Entanglement monogamy and entanglement evolution in multipartite systems

Yan-Kui Bai,[1,2] Ming-Yong Ye,[1,3] and Z. D. Wang[1]
[1]*Department of Physics and Center of Theoretical and Computational Physics, University of Hong Kong, Pokfulam Road, Hong Kong, China*
[2]*College of Physical Science and Information Engineering and Hebei Advance Thin Films Laboratory, Hebei Normal University, Shijiazhuang, Hebei 050016, China*
[3]*School of Physics and Optoelectronics Technology, Fujian Normal University, Fuzhou 350007, China*
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We analyze the entanglement distribution and the two-qubit residual entanglement in multipartite systems. For a composite system consisting of two cavities interacting with independent reservoirs, it is revealed that the entanglement evolution is restricted by an entanglement monogamy relation derived here. Moreover, it is found that the initial cavity-cavity entanglement evolves completely to the genuine four-partite cavities-reservoirs entanglement in the time interval between the sudden death of cavity-cavity entanglement and the birth of reservoir-reservoir entanglement. In addition, we also address the relationship between the genuine block-block entanglement form and qubit-block form in the interval.

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I. INTRODUCTION

As an important physical resource, entanglement has widely been applied to quantum communication [1,2] and quantum computation [3,4]. It is fundamental to characterize entanglement nature of quantum systems, especially at a quantitative level. Until now, although the bipartite entanglement is well understood in many aspects, the multipartite entanglement is far from clear [5] and thus deserves profound understandings. In many-body quantum systems, one of the most important properties is that entanglement is monogamous, which means quantum entanglement cannot be freely shared among many parties. As quantified by the square of the concurrence [6], a three-qubit monogamy inequality was given by Coffman *et al.* [7] as $\mathcal{C}^2_{A|BC} \geq \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC}$. Recently, its N-qubit generalization was made by Osborne and Verstraete [8]. Moreover, using some other entanglement measures, similar monogamy inequalities have also established [9–14]. However, in these monogamous relations, only the single party partition $A_1|A_2A_3\cdots A_n$ is considered. Whether it can be generalized to other partitions, such as two parties cut $A_iA_j|A_kA_l\cdots A_n$, is still an open question to be answered.

On the other hand, the entanglement dynamical behavior under the influence of environment is also an important property of quantum systems. This is because, in realistic situations, quantum systems interact unavoidably with the environment and may lose their coherence. It was reported recently that an entangled state of two qubits interacting, respectively, with two local reservoirs would experience disentanglement in a finite time, even if the coherence is lost asymptotically [15–18]. This phenomenon is referred to as entanglement sudden death (ESD) and has received a lot of attentions both theoretically and experimentally (see a review paper [19] and references therein).

Recently, López *et al.* analyzed the entanglement transfer between two entangled cavity photons and their corresponding reservoirs and showed that the entanglement sudden birth (ESB) of reservoir-reservoir subsystem must happen whenever the ESD of cavity-cavity subsystem occurs [20]. However, in this process, *whether there exists an entanglement monogamy relation restricting the dynamical evolution* is awaited for further studies. Moreover, in the time interval where both the cavity-cavity entanglement and the reservoir-reservoir entanglement are zero, a subtle issue *where the initial entanglement really goes* is yet to be resolved, although the nonzero cavity-reservoir entanglement in this time window was pointed out.

In this paper, based on a new monogamy relation, the entanglement dynamics of two cavities interacting with individual reservoirs is studied. It is found that the genuine multipartite entanglement is involved in the dynamical process. Particularly, at a quantitative level, we show the initial cavity-cavity entanglement evolves completely to the genuine four-partite entanglement in the time interval between the ESD and the ESB. In addition, we also address the property of the genuine multipartite entanglement which exhibits in the block-block form under the bipartite two-qubit partition.

II. TWO-QUBIT RESIDUAL ENTANGLEMENT AND MONOGAMY RELATIONS

Let us first recapitulate the monogamy inequality in bipartite single-qubit partition, which can be written as [8]

$$\mathcal{C}^2_{A_1|A_2A_3\cdots A_n} \geq \mathcal{C}^2_{A_1A_2} + \mathcal{C}^2_{A_1A_3} + \cdots + \mathcal{C}^2_{A_1A_n}. \quad (1)$$

The entanglement between subsystems A_1 and $A_2A_3\cdots A_n$ is quantified by $\mathcal{C}^2_{A_1|A_2A_3\cdots A_n}(\rho_{A_1A_2A_3\cdots A_n}) = \min\sum_x p_x \tau_{A_1}(\rho^x_{A_1})$, where the $\tau_{A_1}(\rho^x_{A_1}) = 2[1-\mathrm{tr}(\rho^x_{A_1})^2]$ is the linear entropy [21,22] and the minimum runs over all the pure state decompositions. For the two-qubit quantum state $\rho_{A_iA_j}$, its entanglement is analytically expressed as $\mathcal{C}^2_{A_iA_j} = [\max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4})]^2$, with the decreasing nonnegative real numbers λ_i being the eigenvalues of the matrix $\rho_{ij}(\sigma_y \otimes \sigma_y)\rho^*_{ij}(\sigma_y \otimes \sigma_y)$ [6]. Based on the sum of the residual entanglements $M_{A_i} = \mathcal{C}^2_{A_i|R(A_i)} - \Sigma_j \mathcal{C}^2_{A_iA_j}$, a multipartite entanglement measure for pure states is introduced [23,24].

Now we analyze the multiqubit entanglement distribution under bipartite two-qubit partition. First, we consider a $2N$-qubit mixed state $\rho_{A_1A_1'A_2A_2'\cdots A_nA_n'}$ with the reduced density matrix $\rho_{A_iA_i'}$ being a rank-2 quantum state. For this quantum state, the following relations hold:

$$\mathcal{C}^2_{A_1A_1'|A_2A_2'\cdots A_nA_n'} \geq \sum_{i=2}^{n} \mathcal{C}^2_{A_1A_1'|A_iA_i'} \tag{2a}$$

$$\geq \sum_{i=2}^{n} \mathcal{C}^2_{A_1|A_iA_i'} + \sum_{i=2}^{n} \mathcal{C}^2_{A_1'|A_iA_i'} \tag{2b}$$

$$\geq \sum_{i=2}^{n} (\mathcal{C}^2_{A_1A_i} + \mathcal{C}^2_{A_1A_i'} + \mathcal{C}^2_{A_1'A_i} + \mathcal{C}^2_{A_1'A_i'}). \tag{2c}$$

In the derivation of the above inequalities, we have used the property that A_iA_i' is equivalent to a single qubit and the monogamy relation in Eq. (1). We here refer to the inequalities (2a) and (2b) as the *strong monogamy relations* and the inequality (2c) as the *weak monogamy relation*. In the rank-2 case, we define the two-qubit residual entanglement as

$$M_{A_iA_i'}(\rho_{A^{\otimes N}A'^{\otimes N}}) = \mathcal{C}^2_{A_iA_i'|R(A_iA_i')} - \sum \mathcal{C}^2_{ij}, \tag{3}$$

where $R(A_iA_i')$ denotes the subset of qubits other than A_iA_i' and i, j in the sum represent the qubit in the subsets $\{A_i, A_i'\}$ and $\{R(A_iA_i')\}$, respectively. It is obvious that the residual entanglement is zero when the $2N$-qubit state is separable under the two-qubit partition. As a nontrivial example, we consider the $2N$-qubit W state, which can be written as $|W\rangle_{2N} = \alpha_1|10\ldots00\rangle + \alpha_2|01\ldots00\rangle + \ldots + \alpha_{2n}|00\ldots01\rangle$. For this quantum state, we have $\mathcal{C}^2_{A_1A_1'|R(A_1A_1')} = 4\sum_{i=1}^{2}\sum_{j=3}^{2n}|\alpha_i|^2|\alpha_j|^2$ and $\mathcal{C}^2_{ij} = 4|\alpha_i|^2|\alpha_j|^2$. Then, according to Eq. (3), the two-qubit residual entanglement is zero. Since the square of the concurrence is a good entanglement measure for two-qubit quantum state, the nonzero residual entanglement $M_{A_iA_i'}$ implies the existence of multipartite entanglement. While for the two-qubit partition of rank-3 and rank-4 cases, the monogamy relation in Eq. (2) may not hold [25].

III. ENTANGLEMENT EVOLUTION IN MULTIPARTITE CAVITY-RESERVOIR SYSTEMS

In Ref. [20], López *et al.* analyzed the entanglement dynamics of two cavities interacting with independent reservoirs. The initial quantum state of the composite system is $|\Phi_0\rangle = (\alpha|00\rangle + \beta|11\rangle)_{c_1c_2}|00\rangle_{r_1r_2}$, where the two entangled cavity photons are in a Bell-like state and their corresponding dissipation reservoirs are in the vacuum states. The interaction Hamiltonian of a single cavity and an N-mode reservoir is $H = \hbar\omega a^\dagger a + \hbar\sum_{k=1}^{N}\omega_k b_k^\dagger b_k + \hbar\sum_{k=1}^{N} g_k(ab_k^\dagger + b_k a^\dagger)$. Under the unitary evolution $U(H,t) = U_{c_1r_1}(H,t) \otimes U_{c_2r_2}(H,t)$, the output state is given by

$$|\Phi_t\rangle = \alpha|0000\rangle_{c_1r_1c_2r_2} + \beta|\phi_t\rangle_{c_1r_1}|\phi_t\rangle_{c_2r_2}, \tag{4}$$

where $|\phi_t\rangle = \xi(t)|10\rangle_{cr} + \chi(t)|01\rangle_{cr}$ and the amplitudes $\xi(t) = \exp(-\kappa t/2)$ and $\chi(t) = [1-\exp(-\kappa t)]^{1/2}$ in the large N limit. For this dynamical process, López *et al.* disclosed an intrinsic connection between the ESD of the cavities and the ESB of the reservoirs. However, it is not clear whether one can establish a quantitative relation of the entanglements in different subsystems in the process. Furthermore, it is still a subtle issue where the entanglement really goes in the time window between the ESD and the ESB.

We first show that an entanglement monogamy relation exists and restricts the dynamical process of the multipartite systems. The reduced density matrix of a single cavity with its reservoir is $\rho_{c_1r_1}(t) = U_{c_1r_1}[\rho_{c_1r_1}(0)]U^\dagger_{c_1r_1}$, where $\rho_{c_1r_1}(0) = |\alpha|^2|00\rangle\langle 00| + |\beta|^2|10\rangle\langle 10|$ is a rank-2 two-qubit state. Since the unitary operation does not change the rank of the matrix, the $\rho_{c_1r_1}(t)$ is also a rank-2 density matrix. Therefore, the entanglement monogamy relations under the partition $c_1r_1|c_2r_2$ always hold in the dynamical procedure. Particularly, we have

$$\mathcal{C}^2_{c_1r_1|c_2r_2}(t) \geq \mathcal{C}^2_{c_1c_2}(t) + \mathcal{C}^2_{r_1r_2}(t) + \mathcal{C}^2_{c_1r_2}(t) + \mathcal{C}^2_{c_2r_1}(t), \tag{5}$$

where the two-qubit entanglements are

$$\mathcal{C}^2_{c_1c_2}(t) = 4[\max(|\alpha\beta\xi^2| - |\beta\xi\chi|^2, 0)]^2,$$

$$\mathcal{C}^2_{r_1r_2}(t) = 4[\max(|\alpha\beta\chi^2| - |\beta\xi\chi|^2, 0)]^2,$$

$$\mathcal{C}^2_{c_1r_2}(t) = \mathcal{C}^2_{c_2r_1}(t) = 4[\max(|\alpha\beta\xi\chi| - |\beta\xi\chi|^2, 0)]^2. \tag{6}$$

Here, the bipartite entanglements are quantified by the square of the concurrence rather than the concurrence in the analysis of López *et al.* It should be emphasized that, once the initial state is given, the bipartite entanglement $\mathcal{C}^2_{c_1r_1|c_2r_2}(\Phi_t) = 4|\alpha\beta|^2$ is invariant in the entanglement evolution, where the invariance property of entanglement under local unitary operations is used.

In Ref. [20], the multipartite entanglement is quantified by the multipartite concurrence C_N [26]. However, C_N is unable to characterize completely the genuine multipartite entanglement. For example, when the quantum state is a tensor product of two Bell states, C_N is nonzero. In this paper, we consider the two-qubit residual entanglement

$$M_{c_1r_1}(\Phi_t) = \mathcal{C}^2_{c_1r_1|c_2r_2}(t) - \sum \mathcal{C}^2_{ij}(t), \tag{7}$$

where $i \in \{c_1, r_1\}$ and $j \in \{c_2, r_2\}$. This quantity cannot only validate the monogamy relation, but also serve as an *indicator* of genuine multipartite entanglement in the dynamical process. According to the expression of $|\Phi_t\rangle$ in Eq. (4), one can deduce that all its three-tangles [7] $\tau_3(\rho_{ijk}) = 0$ because ρ_{ijk} can be written as the mix of a W state and a product state. Therefore, the nonzero $M_{c_1r_1}(\Phi_t)$ indicates only the genuine four-qubit entanglement. In Fig. 1, we plot the residual entanglement $M_{c_1r_1}$ as a function of the initial-state amplitude $|\alpha|$ and the dissipation time κt. For a given value of the α, the $M_{c_1r_1}(\kappa t)$ changes from zero to a maximum value and then decreases asymptotically to zero when $\kappa t \to \infty$. Moreover, the maximum values of $M_{c_1r_1}(\kappa t)$ occur in the time $\kappa t = \ln 2$ being independent of the amplitude α. For all possible α, the maximum of the residual entanglement is

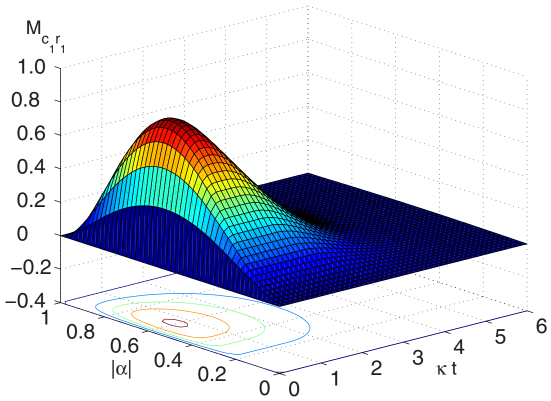

FIG. 1. (Color online) Two-qubit residual entanglement $M_{c_1r_1}(\Phi_t)$ vs the real parameters $|\alpha|$ and κt in the entanglement evolution.

$M_{c_1r_1}(\alpha,\ln 2)=(13\sqrt{13}-19)/34\approx 0.819\,77$, where $|\alpha|=[(9+\sqrt{13})/34]^{1/2}\approx 0.608\,89$.

Now, we look into the subtle issue where the initial entanglement goes in the time interval when both cavity-cavity and reservoir-reservoir entanglements are zero. We choose the initial-state parameter $\alpha=1/\sqrt{10}$ which is the same as that in Ref. [20] and for this value, there is such a time window. In Fig. 2, we plot the two-qubit residual entanglement $M_{c_1r_1}$ and related bipartite concurrences $\mathcal{C}^2$ against the parameter κt. The bipartite entanglement $\mathcal{C}^2_{c_1r_1|c_2r_2}(\kappa t)$ in the process is a conserved quantity (=0.36) and the monogamy relation in Eq. (5) restricts the entanglement evolution. The two-qubit residual entanglement $M_{c_1r_1}$ changes from zero to the maximum 0.36 in the time $[0,-\ln(2/3)]$, then the value keeps unchanged until $\kappa t=\ln 3$, and finally the $M_{c_1r_1}$ decreases asymptotically to zero as the time $\kappa t\to\infty$. This indicates that the genuine multipartite entanglement is always involved in the dynamical process. Particularly, in the plateau of $\kappa t\in[-\ln(2/3),\ln 3]$ where all the $\mathcal{C}^2_{ij}(t)$ in Eq. (7) are zero, the initial entanglement $\mathcal{C}^2_{c_1c_2}(0)=0.36$ transfers completely to the genuine four-qubit entanglement in the composite system (note that all the three-tangles are zero). In this region, the $M_{c_1r_1}$ is just the $\mathcal{C}^2_{c_1r_1|c_2r_2}$ and is entanglement monotone, being able to characterize the genuine four-qubit entanglement. For other initial-state amplitudes satisfying $|\alpha|<|\beta|/2$, there is also a plateau of $M_{c_1r_1}(\kappa t)$ (see Fig. 1) whose width and value are $\kappa t_{\mathrm{w}}=\ln(|\beta/\alpha|-1)$ and $M_{c_1r_1}=4|\alpha\beta|^2$, respectively. After a direct comparison, we can get that the value is equal to the initial cavity-cavity entanglement ($\mathcal{C}^2_{c_1c_2}(0)=4|\alpha\beta|^2$) and the width is just the time window [20] between the ESD of cavities and the ESB of reservoirs. Here, according to Eq. (6), one can prove further $\mathcal{C}^2_{c_1r_2}(t)=\mathcal{C}^2_{c_2r_1}(t)=0$ in the interval. Therefore, we conclude that the initial entanglement evolves completely to the genuine four-partite entanglement in the time window between the ESD of cavity subsystem and the ESB of reservoir subsystem. We also wish to indicate that the nonzero $\mathcal{C}^2_{c_1r_1}(t)$ in Fig. 2 does not come from the initial entanglement $\mathcal{C}^2_{c_1c_2}(0)$, but is generated by a "local" unitary operation $U_{c_1r_1}(H,t)$ with the partition $c_1r_1|c_2r_2$.

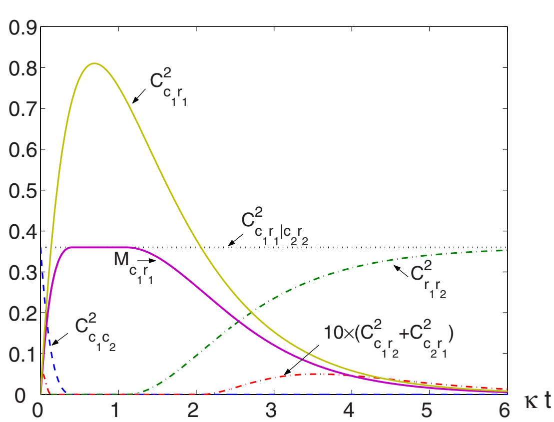

FIG. 2. (Color online) The two-qubit residual entanglement $M_{c_1r_1}$ (purple solid line) vs the time evolution parameter κt in comparison to bipartite entanglements $\mathcal{C}^2_{c_1c_2}$ (blue dashed line), $\mathcal{C}^2_{r_1r_2}$ (green dot-dashed line), $10(\mathcal{C}^2_{c_1r_2}+\mathcal{C}^2_{c_2r_1})$ (red dot-dashed line), $\mathcal{C}^2_{c_1r_1|c_2r_2}$ (black doted line), and $\mathcal{C}^2_{c_1r_1}$ (yellow solid line) in quantum state $|\Psi_t\rangle$ for which $\alpha=1/\sqrt{10}$ [20].

IV. BLOCK-BLOCK ENTANGLEMENT VERSUS GENUINE MULTIPARTITE ENTANGLEMENT

The multiqubit entanglement property in the plateau region is worthy of a further analysis. For the initial state with $\alpha=1/\sqrt{10}$, the output state of the evolution can be written as

$$|\Psi_t\rangle=\frac{1}{\sqrt{10}}|0000\rangle_{c_1r_1c_2r_2}+\frac{3}{\sqrt{10}}|\psi_t\rangle_{c_1r_1}|\psi_t\rangle_{c_2r_2}, \qquad (8)$$

where $|\psi_t\rangle=\xi(t)|10\rangle+\chi(t)|01\rangle$. Its genuine four-qubit entanglement is evaluated in bipartite block-block form, i.e., the entanglement measure $M_{c_1r_1}(\Psi_t)=\mathcal{C}^2_{c_1r_1|c_2r_2}(\Psi_t)=0.36$ characterizes the *genuine block-block entanglement* between subsystems c_1r_1 and c_2r_2. The case for other α with plateau region is similar.

Although the three-tangles $\tau_3(\rho_{ijk})$ and the related $\mathcal{C}^2_{ij}$ in the plateau region are zero, the three-qubit subsystems exhibit *genuine qubit-block* entanglements and the relation

$$\mathcal{C}^2_{c_1r_1|c_2r_2}(t)=\mathcal{C}^2_{c_1|c_2r_2}(t)+\mathcal{C}^2_{r_1|c_2r_2}(t) \qquad (9)$$

holds, in which $\mathcal{C}^2_{c_1|c_2r_2}(t)=4|\alpha\beta|^2|\xi(t)|^2$ and $\mathcal{C}^2_{r_1|c_2r_2}(t)=4|\alpha\beta|^2|\chi(t)|^2$ being equivalent to the mixed state one-tangle [8]. This qubit-block entanglement is similar to that of mixed states in Refs. [27,28] which are entangled but without the (two-qubit) concurrences and three-tangles. For this kind of entanglement, our understanding is that it comes from the genuine multipartite entanglement in its purified state [28]. Here, Eq. (9) actually presents for the first time a quantitative relation for understanding the qubit-block entanglement, with a schematic diagram being depicted in Fig. 3.

V. DISCUSSION AND CONCLUSION

Entanglement monogamy is a fundamental property of multipartite entangled states. We argue that the violation of the monogamy relations in Eq. (2) for higher rank cases is

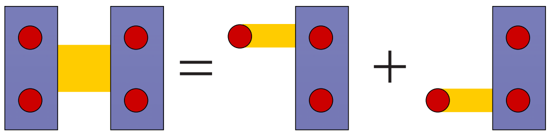

FIG. 3. (Color online) The relation between the block-block entanglement and qubit-block entanglement in the plateau region.

because the square of the concurrence does not have the additivity, i.e., $\mathcal{C}^2_{A_1A_1'|A_2A_2'} \neq \mathcal{C}^2_{A_1A_2} + \mathcal{C}^2_{A_1'A_2'}$ for the tensor product of two Bell states. The von Neumann entropy has this additivity property, however, it has the negative residual entanglement for multipartite systems [29]. How to define an additive entanglement measure with nonnegative residual entanglement is still challenging.

The monogamy relations in Eq. (2) can be applied to other systems [30] only if the individual system environment is in a rank-2 quantum state and the evolution has a tensor structure $U(H,t) = U_{S_1E_1}(H,t) \otimes U_{S_2E_2}(H,t) \otimes \cdots \otimes U_{S_nE_n}(H,t)$. Moreover, based on this relation, one can derive other useful monogamy inequality. For example, if the initial state of a three cavity-reservoir composite system is $|\Psi_0\rangle = (\alpha|000\rangle + \beta|111\rangle)_c|000\rangle_r$ and the individual cavity-reservoir interaction is the same as the previous one, we can derive

$$\mathcal{C}^2_{c_1r_1|c_2r_2c_3r_3}(0) \geq \tau_3(\rho_{c_1c_2c_3}(t)) + \tau_3(\rho_{r_1r_2r_3}(t)), \quad (10)$$

where $\mathcal{C}^2_{c_1r_1|c_2r_2c_3r_3}(0) = 4|\alpha\beta|^2$ gives an upper bound for the three-tangles in the entanglement evolution.

In conclusion, we show that a monogamy relation restricts the entanglement evolution of two cavities with individual reservoirs. Moreover, based on the relation, we find the initial-state entanglement evolves completely to the genuine four-partite entanglement in the time interval between the ESD of cavity-cavity entanglement and the ESB of the reservoir-reservoir entanglement. In addition, we give a quantitative relation between the block-block entanglement and the qubit-block entanglement.

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