Conformally symmetric traversable wormholes

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Exact solutions of traversable wormholes are found under the assumption of spherical symmetry and the existence of a *nonstatic* conformal symmetry, which presents a more systematic approach in searching for exact wormhole solutions. In this work, a wide variety of solutions are deduced by considering choices for the form function, a specific linear equation of state relating the energy density and the pressure anisotropy, and various phantom wormhole geometries are explored. A large class of solutions impose that the spatial distribution of the exotic matter is restricted to the throat neighborhood, with a cutoff of the stress-energy tensor at a finite junction interface, although asymptotically flat exact solutions are also found. Using the "volume integral quantifier," it is found that the conformally symmetric phantom wormhole geometries may, in principle, be constructed by infinitesimally small amounts of averaged null energy condition violating matter. Considering the tidal acceleration traversability conditions for the phantom wormhole geometry, specific wormhole dimensions and the traversal velocity are also deduced.

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I. INTRODUCTION

Traversable wormholes are hypothetical tunnels in space and time [1], permitting effective "superluminal travel," although the speed of light is not *locally* surpassed [2], and induce closed timelike curves [3], apparently violating causality. These geometries are supported by material, denoted as *exotic matter*, that violates the null energy condition. In fact, they violate all the known pointwise energy conditions and averaged energy conditions, which are fundamental to the singularity theorems and theorems of classical black hole thermodynamics [4]. Although classical forms of matter are believed to obey these energy conditions [5], it is a well-known fact that they are violated by certain quantum fields, amongst which we may refer to the Casimir effect. The literature is rather extensive in candidates for wormhole spacetimes, and one may mention several cases, ranging from scalar fields [6], wormhole geometries in higher dimensions [7,8], spacetimes in Brans-Dicke theory [9], solutions supported by semiclassical gravity (see Ref. [10] and references therein), geometries in the context of nonlinear electrodynamics [11], to wormholes supported by equations of state responsible for the accelerated expansion of the universe [12–15], etc. We refer the reader to Refs. [4,16] and references therein for more details.

Essentially, wormhole physics is a specific example of adopting the reverse philosophy of solving the Einstein field equations, by first constructing the spacetime metric, then deducing the stress-energy tensor components. It is interesting to consider a more systematic approach in searching for exact solutions. For instance, one may adopt the approach outlined in Refs. [17,18], where the spacetime is assumed to be spherically symmetric and to possess a conformal symmetry. If the vector ξ generates this conformal symmetry, then the metric $\mathbf{g}$ is conformally mapped onto itself along ξ. This is translated by the following relationship:

$$\mathcal{L}_{\xi}\mathbf{g} = \psi\mathbf{g}, \tag{1}$$

where $\mathcal{L}$ is the Lie derivative operator and ψ is the conformal factor.

As emphasized in Ref. [18], despite being essentially geometric in character, this approach is physically justifiable, namely, it is a generalization of self-similarity in hydrodynamics; and it also generalizes the property of the incompressible Schwarzschild interior solution which is conformally flat and therefore is characterized by an additional symmetry. Indeed, if the energy density is nonconstant the spacetime is no longer conformally flat; see e.g. [19]. The above conformal symmetry has been of particular interest in the context of static and spherically symmetric perfect fluid solutions. In general, an equation of state needs to be specified to close the system of differential equations; this choice, however, is an essentially

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nongeometric one. On the other hand, the conformal symmetry can be regarded as a geometrical equation of state closing the system of equations.

The approach outlined in Ref. [17] considers a static conformal symmetry, i.e., a static ξ, which is essentially responsible for the singular solutions at the stellar centers found. For this reason, nonstatic conformal symmetries, i.e., nonstatic ξ and static ψ, were considered in Ref. [18], and we shall essentially follow this approach in this work, as it provides a wider range of exact wormhole solutions. We do, however, emphasize that the singular character of the solutions at the stellar center in Ref. [17] need not be problematic in wormhole physics, due to the absence of a center, as the radial coordinate possesses a minimum value, r_0, denoted as the wormhole throat. Thus, we shall also briefly analyze the static ξ solution, namely, in the phantom wormhole context. It is interesting to note that an exact analytical solution describing the interior of a charged strange quark star was found [20]; solutions were also explored in brane worlds [21], and also in the context of the galactic rotation curves [22].

This paper is outlined in the following manner: In Sec. II, the field equations are presented; in Sec. III, exact general solutions are deduced using nonstatic conformal symmetries; in Sec. IV, a wide variety of solutions are deduced by considering choices for the form function, a specific linear equation of state relating the energy density and the pressure anisotropy, and various phantom wormhole geometries are explored; and in Sec. V we conclude.

II. FIELD EQUATIONS

The spacetime metric representing a spherically symmetric and static wormhole is given by

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-b(r)/r} + r^2 d\Omega^2, \tag{2}$$

where $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate, r, denoted as the redshift function, and the form function, respectively [1]. The radial coordinate has a range that increases from a minimum value at r_0, corresponding to the wormhole throat, to a, where the interior spacetime will be joined to an exterior vacuum solution. Specific asymptotically flat wormhole geometries will also be considered, where r extends from the throat out to infinity.

To be a wormhole solution, several conditions need to be imposed [1]: First, one needs to verify the absence of event horizons, so that the redshift function $\Phi(r)$ is finite throughout the range of interest. Second, the mathematics of embedding imposes a flaring-out condition, translated by the following condition, $(b'r - b)/b^2 < 0$, which reduces to $b' < 1$ at the throat, and taking into account the field equations, it is this condition that implies the violation of the null energy condition, as shown below. The conditions $(1 - b/r) > 0$ and $b(r_0) = r_0$ at the throat are also imposed.

Using the Einstein field equation, $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$ (with $\kappa^2 = 8\pi$ and $c = G = 1$), we obtain the following nonzero stress-energy tensor components:

$$\rho(r) = \frac{1}{\kappa^2}\frac{b'}{r^2}, \tag{3}$$

$$p_r(r) = \frac{1}{\kappa^2}\left[2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r} - \frac{b}{r^3}\right], \tag{4}$$

$$p_t(r) = \frac{1}{\kappa^2}\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r(r-b)}\Phi' - \frac{b'r-b}{2r^2(r-b)}\right], \tag{5}$$

where $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure, and $p_t(r)$ is the lateral pressure measured orthogonally to the radial direction. Note that the conservation of the stress-energy tensor, $T^{\mu\nu}{}_{;\nu} = 0$, provides the following relationship:

$$p_r' = \frac{2}{r}(p_t - p_r) - (\rho + p_r)\Phi'. \tag{6}$$

A fundamental property of wormholes is the violation of the null energy condition (NEC), $T_{\mu\nu}k^\mu k^\nu \geq 0$, where k^μ is *any* null vector [1]. From Eqs. (3) and (4) one verifies

$$\rho(r) + p_r(r) = \frac{1}{8\pi}\left[\frac{b'r-b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\right]. \tag{7}$$

Taking into account the flaring-out condition and the finite character of $\Phi(r)$, evaluated at the throat r_0, we have $\rho + p_r < 0$. Matter that violates the NEC is denoted *exotic matter*. More specifically, in terms of the form function evaluated at the throat, we have $b'(r_0) < 1$.

III. CONFORMAL SYMMETRY

Applying a systematic approach in order to deduce exact solutions, we shall take into account the method used in Ref. [18], where the static and spherically symmetric spacetime possesses a nonstatic conformal symmetry. It should be emphasized that neither ξ nor ψ need to be static even though one considers a static metric. Note that Eq. (1) takes the following form:

$$g_{\mu\nu,\alpha}\xi^\alpha + g_{\alpha\nu}\xi^\alpha{}_{,\mu} + g_{\mu\alpha}\xi^\alpha{}_{,\nu} = \psi g_{\mu\nu}. \tag{8}$$

We shall follow closely the assumptions made in Ref. [18], where the condition

$$\xi = \alpha(t,r)\partial_t + \beta(t,r)\partial_r \tag{9}$$

is considered, and the conformal factor is static, i.e., $\psi = \psi(r)$.

Taking into account metric (2), then Eq. (8) provides the following solutions:

$$\alpha = A + \frac{kt}{2}, \qquad \beta = \frac{1}{2}Br\sqrt{1 - \frac{b(r)}{r}} \tag{10}$$

and

$$\psi(r) = B\sqrt{1 - \frac{b(r)}{r}}, \tag{11}$$

$$e^{2\Phi(r)} = C^2 r^2 \exp\left(-\frac{2k}{B}\int \frac{dr}{r\sqrt{1-\frac{b(r)}{r}}}\right), \tag{12}$$

where A, B, C, and k are constants. Note that, without a loss of generality, one may consider $A = 0$ as $A\partial_t$ is a Killing vector, and $B = 1$ by rescaling ξ and ψ in the following manner: $\xi \to B^{-1}\xi$ and $\psi \to B^{-1}\psi$, which leaves Eq. (8) invariant. Thus, Eqs. (9) and (10) reduce to

$$\xi = \tfrac{1}{2}kt\partial_t + \tfrac{1}{2}\psi(r)r\partial_r \tag{13}$$

and Eqs. (11) and (12) take the form

$$b(r) = r[1 - \psi^2(r)], \tag{14}$$

$$\Phi(r) = \frac{1}{2}\ln(C^2r^2) - k\int \frac{dr}{r\sqrt{1-\frac{b(r)}{r}}}. \tag{15}$$

An interesting feature of these solutions that immediately stands out, by taking into account Eq. (14), is that the conformal factor is zero at the throat, i.e., $\psi(r_0) = 0$.

The existence of conformal motions imposes strong constraints on the wormhole geometry, so that the stress-energy tensor components are written solely in terms of the conformal function, and take the following form:

$$\rho(r) = \frac{1}{\kappa^2 r^2}(1 - \psi^2 - 2r\psi\psi'), \tag{16}$$

$$p_r(r) = \frac{1}{\kappa^2 r^2}(3\psi^2 - 2k\psi - 1), \tag{17}$$

$$p_t(r) = \frac{1}{\kappa^2 r^2}(\psi^2 - 2k\psi + k^2 + 2r\psi\psi'). \tag{18}$$

The NEC violation, Eq. (7), for this case is given by

$$\rho(r) + p_r(r) = \frac{1}{\kappa^2 r^2}[2\psi(\psi - k) - r(\psi^2)'], \tag{19}$$

which, evaluated at the throat, imposes the following condition: $(\psi^2)' > 0$.

Several solutions analyzed in this work are not asymptotically flat, so that one needs to match these interior geometries to an exterior vacuum spacetime. The spatial distribution of the exotic matter is restricted to the throat neighborhood, so that the dimensions of these wormholes are not arbitrarily large. For simplicity, consider that the exterior vacuum solution is the Schwarzschild spacetime, given by the following metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2. \tag{20}$$

Note that the matching occurs at a radius greater than the event horizon $r_b = 2M$, i.e., $a > 2M$. The Darmois-Israel formalism [23] then provides the following expressions for the surface stresses of a dynamic thin shell [12,24]:

$$\sigma = -\frac{2}{\kappa^2 a}\left(\sqrt{1 - \frac{2M}{a} + \dot{a}^2} - \sqrt{\psi^2(a) + \dot{a}^2}\right), \tag{21}$$

$$\mathcal{P} = \frac{1}{\kappa^2 a}\left[\frac{1 - \frac{M}{a} + \dot{a}^2 + a\ddot{a}}{\sqrt{1 - \frac{2M}{a} + \dot{a}^2}} - \frac{(2 + \frac{k}{\psi(a)})(\psi^2(a) + \dot{a}^2) + a\ddot{a} - \frac{\psi'(a)a\dot{a}^2}{\psi(a)}}{\sqrt{\psi^2(a) + \dot{a}^2}}\right], \tag{22}$$

where the overdot denotes a derivative with respect to the proper time τ. σ and $\mathcal{P}$ are the surface energy density and the tangential surface pressure, respectively. The static case is given by taking into account $\dot{a} = \ddot{a} = 0$ [25].

IV. SPECIFIC SOLUTIONS

A. Specific form functions

In this section, we explore a wide variety of wormhole geometries by considering specific form functions.

1. $b(r) = r_0$

A particularly interesting case is considering a zero energy density, which implies a constant form function. Considering $b(r) = r_0$, and taking into account Eq. (15), we have

$$e^{2\Phi(r)} = C^2 r^2 \left(r - \frac{r_0}{2} + r\sqrt{1 - \frac{r_0}{r}}\right)^{-2k}. \tag{23}$$

An interesting feature of this geometry is the specific case of $k = 1$, which reflects an asymptotically flat spacetime, by normalizing the constant $C^2 = 2$, i.e., $e^{2\Phi(r)} \to 1$ as $r \to +\infty$. For $k \neq 1$, we need to match this interior wormhole geometry to an exterior vacuum spacetime at a junction interface, where the surface stresses are provided by Eqs. (21) and (22).

The stress-energy tensor components are given by $\rho = 0$ and

$$p_r = -\frac{1}{\kappa^2}\left(\frac{3r_0 - 2r}{r^3} + \frac{2k}{r^2}\sqrt{1 - \frac{r_0}{r}}\right), \tag{24}$$

$$p_t = \frac{1}{\kappa^2}\left(\frac{1 + k^2}{r^2} - \frac{2k}{r^2}\sqrt{1 - \frac{r_0}{r}}\right), \tag{25}$$

where $p_r \to 0$ and $p_t \to 0$ as $r \to +\infty$.

2. $b(r) = r_0^2/r$

Consider the case of $b(r) = r_0^2/r$, which implies a negative energy density. From Eq. (15) we have

$$e^{2\Phi(r)} = C^2 r^2 (r + \sqrt{r^2 - r_0^2})^{-2k}. \tag{26}$$

Note that, as in the previous example, an asymptotically flat spacetime is found by considering $k = 1$ and $C^2 = 2$.

The stress-energy tensor components are given by

$$\rho = -\frac{1}{\kappa^2}\frac{r_0^2}{r^4}, \tag{27}$$

$$p_r = -\frac{1}{\kappa^2}\left(\frac{3r_0^2 - 2r^2}{r^4} + \frac{2k}{r^2}\sqrt{1 - \frac{r_0^2}{r^2}}\right), \tag{28}$$

$$p_t = \frac{1}{\kappa^2}\left[\frac{1}{r^2}\left(1 + k^2 + \frac{r_0^2}{r^2}\right) - \frac{2k}{r^2}\sqrt{1 - \frac{r_0^2}{r^2}}\right], \tag{29}$$

which tend to zero as $r \to +\infty$.

3. $b(r) = r_0 + \gamma^2 r_0(1 - r_0/r)$

An interesting form function is $b(r) = r_0 + \gamma^2 r_0(1 - r_0/r)$ [10], with $0 < \gamma^2 < 1$, which provides a positive energy density. From Eq. (15) we have

$$e^{2\Phi(r)} = C^2 r^2\left[r - \frac{r_0}{2}(1 + \gamma^2) + \sqrt{(r - r_0)(r - \gamma^2 r_0)}\right]^{-2k}, \tag{30}$$

and as before, an asymptotically flat spacetime is found by considering $k = 1$ and $C^2 = 2$.

The stress-energy tensor components are given by

$$\rho = \frac{1}{\kappa^2}\frac{\gamma^2 r_0^2}{r^4}, \tag{31}$$

$$p_r = -\frac{1}{\kappa^2}\left\{\frac{3r_0 r(1 + \gamma^2) - 3\gamma^2 r_0^2 - 2r^2}{r^4} + \frac{2k}{r^2}\sqrt{1 - \frac{r_0}{r}[1 + \gamma^2(1 - \frac{r_0}{r})]}\right\}, \tag{32}$$

$$p_t = \frac{1}{\kappa^2}\left\{\frac{1}{r^2}\left(1 + k^2 - \gamma^2\frac{r_0^2}{r^2}\right) - \frac{2k}{r^2}\sqrt{1 - \frac{r_0}{r}\left[1 + \gamma^2\left(1 - \frac{r_0}{r}\right)\right]}\right\}, \tag{33}$$

which tend to zero as $r \to +\infty$.

B. Specific equation of state: $\rho = \alpha(p_t - p_r)$

As was pointed out in the Introduction, wormhole spacetimes are constructed mainly by designing an appropriate metric and followed by deducing the stress-energy tensor. An interesting solution is obtained by assuming that the anisotropy and the energy density are related by a linear equation of state. It turns out that an equation of state of the form $\rho = \alpha(p_t - p_r)$ indeed yields exact solutions.

In this case the resulting differential equation for the conformal factor yields a rather complicated expression. However, the form function admits the following solution:

$$b(r) = \frac{1 - \alpha(1 + k^2)}{(1 - 2\alpha)} r_0^{(1-2\alpha)/(1+\alpha)} r^{3\alpha/(1+\alpha)} - \frac{(1 - k^2)\alpha}{(1 - 2\alpha)} r. \tag{34}$$

It is particularly interesting to note that this form function satisfies the required wormhole conditions for a wide class of parameters α, which makes this model quite generic. Such wormhole models, where one assumes the above equation of state for the matter, have not been studied previously.

C. Conformally symmetric phantom wormhole

An interesting case is that of traversable wormholes supported by the dark energy equation of state in the phantom regime, $\omega = p_r/\rho < -1$. Several physical properties and characteristics have been extensively explored, and we refer the reader to Refs. [12,13,15]. Note that, taking into account the field equations (3) and (4), the phantom energy equation of state imposes the following relationship:

$$\Phi'(r) = \frac{b + \omega r b'}{2r^2(1 - b/r)}. \tag{35}$$

For the case of the nonstatic conformal factor we could only find one exact solution in terms of ψ, namely, if $\omega = -3$, which is treated in more detail below.

1. Static conformal symmetry

Let us first consider that the components of Eq. (9) depend only on the radial coordinate r. This corresponds to imposing $k = 0$ in the solutions given by Eqs. (13) and (15). Note that the relationship (14) still holds, but the expression (15) now provides the following redshift function:

$$e^{2\Phi(r)} = C^2 r^2. \tag{36}$$

Thus, the phantom energy differential equation (35), reduces to

$$\psi' = \frac{(1 + \omega) - (3 + \omega)\psi^2}{2\omega r\psi}. \tag{37}$$

The solution for $\psi(r)$ is given by

$$\psi(r) = \pm\sqrt{C r^{-(3+\omega)/\omega} + \left(\frac{1 + \omega}{3 + \omega}\right)} \tag{38}$$

where C is a constant of integration. Note that the solution given by (38) holds for $\omega \neq 3$ only. This case must be considered separately (see below).

Taking into account Eq. (14) provides the following form function:

$$b(r) = \frac{r_0(1+\omega)}{(3+\omega)}\left(\frac{r}{r_0}\right)^{-3/\omega} + \frac{2r}{3+\omega}. \tag{39}$$

The constant of integration C is determined by imposing $b(r_0) = r_0$. Note that $b'(r_0) = 1/|\omega| < 1$, which obeys the flaring-out condition at the throat. One also needs to impose $b(r) > 0$ (see Ref. [16] for details) and $b(r) < r$, which is represented by the surface in Fig. 1. Thus, one needs to match this solution to an exterior spacetime at a junction interface, $a > 2M$.

For $\omega = -3$ the phantom energy differential equation simplifies to

$$\psi' = \frac{1}{3r\psi}, \tag{40}$$

and yields the following solution for the conformal factor:

$$\psi(r) = \pm\sqrt{\frac{2}{3}\log\left(\frac{r}{r_0}\right)}. \tag{41}$$

This gives the following form function:

$$b(r) = r\left[1 - \frac{2}{3}\log\left(\frac{r}{r_0}\right)\right], \tag{42}$$

where we have fixed the constant of integration as in the above. We verify that $b' = 1/3$, and since for a wormhole solution $b(r) > 0$ has to be imposed, we find that this solution needs to be matched to an exterior spacetime at some surface $r_0 < a < r_0 e^{3/2}$.

2. Nonstatic conformal symmetry

Let us now consider the specific nonstatic case where the energy density and the radial pressure given by (16) and (17), respectively, are related by the equation of state $\omega = -3$. In this case the conformal factor is given by

$$\psi(r) = \frac{1 + \mathcal{W}(x)}{k}, \qquad x = \frac{1}{e}\left(\frac{r_0}{r}\right)^{k^2/3}, \tag{43}$$

where $\mathcal{W}(x)$ is the Lambert $\mathcal{W}$ function [26]. As in the above, the constant of integration was chosen such that $\psi(r_0) = 0$; therefore, at the throat r_0 the form function satisfies $b(r_0) = r_0$. With the above solution one also verifies that the form function obeys $b'(r_0) < 1$. Moreover, let us also consider the case $k = 1$. The limit $r \to \infty$ corresponds to $x \to 0$, which by definition of the Lambert $\mathcal{W}$ function implies $\mathcal{W}(0) = 0$. Hence, we conclude that the spacetime is also asymptotically flat and no further exterior matching is required for this conformal wormhole solution.

3. Volume integral quantifier

It is also interesting to consider the "volume integral quantifier," which provides information on the total amount of matter violating the averaged null energy condition in the spacetime. This is defined by $I_V = \int[\rho(r) + p_r(r)]dV$ (see Ref. [27] for details), and with a cutoff of the stress-energy at a, is given by

$$\begin{aligned} I_V &= \left[r\left(1 - \frac{b}{r}\right)\ln\left(\frac{e^{2\Phi}}{1 - b/r}\right)\right]_{r_0}^{a} \\ &\quad - \int_{r_0}^{a}(1 - b')\left[\ln\left(\frac{e^{2\Phi}}{1 - b/r}\right)\right]dr \\ &= \int_{r_0}^{a}(r - b)\left[\ln\left(\frac{e^{2\Phi}}{1 - b/r}\right)\right]' dr. \end{aligned} \tag{44}$$

Taking into account the redshift function, Eq. (36), and the form function, Eq. (39), and recalling that $\omega < -1$, one obtains the following solution for the volume integral quantifier:

$$I_V = \left(\frac{1+\omega}{3+\omega}\right)\left[2a - r_0(3+\omega) + r_0(1+\omega)\left(\frac{a}{r_0}\right)^{-3/\omega}\right]. \tag{45}$$

Note that, for a parameter ω arbitrary close to -1, the volume integral quantifier would by itself become infinitesimally small, independently of the value of the matching radius a, although for this case the wormhole would flare out very slowly. Now taking the limit $a \to r_0$, one verifies that $I_V \to 0$. Therefore, as in the examples presented in Refs. [12,14], one verifies that, in principle, one may construct conformally symmetric phantom wormholes with vanishingly small amounts of phantom energy violating the averaged null energy condition.

4. Tidal acceleration restrictions

An interesting constraint on the wormhole dimensions, in particular, on the throat radius, may be inferred from the

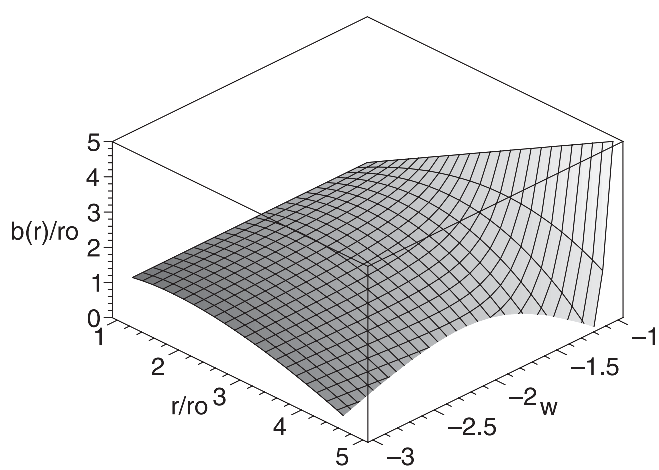

FIG. 1. The surface represents the adimensional form function $b(r)/r_0$. Note that to be a wormhole solution one needs to impose $0 < b(r) < r$.

tidal acceleration restrictions [1]. The latter constraints, as measured by a traveler moving radially through the wormhole, are given by the following inequalities:

$$\left|\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r-b}{2r(r-b)}\Phi'\right]\right| |\eta^{\hat{1}'}| \leq g_{\oplus}, \tag{46}$$

$$\left|\frac{\gamma^2}{2r^2}\left[v^2\left(b' - \frac{b}{r}\right) + 2(r-b)\Phi'\right]\right| |\eta^{\hat{2}'}| \leq g_{\oplus}, \tag{47}$$

where $\eta^{\hat{i}'}$ is the separation between two arbitrary parts of his body measured in the traveler's reference frame. We shall consider $|\eta^{\hat{i}'}| = |\eta|$, for simplicity. We refer the reader to Ref. [1] for details. The radial tidal constraint, inequality (46), constrains the redshift function; and the lateral tidal constraint, inequality (47), constrains the velocity with which observers traverse the wormhole. These inequalities are particularly simple at the throat, r_0,

$$|\Phi'(r_0)| \leq \frac{2g_{\oplus} r_0}{(1-b')|\eta|}, \tag{48}$$

$$\gamma^2 v^2 \leq \frac{2g_{\oplus} r_0^2}{(1-b')|\eta|}. \tag{49}$$

From the radial tidal condition (48) one verifies

$$r_0^2 \geq \left(\frac{1+\omega}{\omega}\right)\frac{|\eta|}{2g_{\oplus}}. \tag{50}$$

Now, considering the equality case for simplicity, assuming that $|\eta| \approx 2$ m along any spatial direction in the traveler's reference frame, and inserting c for clarity, one verifies that $r_0 \simeq c[(1+\omega)/(10\omega)]^{1/2}$. Note that one may obtain an arbitrary small wormhole throat radius by imposing that $\omega \rightarrow -1$.

Analogously, from the lateral tidal condition (49), and considering nonrelativistic velocities, i.e., $\gamma \approx 1$, one has

$$v^2 \leq \left(\frac{\omega}{1+\omega}\right)\frac{2r_0^2 g_{\oplus}}{|\eta|}. \tag{51}$$

Considering the equality case in Eq. (50), one immediately verifies the following consistency relationship, $v \leq 1$.

V. SUMMARY AND CONCLUSION

The conventional manner of finding wormhole solutions is essentially to consider an interesting spacetime metric, and then deduce the stress-energy tensor components. In this work, we have considered a more systematic approach in searching for exact solutions, namely, by assuming spherical symmetry and the existence of a *nonstatic* conformal symmetry. A wide variety of solutions with the exotic matter restricted to the throat neighborhood and with a cutoff of the stress-energy tensor at a junction interface were deduced, and particular asymptotically flat geometries were also found. The specific solutions were deduced by considering choices for the form function, an equation of state relating the energy density and the anisotropy, and phantom wormhole geometries were also explored.

Although the assumption of a static conformal symmetry, i.e., with a static vector ξ, was found responsible for the singular solutions at the center, we emphasize that this is not problematic to wormhole physics, due to the absence of a center. A wide variety of the solutions found in this work were considered by choosing a nonstatic conformal symmetry, i.e., with a nonstatic ξ and static ψ. Note that this analysis could be generalized by imposing a nonstatic conformal function $\psi(r, t)$, where a wider variety of exact solutions may be found. However, this shall be analyzed in a future work.

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