}{\partial V_i}\frac{\partial V_i}{\partial s}\Bigg]ds\,dt\\
&=\int_a^b\int_c^d\frac{\partial^2}{\partial s\partial t}\left[F(U_1(s,t),\dots,U_n(s,t))\cdot G(V_1(s,t),\dots,V_n(s,t))\right]ds\,dt\\
&=F(U_1(b,d),\dots,U_n(b,d))\cdot G(V_1(b,d),\dots,V_n(b,d))\\
&=F\left(\int_a^b\int_c^d\left|\frac{\partial^2 u_1}{\partial s\partial t}\right|ds\,dt,\dots,\int_a^b\int_c^d\left|\frac{\partial^2 u_n}{\partial s\partial t}\right|ds\,dt\right)\\
&\quad\cdot G\left(\int_a^b\int_c^d\left|\frac{\partial^2 v_1}{\partial s\partial t}\right|ds\,dt,\dots,\int_a^b\int_c^d\left|\frac{\partial^2 v_n}{\partial s\partial t}\right|ds\,dt\right).
\end{aligned}\tag{2.5}$$

This completes the proof of inequality (2.1). □

Remark 2.2. (i) Taking $G = 1$ in inequality (2.1), and in view of

$$\frac{\partial^2 G}{\partial |v_i|^2} \cdot \frac{\partial |v_i|}{\partial t} \cdot \frac{\partial |v_i|}{\partial s} + \frac{\partial G}{\partial |v_i|} \frac{\partial |v_i|}{\partial s \partial t} = 0, \qquad S(s,t) = 0, \tag{2.6}$$

for $i = 1,\ldots,n$, we have

$$\begin{aligned} &\int_a^b \int_c^d \left[\sum_{i=1}^n \left(\frac{\partial^2 F}{\partial |u_i|^2} \cdot \frac{\partial |u_i|}{\partial t} \cdot \frac{\partial |u_i|}{\partial s} + \frac{\partial F}{\partial |u_i|} \cdot \frac{\partial^2 |u_i|}{\partial s \partial t} \right) \right] ds\,dt \\ &\quad \le F\left(\int_a^b \int_c^d \left| \frac{\partial^2 u_1}{\partial s \partial t} \right| ds\,dt, \ldots, \int_a^b \int_c^d \left| \frac{\partial^2 u_n}{\partial s \partial t} \right| ds\,dt \right), \end{aligned} \tag{2.7}$$

for $i = 1,\ldots,n$.

Let $u_i(s,t)$ reduce to $u_i(t)$, where $i = 1,\ldots,n$ and with suitable modifications, then (2.7) becomes the following inequality:

$$\int_a^b \left[\sum_{i=1}^n F_i'(|u_1(t)|,\ldots,|u_n(t)|)\,|u_i'(t)| \right] dt \le F\left(\int_a^b |u_1'(t)|\,dt, \ldots, \int_a^b |u_n'(t)|\,dt \right). \tag{2.8}$$

This is a recent inequality which was given by Pečarić and Brnetić [18, 19].

Taking $n = 1$, inequality (2.7) reduces to

$$\int_a^b \int_c^d \left(\frac{\partial^2 F}{\partial |u|^2} \cdot \frac{\partial |u|}{\partial t} \cdot \frac{\partial |u|}{\partial s} + \frac{\partial F}{\partial |u|} \cdot \frac{\partial^2 |u|}{\partial s \partial t} \right) ds\,dt \le F\left(\int_a^b \int_c^d \left| \frac{\partial^2 u}{\partial s \partial t} \right| ds\,dt \right). \tag{2.9}$$

Let $u(s,t)$ reduce to $u(t)$ and with suitable modifications, then the above inequality becomes the following inequality:

$$\int_a^b F'(|f(t)|)\,|f'(t)|\,dt \le F\left(\int_a^b |f'(x)|\,dt \right). \tag{2.10}$$

This is an inequality which was given by Godunova and Levin [12].

(ii) Taking $G = F$ and $u_i(s,t) = v_i(s,t)$, $i = 1,\ldots,n$, in inequality (2.1), we have

$$\begin{aligned} &\int_a^b \int_c^d \Bigg[F(|u_1(s,t)|,\ldots,|u_n(s,t)|) \sum_{i=1}^n \left(\frac{\partial^2 G}{\partial |u_i|^2} \cdot \frac{\partial |u_i|}{\partial t} \cdot \frac{\partial |u_i|}{\partial s} + \frac{\partial G}{\partial |u_i|} \frac{\partial |u_i|}{\partial s \partial t} \right) \\ &\qquad + \sum_{i=1}^n \frac{\partial F}{\partial u_i} \frac{\partial u_i}{\partial s} \cdot \sum_{i=1}^n \frac{\partial F}{\partial u_i} \frac{\partial u_i}{\partial t} \Bigg] ds\,dt \\ &\quad \le \frac{1}{2} \cdot F^2\left(\int_a^b \int_c^d \left| \frac{\partial^2 u_1}{\partial s \partial t} \right| ds\,dt, \ldots, \int_a^b \int_c^d \left| \frac{\partial^2 u_n}{\partial s \partial t} \right| ds\,dt \right). \end{aligned} \tag{2.11}$$

Taking $n = 1$, (2.11) reduces to

$$\int_a^b \int_c^d \left[F(|u(s,t)|) \left(\frac{\partial^2 G}{\partial |u|^2} \cdot \frac{\partial |u|}{\partial t} \cdot \frac{\partial |u|}{\partial s} + \frac{\partial G}{\partial |u|} \frac{\partial |u|}{\partial s \partial t} \right) + \frac{\partial F}{\partial u} \frac{\partial u}{\partial s} \frac{\partial F}{\partial u} \frac{\partial u}{\partial t} \right] ds\, dt$$
$$\leq \frac{1}{2} \cdot F^2 \left(\int_a^b \int_c^d \left| \frac{\partial^2 u}{\partial s \partial t} \right| ds\, dt \right). \tag{2.12}$$

Let $u(s,t)$ reduce to $u(t)$ and with suitable modifications, then (2.12) becomes the following inequality:

$$\int_a^b [F(|u(t)|) \cdot F'(|u(t)|) \cdot |u'(t)|] dt \leq \frac{1}{2} F^2 \left(\int_a^b |u'(t)| dt \right). \tag{2.13}$$

This is an inequality given by Pachpatte in [15].

Inequality (2.12) is also a similar form of the following inequality which was given by Yang [22]:

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| f(t_1,t_2) \frac{\partial^2 f}{\partial t_1 \partial t_2} \right| dt_1 dt_2 \leq \frac{(b_1 - a_1)(b_2 - a_2)}{8} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left(\frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1,t_2) \right)^2 dt_1 dt_2. \tag{2.14}$$

(iii) Let $u_i(s,t)$ and $v_i(s,t)$ reduce to $u_i(s)$ and $v_i(s)$, respectively, and with suitable modifications (where $i = 1,\dots,n$), then inequality (2.1) changes to the following inequality:

$$\int_a^b \left[F(|u_1(t)|,\dots,|u_n(t)|) \sum_{i=1}^n G_i'(|v_1(t)|,\dots,|v_n(t)|)|v_i'(t)| \right.$$
$$\left. + G(|v_1(t)|,\dots,|v_n(t)|) \sum_{i=1}^n F_i'(|u_1(t)|,\dots,|u_n(t)|)|u_i'(t)| \right] dt$$
$$\leq F\left(\int_a^b |u_1'(t)| dt,\dots,\int_a^b |u_n'(t)| dt \right) \cdot G\left(\int_a^b |u_1'(t)| dt,\dots,\int_a^b |u_n'(t)| dt \right). \tag{2.15}$$

This is an inequality given by Agarwal and Pang in [2].

Taking $n = 1$, $G = 1$, $F(u) = u^2$, (2.15) changes to

$$\int_a^b |u(t)||u'(t)| dt \leq \frac{1}{2}(b-a) \int_a^b |u'(t)|^2 dt. \tag{2.16}$$

This is another version of the Opial's inequality, (see [13]).

(iv) Taking $G = 1$, $F = (|u_1|,\dots,|u_n|) = \prod_{i=1}^n f_i(|u_i|)$, $i = 1,\dots,n$, in (2.1), (2.1) changes to a general form of the inequality which was given by Pachpatte [16], where the functions f_i must satisfy some suitable conditions, (see [16]).

Theorem 2.3. *Let* $u_i(s,t)$, $v_i(s,t)$, F, G, $\partial^2 F/\partial|u_i|^2$, $\partial F/\partial|u_i|$, $\partial|u_i|/\partial s$, $\partial|u_i|/\partial t$, $\partial^2|u_i|/\partial s\partial t$, $\partial^2 G/\partial|v_i|^2$, $\partial G/\partial|v_i|$, $\partial|v_i|/\partial s$, $\partial|v_i|/\partial t$, $\partial^2|v_i|/\partial s\partial t$, $i=1,\dots,n$, *be as in Theorem 2.1. Let* $p_i(s,t)$, $q_i(s,t)$, $i=1,\dots,n$, *be real-valued positive functions defined on* $[a,b]\times[c,d]$ *satisfying*

$$\int_a^b\int_c^d p_i(s,t)\,ds\,dt=1,\qquad \int_a^b\int_c^d q_i(s,t)\,ds\,dt=1\quad (i=1,\dots,n). \tag{2.17}$$

Let h_i, w_i, $i=1,\dots,n$, *be real-valued positive convex and increasing functions on* $(0,\infty)^2$. *Then the following integral inequality holds:*

$$\begin{aligned}\int_a^b\int_c^d\Bigg[&F(|u_1(s,t)|,\dots,|u_n(s,t)|)\sum_{i=1}^n\left(\frac{\partial^2 G}{\partial|v_i|^2}\cdot\frac{\partial|v_i|}{\partial t}\cdot\frac{\partial|v_i|}{\partial s}+\frac{\partial G}{\partial|v_i|}\frac{\partial^2|v_i|}{\partial s\partial t}\right)\\ &+G(|v_1(s,t)|,\dots,|v_n(s,t)|)\sum_{i=1}^n\left(\frac{\partial^2 F}{\partial|u_i|^2}\cdot\frac{\partial|u_i|}{\partial t}\cdot\frac{\partial|u_i|}{\partial s}+\frac{\partial F}{\partial|u_i|}\frac{\partial^2|u_i|}{\partial s\partial t}\right)\\ &+S(s,t)\Bigg]ds\,dt\\ \le F\Bigg[&h_1^{-1}\left(\int_a^b\int_c^d p_1(s,t)h_1\left(\frac{|\partial^2u_1/\partial s\partial t|}{p_1(s,t)}\right)ds\,dt\right),\dots,\\ &h_n^{-1}\left(\int_a^b\int_c^d p_n(s,t)h_n\left(\frac{|\partial^2u_n/\partial s\partial t|}{p_n(s,t)}ds\,dt\right)\right)\Bigg]\\ \cdot G\Bigg[&w_1^{-1}\left(\int_a^b\int_c^d q_1(s,t)w_1\left(\frac{|\partial^2v_1/\partial s\partial t|}{q_1(s,t)}\right)ds\,dt\right),\dots,\\ &w_n^{-1}\left(\int_a^b\int_c^d q_n(s,t)w_n\left(\frac{|\partial^2v_n/\partial s\partial t|}{q_n(s,t)}ds\,dt\right)\right)\Bigg],\end{aligned} \tag{2.18}$$

where

$$S(s,t)=\sum_{i=1}^n\frac{\partial F}{\partial|u_i|}\frac{\partial|u_i|}{\partial s}\cdot\sum_{i=1}^n\frac{\partial G}{\partial|v_i|}\frac{\partial|v_i|}{\partial t}+\sum_{i=1}^n\frac{\partial F}{\partial|u_i|}\frac{\partial|u_i|}{\partial t}\cdot\sum_{i=1}^n\frac{\partial G}{\partial|v_i|}\frac{\partial|v_i|}{\partial s}. \tag{2.19}$$

Proof. From the hypotheses, we have

$$\begin{aligned}\int_a^b\int_c^d\left|\frac{\partial u_i^2}{\partial s\partial t}\right|ds\,dt&=\frac{\int_a^b\int_c^d p_i(s,t)\,|\partial u_i^2/\partial s\partial t|/p_i(s,t)\,ds\,dt}{\int_a^b\int_c^d p_i(s,t)\,ds\,dt},\\ \int_a^b\int_c^d\left|\frac{\partial v_i^2}{\partial s\partial t}\right|ds\,dt&=\frac{\int_a^b\int_c^d q_i(s,t)\,|\partial v_i^2/\partial s\partial t|/q_i(s,t)\,ds\,dt}{\int_a^b\int_c^d q_i(s,t)\,ds\,dt},\end{aligned} \tag{2.20}$$

for $i=1,\dots,n$.

From (2.20), the hypotheses on h_i, w_i, $i = 1,\dots,n$, and in view of Jensen's inequality, we obtain

$$h_i\left(\int_a^b\int_c^d\left|\frac{\partial u_i^2}{\partial s\partial t}\right|ds\,dt\right) \le \int_a^b\int_c^d p_i(s,t)\cdot h_i\left(\frac{|\partial^2 u_i/\partial s\partial t|}{p_i(s,t)}\right)ds\,dt,$$
$$w_i\left(\int_a^b\int_c^d\left|\frac{\partial v_i^2}{\partial s\partial t}\right|ds\,dt\right) \le \int_a^b\int_c^d q_i(s,t)\cdot w_i\left(\frac{|\partial^2 v_i/\partial s\partial t|}{q_i(s,t)}\right)ds\,dt, \tag{2.21}$$

for $i = 1,\dots,n$.

From (2.21), we observe that

$$\int_a^b\int_c^d\left|\frac{\partial u_i^2}{\partial s\partial t}\right|ds\,dt \le h_{(i)^{-1}}\left(\int_a^b\int_c^d p_i(s,t)\cdot h_i\left(\frac{|\partial^2 u_i/\partial s\partial t|}{p_i(s,t)}\right)ds\,dt\right),$$
$$\int_a^b\int_c^d\left|\frac{\partial v_i^2}{\partial s\partial t}\right|ds\,dt \le w_{(i)^{-1}}\left(\int_a^b\int_c^d q_i(s,t)\cdot w_i\left(\frac{|\partial^2 v_i/\partial s\partial t|}{q_i(s,t)}\right)ds\,dt\right), \tag{2.22}$$

for $i = 1,\dots,n$.

From (2.22) and in view of inequality (2.1), we get inequality (2.18) and the proof is complete. □

Remark 2.4. (i) Taking $G = 1$ in inequality (2.18), and in view of

$$\frac{\partial^2 G}{\partial|v_i|^2}\cdot\frac{\partial|v_i|}{\partial t}\cdot\frac{\partial|v_i|}{\partial s}+\frac{\partial G}{\partial|v_i|}\frac{\partial|v_i|}{\partial s\partial t} = 0, \tag{2.23}$$

for $i = 1,\dots,n$, and

$$S(s,t) = 0, \tag{2.24}$$

(2.18) becomes

$$\int_a^b\int_c^d\left[\sum_{i=1}^n\left(\frac{\partial^2 F}{\partial|u_i|^2}\cdot\frac{\partial|u_i|}{\partial t}\cdot\frac{\partial|u_i|}{\partial s}+\frac{\partial F}{\partial|u_i|}\cdot\frac{\partial^2|u_i|}{\partial s\partial t}\right)\right]ds\,dt$$
$$\le F\left[h_1^{-1}\left(\int_a^b\int_c^d p_1(s,t)h_1\left(\frac{|\partial^2 u_1/\partial s\partial t|}{p_1(s,t)}\right)ds\,dt\right),\dots,\right. \tag{2.25}$$
$$\left.h_n^{-1}\left(\int_a^b\int_c^d p_n(s,t)h_n\left(\frac{|\partial^2 u_n/\partial s\partial t|}{p_n(s,t)}\right)ds\,dt\right)\right],$$

for $i = 1,\dots,n$.

Let $u_i(s,t)$, $h_i(s,t)$, and $p_i(s,t)$ change to $f_i(t)$, $h_i(t)$, and $p_i(t)$, respectively, where $i = 1,\dots,n$, then (2.25) reduces to the following inequality:

$$\begin{aligned}
&\int_a^b \left(\sum_{i=1}^{n} D_i F(|f_1(t)|,\dots,|f_n(t)|)\,|f_i'(t)|\right)dt \\
&\quad \le F\left(h_1^{-1}\left(\int_a^b p_1(t)h_1\left(\frac{|f_1'(t)|}{p_1(t)}\right)dt\right),\dots,h_n^{-1}\left(\int_a^b p_n(t)h_n\left(\frac{|f_n'(t)|}{p_n(t)}\right)dt\right)\right),
\end{aligned} \tag{2.26}$$

where D_iF is as in [18]. This is an inequality given by Pečarić in [18].

Taking $F(x_1,\dots,x_n) = \prod_{i=1}^{n} F_i(x_i)$, $i = 1,\dots,n$, (2.25) changes to a general form of the inequality which was given by Pachpatte [16]. Taking $n = 1$, (2.25) reduces to a general form of the inequality which was given by Godunova and Levin [12].

On the other hand, inequality (2.18) is also a general form of another inequality in Pečarić and Brnetić [20, Theorem 1].

(ii) Taking $G = F$ and $u_i(s,t) = v_i(s,t)$, $i = 1,\dots,n$, in inequality (2.18), we have

$$\begin{aligned}
&\int_a^b\int_c^d \Bigg[F(|u_1(s,t)|,\dots,|u_n(s,t)|)\sum_{i=1}^{n}\left(\frac{\partial^2 G}{\partial|u_i|^2}\cdot\frac{\partial|u_i|}{\partial t}\cdot\frac{\partial|u_i|}{\partial s}+\frac{\partial G}{\partial|u_i|}\frac{\partial|u_i|}{\partial s\partial t}\right) \\
&\qquad +\sum_{i=1}^{n}\frac{\partial F}{\partial u_i}\frac{\partial u_i}{\partial s}\cdot\sum_{i=1}^{n}\frac{\partial F}{\partial u_i}\frac{\partial u_i}{\partial t}\Bigg]ds\,dt \\
&\quad \le \frac{1}{2}\cdot F^2\Bigg[h_1^{-1}\left(\int_a^b\int_c^d p_1(s,t)h_1\left(\frac{|\partial^2 u_1/\partial s\partial t|}{p_1(s,t)}\right)ds\,dt\right),\dots, \\
&\qquad\qquad h_n^{-1}\left(\int_a^b\int_c^d p_n(s,t)h_n\left(\frac{|\partial^2 u_n/\partial s\partial t|}{p_n(s,t)}\right)ds\,dt\right)\Bigg].
\end{aligned} \tag{2.27}$$

Taking $n = 1$, (2.27) reduces to

$$\begin{aligned}
&\int_a^b\int_c^d \left[F(|u(s,t)|)\left(\frac{\partial^2 G}{\partial|u|^2}\cdot\frac{\partial|u|}{\partial t}\cdot\frac{\partial|u|}{\partial s}+\frac{\partial G}{\partial|u|}\frac{\partial|u|}{\partial s\partial t}\right)+\frac{\partial F}{\partial u}\frac{\partial u}{\partial s}\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}\right]ds\,dt \\
&\quad \le \frac{1}{2}\cdot F^2\left(h^{-1}\left(\int_a^b\int_c^d p(s,t)h\left(\frac{|\partial^2 u/\partial s\partial t|}{p(s,t)}\right)ds\,dt\right)\right).
\end{aligned} \tag{2.28}$$

This is a general form of the inequality which was given by Pachpatte [14].

(iii) Let $u_i(s,t)$, $v_i(s,t)$, $h_i(s,t)$, $w_i(s,t)$, $p_i(s,t)$, and $q_i(s,t)$ reduce to $u_i(t)$, $v_i(t)$, $h_i(t)$, $w_i(t)$, $p_i(t)$, and $q_i(t)$, respectively, and with suitable modifications (where $i = 1,\dots,n$), then inequality (2.18) changes to the following inequality:

$$\begin{aligned}
&\int_a^b \bigg[F(|u_1(t)|,\dots,|u_n(t)|) \sum_{i=1}^n G_i'(|v_1(t)|,\dots,|v_n(t)|)\,|v_i'(t)| \\
&\qquad + G(|v_1(t)|,\dots,|v_n(t)|) \sum_{i=1}^n F_i'(|u_1(t)|,\dots,|u_n(t)|)\,|u_i'(t)| \bigg] dt \\
&\qquad \le F\bigg(h_1^{-1}\bigg(\int_a^b p_1(t) h_1\bigg(\frac{|u_1'(t)|}{p_1(t)}\bigg) dt\bigg),\dots, h_n^{-1}\bigg(\int_a^b p_n(t) h_n\bigg(\frac{|u_n'(t)|}{p_n(t)}\bigg) dt\bigg)\bigg) \\
&\qquad\quad \cdot G\bigg(w_1^{-1}\bigg(\int_a^b q_1(t) w_1\bigg(\frac{|v_1'(t)|}{q_1(t)}\bigg) dt\bigg),\dots, w_n^{-1}\bigg(\int_a^b q_n(t) w_n\bigg(\frac{|v_n'(t)|}{q_n(t)}\bigg) dt\bigg)\bigg).
\end{aligned} \tag{2.29}$$

This is just an inequality given by Agarwal and Pang in [2].

Theorem 2.5. *Let $u_i(s,t)$, $v_i(s,t)$, F, G, be as in Theorem 2.1. Let ϕ_i, ψ_i, $i = 1,\dots,n$, be real-valued positive convex and increasing functions on $(0,\infty)^2$. Let $r_i(s,t) \ge 0$, $\partial^2 r_i/\partial s\partial t > 0$, $r_i(s,c) = r_i(a,t) = r_i(a,c) = 0$, $\partial^2 e_i/\partial s\partial t > 0$, $e_i(s,c) = e_i(a,t) = e_i(a,c) = 0$, $i = 1,\dots,n$. Let $\partial^2 F/\partial \overline{M}_i^2$, $\partial F/\partial \overline{M}_i$, $\partial^2 G/\partial \overline{N}_i^2$, $\partial G/\partial \overline{N}_i$, $i = 1,\dots,n$, be nonnegative continuous and nondecreasing functions on $[0,\infty)^n$. Let $\partial\overline{M}_i/\partial s$, $\partial\overline{M}_i/\partial t$, $\partial^2\overline{M}_i/\partial s\partial t$, $\partial\overline{N}_i\partial s$, $\partial\overline{N}_i/\partial t$, $\partial^2\overline{N}_i/\partial s\partial t$, $i = 1,\dots,n$, be nonnegative continuous and nondecreasing functions on $[a,b]\times[c,d]$. Then the following inequality holds:*

$$\begin{aligned}
&\int_a^b\int_c^d \bigg[F(\overline{M}_1(s,t),\dots,\overline{M}_n(s,t)) \cdot \sum_{i=1}^n \bigg(\frac{\partial^2 G}{\partial \overline{N}_i^2}\cdot\frac{\partial \overline{N}_i}{\partial t}\cdot\frac{\partial \overline{N}_i}{\partial s} + \frac{\partial G}{\partial \overline{N}_i}\cdot\frac{\partial^2 \overline{N}_i}{\partial s\partial t}\bigg) \\
&\qquad + G(\overline{N}_1(s,t),\dots,\overline{N}_n(s,t)) \cdot \sum_{i=1}^n \bigg(\frac{\partial^2 F}{\partial \overline{M}_i^2}\cdot\frac{\partial \overline{M}_i}{\partial t}\cdot\frac{\partial \overline{M}_i}{\partial s} + \frac{\partial F}{\partial \overline{M}_i}\cdot\frac{\partial^2 \overline{M}_i}{\partial s\partial t}\bigg) + \overline{S}(s,t) \bigg] ds\,dt \\
&\quad \le F\bigg(\int_a^b\int_c^d \frac{\partial^2 r_1}{\partial s\partial t}\cdot\phi_1\bigg(\frac{|\partial^2 u_1/\partial s\partial t|}{\partial^2 r_1/\partial s\partial t}\bigg) ds\,dt,\dots, \int_a^b\int_c^d \frac{\partial^2 r_n}{\partial s\partial t}\cdot\phi_n\bigg(\frac{|\partial^2 u_n/\partial s\partial t|}{\partial^2 r_n/\partial s\partial t}\bigg) ds\,dt\bigg) \\
&\qquad \cdot G\bigg(\int_a^b\int_c^d \frac{\partial^2 e_1}{\partial s\partial t}\cdot\psi_1\bigg(\frac{|\partial^2 v_1/\partial s\partial t|}{\partial^2 e_1/\partial s\partial t}\bigg) ds\,dt,\dots, \int_a^b\int_c^d \frac{\partial^2 e_n}{\partial s\partial t}\cdot\psi_n\bigg(\frac{|\partial^2 v_n/\partial s\partial t|}{\partial^2 e_n/\partial s\partial t}\bigg) ds\,dt\bigg),
\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}
\overline{M_i}(s,t) &= r_i(s,t)\cdot\phi_i\bigg(\frac{|u_i(s,t)|}{r_i(s,t)}\bigg), \\
\overline{N_i}(s,t) &= e_i(s,t)\cdot\psi_i\bigg(\frac{|v_i(s,t)|}{e_i(s,t)}\bigg),
\end{aligned} \tag{2.31}$$

for $i = 1,\dots,n$, *and*

$$\overline{S}(s,t) = \sum_{i=1}^{n} \frac{\partial F}{\partial \overline{M}_i}\frac{\partial \overline{M}_i}{\partial s} \cdot \sum_{i=1}^{n} \frac{\partial G}{\partial \overline{N}_i}\frac{\partial \overline{N}_i}{\partial t} + \sum_{i=1}^{n} \frac{\partial F}{\partial \overline{M}_i}\frac{\partial \overline{M}_i}{\partial t} \cdot \sum_{i=1}^{n} \frac{\partial G}{\partial \overline{N}_i}\frac{\partial \overline{N}_i}{\partial s}, \tag{2.32}$$

for $i = 1,\dots,n$.

Proof. From the hypotheses on $u_i(s,t)$, $v_i(s,t)$, $r_i(s,t)$, $e_i(s,t)$, $i = 1,\dots,n$, we have

$$\begin{aligned}
|u_i(s,t)| &\le \int_a^s\int_c^t \left|\frac{\partial^2 u_i}{\partial\sigma\partial\tau}(\sigma,\tau)\right| d\sigma d\tau,\\
|v_i(s,t)| &\le \int_a^s\int_c^t \left|\frac{\partial^2 v_i}{\partial\sigma\partial\tau}(\sigma,\tau)\right| d\sigma d\tau,\\
r_i(s,t) &= \int_a^s\int_c^t \frac{\partial^2 r_i}{\partial\sigma\partial\tau}(\sigma,\tau) d\sigma d\tau,\\
e_i(s,t) &= \int_a^s\int_c^t \frac{\partial^2 e_i}{\partial\sigma\partial\tau}(\sigma,\tau) d\sigma d\tau,
\end{aligned} \tag{2.33}$$

for $s \in [a,b]$, $t \in [c,d]$.

From (2.33) and using the hypotheses on ϕ_i, ψ_i, $i = 1,\dots,n$, and Jensen's inequality, we have

$$\begin{aligned}
\overline{M}_i(s,t) &\le \int_a^s\int_c^t \frac{\partial^2 r_i}{\partial\sigma\partial\tau} \cdot \phi_i\left(\frac{|(\partial^2 u_i/\partial\sigma\partial\tau)(\sigma,\tau)|}{(\partial^2 r_i/\partial\sigma\partial\tau)(\sigma,\tau)}\right) d\sigma d\tau,\\
\overline{N}_i(s,t) &\le \int_a^s\int_c^t \frac{\partial^2 e_i}{\partial\sigma\partial\tau} \cdot \psi_i\left(\frac{|(\partial^2 v_i/\partial\sigma\partial\tau)(\sigma,\tau)|}{(\partial^2 e_i/\partial\sigma\partial\tau)(\sigma,\tau)}\right) d\sigma d\tau,
\end{aligned} \tag{2.34}$$

for $s \in [a,b]$, $t \in [c,d]$.

From (2.34), using the hopytheses on all partial derivatives and in view of

$$\begin{aligned}
M_i(s,t) &= \int_a^s\int_c^t \frac{\partial^2 r_i}{\partial\sigma\partial\tau} \cdot \phi_i\left(\frac{|\partial^2 u_i/\partial\sigma\partial\tau|}{\partial^2 r_i/\partial\sigma\partial\tau}\right) d\sigma d\tau,\\
N_i(s,t) &= \int_a^s\int_c^t \frac{\partial^2 e_i}{\partial\sigma\partial\tau} \cdot \psi_i\left(\frac{|\partial^2 v_i/\partial\sigma\partial\tau|}{\partial^2 e_i/\partial\sigma\partial\tau}\right) d\sigma d\tau,
\end{aligned} \tag{2.35}$$

we have

$$
\begin{aligned}
&\int_a^b\int_c^d\Bigg[F(\overline{M}_1(s,t),\dots,\overline{M}_n(s,t))\cdot\sum_{i=1}^n\left(\frac{\partial^2 G}{\partial\overline{N}_i^2}\cdot\frac{\partial\overline{N}_i}{\partial t}\cdot\frac{\partial\overline{N}_i}{\partial s}+\frac{\partial G}{\partial\overline{N}_i}\cdot\frac{\partial^2\overline{N}_i}{\partial s\partial t}\right)\\
&\qquad+G(\overline{N}_1(s,t),\dots,\overline{N}_n(s,t))\cdot\sum_{i=1}^n\left(\frac{\partial^2 F}{\partial\overline{M}_i^2}\cdot\frac{\partial\overline{M}_i}{\partial t}\cdot\frac{\partial\overline{M}_i}{\partial s}+\frac{\partial F}{\partial\overline{M}_i}\cdot\frac{\partial^2\overline{M}_i}{\partial s\partial t}\right)+\overline{S}(s,t)\Bigg]ds\,dt\\
&\quad\le\int_a^b\int_c^d\Bigg[F(M_1(s,t),\dots,M_n(s,t))\cdot\sum_{i=1}^n\left(\frac{\partial^2 G}{\partial N_i^2}\cdot\frac{\partial N_i}{\partial t}\cdot\frac{\partial N_i}{\partial s}+\frac{\partial G}{\partial N_i}\cdot\frac{\partial^2 N_i}{\partial s\partial t}\right)\\
&\qquad\qquad+G(N_1(s,t),\dots,N_n(s,t))\cdot\sum_{i=1}^n\left(\frac{\partial^2 F}{\partial M_i^2}\cdot\frac{\partial M_i}{\partial t}\cdot\frac{\partial M_i}{\partial s}+\frac{\partial F}{\partial M_i}\cdot\frac{\partial^2 M_i}{\partial s\partial t}\right)\\
&\qquad\qquad+\sum_{i=1}^n\frac{\partial F}{\partial M_i}\frac{\partial M_i}{\partial s}\cdot\sum_{i=1}^n\frac{\partial G}{\partial N_i}\frac{\partial N_i}{\partial t}+\sum_{i=1}^n\frac{\partial F}{\partial M_i}\frac{\partial M_i}{\partial t}\cdot\sum_{i=1}^n\frac{\partial G}{\partial N_i}\frac{\partial N_i}{\partial s}\Bigg]ds\,dt\\
&\quad=\int_a^b\int_c^d\frac{\partial^2}{\partial s\partial t}\Big[F(M_1(s,t),\dots,M_n(s,t))\cdot G(N_1(s,t),\dots,N_n(s,t))\Big]ds\,dt\\
&\quad=F(M_1(b,d),\dots,M_n(b,d))\cdot G(N_1(b,d),\dots,N_n(b,d))\\
&\quad=F\left(\int_a^b\int_c^d\frac{\partial^2 r_1}{\partial s\partial t}\cdot\phi_1\left(\frac{|\partial^2 u_1/\partial s\partial t|}{\partial^2 r_1/\partial s\partial t}\right)ds\,dt,\dots,\int_a^b\int_c^d\frac{\partial^2 r_n}{\partial s\partial t}\cdot\phi_n\left(\frac{|\partial^2 u_n/\partial s\partial t|}{\partial^2 r_n/\partial s\partial t}\right)ds\,dt\right)\\
&\qquad\cdot G\left(\int_a^b\int_c^d\frac{\partial^2 e_1}{\partial s\partial t}\cdot\psi_1\left(\frac{|\partial^2 v_1/\partial s\partial t|}{\partial^2 e_1/\partial s\partial t}\right)ds\,dt,\dots,\int_a^b\int_c^d\frac{\partial^2 e_n}{\partial s\partial t}\cdot\psi_n\left(\frac{|\partial^2 v_n/\partial s\partial t|}{\partial^2 e_n/\partial s\partial t}\right)ds\,dt\right).
\end{aligned}
\tag{2.36}
$$

This completes the proof. □

Remark 2.6. (i) Taking $n=1$, (2.30) changes to a general form of the inequality which was given by Pachpatte [17].

(ii) Taking $G=1$, (2.30) changes to a general form of the inequality which was given by Pečarić and Brnetić [19].

(iii) Taking $n=1$, $G=1$, (2.30) changes to the following inequality:

$$
\int_a^b\int_c^d\left(\frac{\partial^2 F}{\partial\overline{M}^2}\cdot\frac{\partial\overline{M}}{\partial t}\cdot\frac{\partial\overline{M}}{\partial s}+\frac{\partial F}{\partial\overline{M}}\cdot\frac{\partial^2\overline{M}}{\partial s\partial t}\right)ds\,dt\le F\left(\int_a^b\int_c^d\frac{\partial^2 r}{\partial s\partial t}\cdot\phi\left(\frac{|\partial^2 u/\partial s\partial t|}{\partial^2 r/\partial s\partial t}\right)ds\,dt\right),
\tag{2.37}
$$

which is a general form of the follwing inequality established by Rozanova [21]:

$$
\int_a^b F'\left(r(t)\phi\left(\frac{|f(t)|}{r(t)}\right)\right)r'(t)\phi\left(\frac{|f'(t)|}{r'(t)}\right)dt\le F\left(\int_a^b r'(t)\phi\left(\frac{|f'(x)|}{r'(t)}\right)dt\right).
\tag{2.38}
$$

(iv) Let $u_i(s,t)$, $v_i(s,t)$, $r_i(s,t)$, and $e_i(s,t)$ reduce to $u_i(t)$, $v_i(t)$, $r_i(t)$, and $e_i(t)$, respectively, and with suitable modifications (where $i=1,\dots,n$), the inequality in Theorem 2.5 changes to the inequality in Agarwal and Pang [2, Theorem 3, page 305].

Theorem 2.7. *Let $u_i(s,t)$, $v_i(s,t)$, F, G, be as in Theorem 2.1. Let p_i, q_i, h_i, w_i, $i=1,\dots,n$, be as in Theorem 2.3. Let $\overline{S}(s,t)$, $\overline{M}_i$, $\overline{N}_i$, $\partial^2F/\partial\overline{M}_i^2$, $\partial F/\partial\overline{M}_i$, $\partial^2G/\partial\overline{N}_i^2$, $\partial G/\partial\overline{N}_i$, $i=1,\dots,n$, $\partial\overline{M}_i/\partial s$, $\partial\overline{M}_i/\partial t$, $\partial^2\overline{M}_i/\partial s\partial t$, $\partial\overline{N}_i/\partial s$, $\partial\overline{N}_i/\partial t$, $\partial^2\overline{N}_i/\partial s\partial t$, $i=1,\dots,n$, be as in Theorem 2.5. Then the following integral inequality holds:*

$$\begin{aligned}
&\int_a^b\int_c^d\Bigg[F(\overline{M}_1(s,t),\dots,\overline{M}_n(s,t))\cdot\sum_{i=1}^n\left(\frac{\partial^2G}{\partial\overline{N}_i^2}\cdot\frac{\partial\overline{N}_i}{\partial t}\cdot\frac{\partial\overline{N}_i}{\partial s}+\frac{\partial G}{\partial\overline{N}_i}\cdot\frac{\partial^2\overline{N}_i}{\partial s\partial t}\right)\\
&\qquad+G(\overline{N}_1(s,t),\dots,\overline{N}_n(s,t))\cdot\sum_{i=1}^n\left(\frac{\partial^2F}{\partial\overline{M}_i^2}\cdot\frac{\partial\overline{M}_i}{\partial t}\cdot\frac{\partial\overline{M}_i}{\partial s}+\frac{\partial F}{\partial\overline{M}_i}\cdot\frac{\partial^2\overline{M}_i}{\partial s\partial t}\right)+\overline{S}(s,t)\Bigg]ds\,dt\\
&\quad\le F\Bigg[h_1^{-1}\left(\int_a^b\int_c^d p_1(s,t)h_1\left(\frac{\partial^2r_1}{\partial s\partial t}\cdot\phi_1\left(\frac{|\partial^2u_1/\partial s\partial t|}{\partial^2r_1}\partial s\partial t\right)p_1(s,t)\right)ds\,dt\right),\dots,\\
&\qquad h_n^{-1}\left(\int_a^b\int_c^d p_n(s,t)h_n\left(\frac{\partial^2r_n}{\partial s\partial t}\cdot\phi_n\left(\frac{|\partial^2u_n/\partial s\partial t|}{\partial^2r_n/\partial s\partial t}\right)p_n(s,t)\right)ds\,dt\right)\Bigg]\\
&\qquad\cdot G\Bigg[w_1^{-1}\left(\int_a^b\int_c^d q_1(s,t)w_1\left(\frac{\partial^2e_1}{\partial s\partial t}\cdot\psi_1\left(\frac{|\partial^2v_1/\partial s\partial t|}{\partial^2e_1/\partial s\partial t}\right)q_1(s,t)\right)ds\,dt\right),\dots,\\
&\qquad w_n^{-1}\left(\int_a^b\int_c^d q_n(s,t)w_n\left(\frac{\partial^2e_n}{\partial s\partial t}\cdot\psi_n\left(\frac{|\partial^2v_n/\partial s\partial t|}{\partial^2e_n/\partial s\partial t}\right)q_n(s,t)\right)ds\,dt\right)\Bigg].
\end{aligned}\tag{2.39}$$

Proof. From the hypotheses of Theorem 2.7, we have

$$\begin{aligned}
&\int_a^b\int_c^d\frac{\partial^2r_i}{\partial s\partial t}\cdot\phi_i\left(\frac{|\partial^2u_i/\partial s\partial t|}{\partial^2r_i/\partial s\partial t}\right)ds\,dt\\
&\qquad=\frac{\int_a^b\int_c^d p_i(s,t)((((\partial^2r_i/\partial s\partial t)\cdot\phi_i)(|\partial^2u_i/\partial s\partial t|/(\partial^2r_i/\partial s\partial t)))/p_i(s,t))ds\,dt}{\int_a^b\int_c^d p_i(s,t)ds\,dt},\\
&\int_a^b\int_c^d\frac{\partial^2e_i}{\partial s\partial t}\cdot\psi_i\left(\frac{|\partial^2v_i/\partial s\partial t|}{\partial^2e_i/\partial s\partial t}\right)ds\,dt\\
&\qquad=\frac{\int_a^b\int_c^d q_i(s,t)((((\partial^2e_i/\partial s\partial t)\cdot\psi_i)(|\partial^2v_i/\partial s\partial t|/(\partial^2e_i/\partial s\partial t)))/q_i(s,t))ds\,dt}{\int_a^b\int_c^d q_i(s,t)ds\,dt},
\end{aligned}\tag{2.40}$$

for $i=1,\dots,n$.

From (2.40) and using the hypotheses on h_i, w_i, $i = 1,\dots,n$, and Jensen's inequality, we obtain

$$
\begin{aligned}
&h_i\left(\int_a^b\int_c^d\left(\frac{\partial^2 r_i}{\partial s\partial t}\cdot\phi_i\left(\frac{|\partial^2 u_i/\partial s\partial t|}{\partial^2 r_i/\partial s\partial t}\right)\right)ds\,dt\right)\\
&\qquad\leq\int_a^b\int_c^d p_i(s,t)h_i\left(\frac{(\partial^2 r_i/\partial s\partial t)\cdot\phi_i(|\partial^2 u_i/\partial s\partial t|/\partial^2 r_i/\partial s\partial t)}{p_i(s,t)}\right)ds\,dt,
\end{aligned}
\tag{2.41}
$$

$$
\begin{aligned}
&w_i\left(\int_a^b\int_c^d\left(\frac{\partial^2 e_i}{\partial s\partial t}\cdot\psi_i\left(\frac{|\partial^2 v_i/\partial s\partial t|}{\partial^2 e_i/\partial s\partial t}\right)\right)ds\,dt\right)\\
&\qquad\leq\int_a^b\int_c^d q_i(s,t)w_i\left(\frac{(\partial^2 e_i/\partial s\partial t)\cdot\psi_i(|\partial^2 v_i/\partial s\partial t|/\partial^2 e_i/\partial s\partial t)}{q_i(s,t)}\right)ds\,dt,
\end{aligned}
\tag{2.42}
$$

for $i = 1,\dots,n$.

Then

$$
\begin{aligned}
&\int_a^b\int_c^d\frac{\partial^2 r_i}{\partial s\partial t}\cdot\phi_i\left(\frac{|\partial^2 u_i/\partial s\partial t|}{\partial^2 r_i/\partial s\partial t}\right)ds\,dt\\
&\qquad\leq h_i^{-1}\left(\int_a^b\int_c^d p_i(s,t)\cdot h_i\left(\frac{(\partial^2 r_i/\partial s\partial t)\cdot\phi_i(|\partial^2 u_i/\partial s\partial t|/\partial^2 r_i/\partial s\partial t)}{p_i(s,t)}\right)ds\,dt\right),
\end{aligned}
\tag{2.43}
$$

$$
\begin{aligned}
&\int_a^b\int_c^d\frac{\partial^2 e_i}{\partial s\partial t}\cdot\psi_i\left(\frac{|\partial^2 v_i/\partial s\partial t|}{\partial^2 e_i/\partial s\partial t}\right)ds\,dt\\
&\qquad\leq w_i^{-1}\left(\int_a^b\int_c^d q_i(s,t)\cdot w_i\left(\frac{(\partial^2 e_i/\partial s\partial t)\cdot\psi_i(|\partial^2 v_i/\partial s\partial t|/\partial^2 e_i/\partial s\partial t)}{q_i(s,t)}\right)ds\,dt\right).
\end{aligned}
\tag{2.44}
$$

By applying (2.43) and (2.44) to the right-hand side of inequality (2.30), we get the desired inequality (2.39) and the proof is complete. □

Remark 2.8. (i) Taking $n = 1$, (2.39) changes to a general form of the inequality which was given by Pachpatte [17].

(ii) Taking $G = 1$, (2.39) changes to a general form of the inequality which was given by Pečarić and Brnetić [19].

(iii) Let $u_i(s,t)$, $v_i(s,t)$, $h_i(s,t)$, $w_i(s,t)$, $r_i(s,t)$, and $e_i(s,t)$ reduce to $u_i(t)$, $v_i(t)$, $h_i(t)$, $w_i(t)$, $r_i(t)$, and $e_i(t)$, respectively, and with suitable modifications (where $i = 1,\dots,n$), then inequality (2.39) changes to the inequality in Agarwal and Pang [2, Theorem 4, page 308].

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